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T. Y. Lam

Lectures on Modules and Rings

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T. Y. Lam

Lectures on Modules and Rings

With 43 Figures

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To Chee King

Juwen, Fumei, Juleen, Tsai Yu

Preface

Textbook writing must be one of the
cruelest of self-inflicted tortures.
– Carl Faith
Math Reviews **54**: 5281

So why didn't I heed the warning of a wise colleague, especially one who is a great expert in the subject of modules and rings? The answer is simple: I did not learn about it until it was too late!

My writing project in ring theory started in 1983 after I taught a year-long course in the subject at Berkeley. My original plan was to write up my lectures and publish them as a graduate text in a couple of years. My hopes of carrying out this plan on schedule were, however, quickly dashed as I began to realize how much material was at hand and how little time I had at my disposal. As the years went by, I added further material to my notes, and used them to teach different versions of the course. Eventually, I came to the realization that writing a single volume would not fully accomplish my original goal of giving a comprehensive treatment of basic ring theory.

At the suggestion of Ulrike Schmickler-Hirzebruch, then Mathematics Editor of Springer-Verlag, I completed the first part of my project and published the write-up in 1991 as *A First Course in Noncommutative Rings*, GTM 131, hereafter referred to as *First Course* (or simply *FC*). This volume contained a treatment of the Wedderburn-Artin theory of semisimple rings, Jacobson's theory of the radical, representation theory of groups and algebras, prime and semiprime rings, division rings, ordered rings, local and semilocal rings, culminating in the theory of perfect and semiperfect rings. The publication of this volume was accompanied several years later by that of *Exercises in Classical Ring Theory*, which contained full solutions of (and additional commentary on) all exercises in *FC*. For further topics in ring theory not yet treated in *FC*, the reader was referred to a forthcoming second volume, which, for lack of a better name, was tentatively billed as *A Second Course in Noncommutative Rings*.

One primary subject matter I had in mind for the second volume was that part of ring theory in which the consideration of modules plays a crucial role. While

an early chapter of *FC* on representation theory dealt with modules over finite-dimensional algebras (such as group algebras of finite groups over fields), the theory of modules over more general rings did not receive a full treatment in that text. This second volume, therefore, begins with the theory of special classes of modules (free, projective, injective, and flat modules) and the theory of homological dimensions of modules and rings. This material occupies the first two chapters. We then go on to present, in Chapter 3, the theory of uniform dimensions, complements, singular submodules and rational hulls; here, the notions of essentiality and denseness of submodules play a key role. In this chapter, we also encounter several interesting classes of rings, notably Rickart rings and Baer rings, Johnson's nonsingular rings, and Kasch rings, not to mention the hereditary and semihereditary rings that have already figured in the first two chapters.

Another important topic in classical ring theory not yet treated in *FC* was the theory of rings of quotients. This topic is taken up in Chapter 4 of the present text, in which we present Ore's theory of noncommutative localization, followed by a treatment of Goldie's all-important theorem characterizing semiprime right Goldie rings as right orders in semisimple rings. The latter theorem, truly a landmark in ring theory, brought the subject into its modern age, and laid new firm foundations for the theory of noncommutative noetherian rings. Another closely allied theory is that of maximal rings of quotients, due to Findlay, Lambek and Utumi. This theory has a universal appeal, since every ring has a maximal (left, right) ring of quotients. Chapter 5 develops this theory, taking full advantage of the material on injective and rational hulls of modules presented in the previous chapters. In this theory, the theorems of Johnson and Gabriel characterizing rings whose maximal right rings of quotients are von Neumann regular or semisimple may be viewed as analogues of Goldie's theorem mentioned earlier.

One theme that runs like a red thread through Chapters 1–5 is that of self-injective rings. The noetherian self-injective rings, commonly known as quasi-Frobenius (or QF) rings, occupy an especially important place in ring theory. Group algebras of finite groups provided the earliest nontrivial examples of QF rings; in fact, they are examples of finite-dimensional Frobenius algebras that were studied already in the first chapter. The general theory of Frobenius and quasi-Frobenius rings is developed in considerable detail in Chapter 6. Over such rings, we witness a remarkable "perfect duality" between finitely generated left and right modules. Much of the beautiful mathematics here goes back to Dieudonné, Nakayama, Nesbitt, Brauer, and Frobenius. This theory served eventually as the model for the general theory of duality between module categories developed by Kiiti Morita in his classical paper in 1958. Our text concludes with an exposition, in Chapter 7, of this duality theory, along with the equally significant theory of module category equivalences developed concomitantly by Kiiti Morita.

Although the present text was originally conceived as a sequel to *FC*, the material covered here is largely independent of that in *First Course*, and can be used as a text in its own right for a course in ring theory stressing the role of modules over rings. In fact, I have myself used the material in this manner in a couple of courses at Berkeley. For this reason, it is deemed appropriate to rename the book so as to

decouple it from *First Course*; hence the present title, *Lectures on Modules and Rings*. I am fully conscious of the fact that this title is a permutation of *Lectures on Rings and Modules* by Lambek — and even more conscious of the fact that my name happens to be a subset of his!

For readers using this textbook without having read *FC*, some orienting remarks are in order. While it is true that, in various places, references are made to *First Course*, these references are mostly for really basic material in ring theory, such as the Wedderburn-Artin Theorem, facts about the Jacobson radical, noetherian and artinian rings, local and semilocal rings, or the like. These are topics that a graduate student is likely to have learned from a good first-year graduate course in algebra using a strong text such as that of Lang, Hungerford, or Isaacs. For a student with this kind of background, the present text can be used largely independently of *FC*. For others, an occasional consultation with *FC*, together with a willingness to take some ring-theoretic facts for granted, should be enough to help them navigate through the present text with ease. The *Notes to the Reader* section following the Table of Contents spells out in detail some of the things, mathematical or otherwise, which will be useful to know in working with this text. For the reader's convenience, we have also included a fairly complete list of the notations used in the book, together with a partial list of frequently used abbreviations.

In writing the present text, I was guided by three basic principles. First, I tried to write in the way I give my lectures. This means I took it upon myself to select the most central topics to be taught, and I tried to expound these topics by using the clearest and most efficient approach possible, without the hindrance of heavy machinery or undue abstractions. As a result, all material in the text should be well-suited for direct class presentations. Second, I put a premium on the use of examples. Modules and rings are truly ubiquitous objects, and they are a delight to construct. Yet, a number of current ring theory books were almost totally devoid of examples. To reverse this trend, we did it with a vengeance: an abundance of examples was offered virtually every step of the way, to illustrate everything from concepts, definitions, to theorems. It is hoped that the unusual number of examples included in this text makes it fun to read. Third, I recognized the vital role of problem-solving in the learning process. Thus, I have made a special effort to compile extensive sets of exercises for all sections of the book. Varying from the routine to the most challenging, the compendium of (exactly) 600 exercises greatly extends the scope of the text, and offers a rich additional source of information to novices and experts alike. Also, to maintain a good control over the quality and propriety of these exercises, I made it a point to do each and every one of them myself. Solutions to all exercises in this text, with additional commentary on the majority of them, will hopefully appear later in the form of a separate problem book.

As I came to the end of my arduous writing journey that began as early as 1983, I grimaced over the one-liner of Carl Faith quoted at the beginning of this preface. Torture it no doubt was, and the irony lay indeed in the fact that I had chosen to inflict it upon myself. But surely every author had a compelling reason for writing his or her opus; the labor and pain, however excruciating, were only a part of the price to pay for the joyful creation of a new brain-child!

If I had any regrets about this volume, it would only be that I did not find it possible to treat all of the interesting ring-theoretic topics that I would have liked to include. Among the most glaring omissions are: the dimension theory and torsion theory of rings, noncommutative noetherian rings and PI rings, and the theory of central simple algebras and enveloping algebras. Some of these topics were "promised" in *FC*, but obviously, to treat any of them would have further increased the size of this book. I still fondly remember that, in Professor G.-C. Rota's humorous review of my *First Course*, he mused over some mathematicians' unforgiving (and often vociferous) reactions to omissions of their favorite results in textbooks, and gave the example of a "Professor Neanderthal of Redwood Poly.", who, upon seeing my book, was confirmed in his darkest suspicions that I had failed to "include a mention, let alone a proof, of the Worpitzky-Yamamoto Theorem." Sadly enough, to the Professor Neanderthals of the world, I must shamefully confess that, even in this *second* volume in noncommutative ring theory, I still did not manage to include a mention, let alone a proof, of that omnipotent Worpitzky-Yamamoto Theorem!

Obviously, a book like this could not have been written without the generous help of many others. First, I thank the audiences in several of the ring theory courses I taught at Berkeley in the last 15 years. While it is not possible to name them all, I note that the many talented (former) students who attended my classes included Ka Hin Leung, Tara Smith, David Moulton, Bjorn Poonen, Arthur Drisko, Peter Farbman, Geir Agnarsson, Ioannis Emmanouil, Daniel Isaksen, Romyar Sharifi, Nghi Nguyen, Greg Marks, Will Murray, and Monica Vazirani. They have corrected a number of mistakes in my presentations, and their many pertinent questions and remarks in class have led to various improvements in the text. I also thank heartily all those who have read portions of preliminary versions of the book and offered corrections, suggestions, and other constructive comments. This includes Ioannis Emmanouil, Greg Marks, Will Murray, Monica Vazirani, Scott Annin, Stefan Schmidt, André Leroy, S. K. Jain, Charles Curtis, Rad Dimitrić, Ellen Kirkman, and Dan Shapiro. Other colleagues helped by providing proofs, examples and counterexamples, suggesting exercises, pointing out references, or answering my mathematical queries: among them, I should especially thank George Bergman, Hendrik Lenstra, Jr., Carl Faith, Barbara Osofsky, Lance Small, Susan Montgomery, Joseph Rotman, Richard Swan, David Eisenbud, Craig Huneke, and Birge Huisgen-Zimmermann.

Last, first, and always, I owe the greatest debt to members of my family. At the risk of sounding like a broken record, I must once more thank my wife Chee-King for graciously enduring yet another book project. She can now take comfort in my solemn pledge that there will *not* be a *Third Course*! The company of our four children brings cheers and joy into my life, which keep me going. I thank them fondly for their love, devotion and unstinting support.

Berkeley, California T.Y.L.
July 4, 1998

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Notes to the Reader

This book consists of nineteen sections (§§1–19), which, for ease of reference, are numbered consecutively, independently of the seven chapters. Thus, a cross-reference such as (12.7) refers to the result (lemma, theorem, example, or remark) so labeled in §12. On the other hand, Exercise (12.7) will refer to Exercise 7 in the exercise set appearing at the end of §12. In referring to an exercise appearing (or to appear) in the same section, we shall sometimes drop the section number from the reference. Thus, when we refer to "Exercise 7" *within* §12, we shall mean Exercise (12.7). A reference in brackets, such as Amitsur [72] (or [Amitsur: 72]) shall refer to the 1972 paper/book of Amitsur listed in the reference section at the end of the text.

Throughout the text, some familiarity with elementary ring theory is assumed, so that we can start our discussion at an "intermediate" level. Most (if not all) of the facts we need from commutative and noncommutative ring theory are available from standard first-year graduate algebra texts such as those of Lang, Hungerford, and Isaacs, and certainly from the author's *First Course in Noncommutative Rings* (GTM 131). The latter work will be referred to throughout as *First Course* (or simply *FC*). For the reader's convenience, we summarize below a number of basic ring-theoretic notions and results which will prove to be handy in working with the text.

Unless otherwise stated, a ring R means a ring with an identity element 1, and a subring of R means a subring $S \subseteq R$ with $1 \in S$. The word "ideal" always means a two-sided ideal; an adjective such as "noetherian" likewise means right and left noetherian. A ring homomorphism from R to R' is supposed to take the identity of R to that of R'. Left and right R-modules are always assumed to be unital; homomorphisms between modules are usually written (and composed) on the opposite side of scalars. "Semisimple rings" are in the sense of Wedderburn, Noether and Artin: these are rings that are semisimple as left (right) modules over themselves. We shall use freely the classical Wedderburn-Artin Theorem (*FC*–(3.5)), which states that a ring R is semisimple iff it is isomorphic to a direct product $\mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_r}(D_r)$, where the D_i 's are division rings. The $\mathbb{M}_{n_i}(D_i)$'s are called the *simple components* of R; these are the most typical

simple artinian rings. A classical theorem of Maschke states that the group algebra kG of a finite group G over a field k of characteristic prime to $|G|$ is semisimple.

The *Jacobson radical* of a ring R, denoted by rad R, is the intersection of the maximal left (right) ideals of R; its elements are exactly those which act trivially on all left (right) R-modules. If rad $R = 0$, R is said to be *Jacobson semisimple* (or just *J-semisimple*). Such rings generalize the classical semisimple rings, in that semisimple rings are precisely the *artinian* J-semisimple rings. A ring R is called *semilocal* if $R/\text{rad } R$ is artinian (and hence semisimple); in the case when R is commutative, this amounts to R having only a finite number of maximal ideals. If R is semilocal and rad R is nilpotent, R is said to be *semiprimary.* Over such a ring, the Hopkins-Levitzki Theorem (*FC*–(4.15)) states that any noetherian module has a composition series. This theorem implies that left (right) artinian rings are precisely the semiprimary left (right) noetherian rings.

In a ring R, a *prime ideal* is an ideal $\mathfrak{p} \subsetneq R$ such that $aRb \subseteq \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$; a *semiprime ideal* is an ideal $\mathfrak{C}$ such that $aRa \subseteq \mathfrak{C}$ implies $a \in \mathfrak{C}$. Semiprime ideals are exactly intersections of prime ideals. A ring R is called *prime* (*semiprime*) if the zero ideal is prime (semiprime). The *prime radical* (a.k.a. Baer radical, or lower nilradical[1]) of a ring R is denoted by Nil_*R: it is the smallest semiprime ideal of R (given by the intersection of all of its prime ideals). Thus, R is semiprime iff $\text{Nil}_*R = 0$, iff R has no nonzero nilpotent ideals. In case R is commutative, Nil_*R is just $\text{Nil}(R)$, the set of all nilpotent elements in R; R being semiprime in this case simply means that R is a *reduced ring*, that is, a ring without nonzero nilpotent elements. In general, $\text{Nil}_*R \subseteq \text{rad } R$, with equality in case R is a 1-sided artinian ring.

A *domain* is a nonzero ring in which there is no 0-divisor (other than 0). Domains are prime rings, and reduced rings are semiprime rings. A *local ring* is a ring R in which there is a unique maximal left (right) ideal $\mathfrak{m}$; in this case, we often say that $(R, \mathfrak{m})$ is a local ring. For such rings, rad $R = \mathfrak{m}$, and $R/\text{rad } R$ is a division ring. An element a in a ring R is called *regular* if it is neither a left nor a right 0-divisor, and *von Neumann regular* if $a \in aRa$. The ring R itself is called von Neumann regular if every $a \in R$ is von Neumann regular. Such rings are characterized by the fact that every principal (resp., finitely generated) left ideal is generated by an idempotent element.

A nonzero module M is said to be *simple* if it has no submodules other than (0) and M, and *indecomposable* if it is not a direct sum of two nonzero submodules. The *socle* of a module M, denoted by $\text{soc}(M)$, is the sum of all simple submodules of M. In case M is R_R (R viewed as a right module over itself), the socle is always an ideal of R, and is given by the left annihilator of rad R if R is 1-sided artinian (*FC*–Exer. (4.20)). In general, however, $\text{soc}(R_R) \neq \text{soc}({}_RR)$.

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[1]The upper nilradical Nil^*R (the largest nil ideal in R) will not be needed in this book.

Partial List of Notations

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{F}_q$	finite field with q elements
$\mathbb{Z}_n$, C_n	the cyclic group $\mathbb{Z}/n\mathbb{Z}$
C_{p^∞}	the Prüfer p-group
$\emptyset$	the empty set
$\subset$, $\subseteq$	used interchangeably for inclusion
$\subsetneqq$	strict inclusion
$\lvert A \rvert$, Card A	used interchangeably for the cardinality of the set A
$A \setminus B$	set-theoretic difference
$A \rightarrowtail B$	injective mapping from A into B
$A \twoheadrightarrow B$	surjective mapping from A onto B
δ_{ij}	Kronecker deltas
E_{ij}	standard matrix units
M^t, M^T	transpose of the matrix M
$\mathbb{M}_n(S)$	set of $n \times n$ matrices with entries from S
$\mathrm{GL}_n(S)$	group of invertible $n \times n$ matrices over S
$\mathrm{GL}(V)$	group of linear automorphisms of a vector space V
$Z(G)$	center of the group (or the ring) G
$C_G(A)$	centralizer of A in G
$[G : H]$	index of subgroup H in a group G
$[K : F]$	field extension degree
$\mathfrak{M}_R$, ${}_R\mathfrak{M}$	category of right (left) R-modules
$\mathfrak{M}_R^{fg}$, ${}_R^{fg}\mathfrak{M}$	category of f.g. right (left) R-modules
M_R , ${}_RN$	right R-module M, left R-module N
${}_RM_S$	(R, S)-bimodule M
$M \otimes_R N$	tensor product of M_R and ${}_RN$
$\mathrm{Hom}_R(M, N)$	group of R-homomorphisms from M to N
$\mathrm{End}_R(M)$	ring of R-endomorphisms of M

nM	$M \oplus \cdots \oplus M$ (n times)
$M^{(I)}$	$\sum_{i \in I} M$ (direct sum of I copies of M)
M^I	$\prod_{i \in I} M$ (direct product of I copies of M)
$\Lambda^n(M)$	n-th exterior power of M
$\mathrm{soc}(M)$	socle of M
$\mathrm{rad}(M)$	radical of M
$\mathrm{Ass}(M)$	set of associated primes of M
$E(M)$	injective hull (or envelope) of M
$\tilde{E}(M)$	rational hull (or completion) of M
$\mathcal{Z}(M)$	singular submodule of M
length M	(composition) length of M
u.dim M	uniform dimension of M
rank M	torsion-free rank or (Goldie) reduced rank of M
$\rho(M)$, $\rho_R(M)$	ρ-rank of M_R
M^*	R-dual of an R-module M
M', M^0	character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M_R
$\hat{M}$, $M^\wedge$	k-dual of a k-vector space (or k-algebra) M
N^{**}, $\mathrm{cl}(N)$	Goldie closure of a submodule $N \subseteq M$
$N \subseteq_e M$	N is an essential submodule of M
$N \subseteq_d M$	N is a dense submodule of M
$N \subseteq_c M$	N is a complement submodule (or closed submodule) of M
R^{op}	the opposite ring of R
$\mathrm{U}(R)$, R^*	group of units of the ring R
$\mathrm{U}(D)$, D^*, $\dot{D}$	multiplicative group of the division ring D
$\mathcal{C}_R$	set of regular elements of a ring R
$\mathcal{C}(N)$	set of elements which are regular modulo the ideal N
rad R	Jacobson radical of R
$\mathrm{Nil}^* R$	upper nilradical of R
$\mathrm{Nil}_* R$	lower nilradical (a.k.a. prime radical) of R
$\mathrm{Nil}(R)$	nilradical of a commutative ring R
$A^\ell(R)$, $A^r(R)$	left, right artinian radical of R
$\mathrm{Max}(R)$	set of maximal ideals of a ring R
$\mathrm{Spec}(R)$	set of prime ideals of a ring R
$\mathcal{I}(R)$	set of isomorphism classes of indecomposable injective modules over R
$\mathrm{soc}(R_R)$, $\mathrm{soc}(_RR)$	right (left) socle of R
$\mathcal{Z}(R_R)$, $\mathcal{Z}(_RR)$	right (left) singular ideal of R
$\mathrm{Pic}(R)$	Picard group of a commutative ring R
R_S	universal S-inverting ring for R
RS^{-1}, $S^{-1}R$	right (left) Ore localization of R at S
$R_\mathfrak{p}$	localization of (commutative) R at prime ideal $\mathfrak{p}$
$Q^r_{cl}(R)$, $Q^\ell_{cl}(R)$	classical right (left) ring of quotients for R
$Q_{cl}(R)$, $Q(R)$	the above when R is commutative

$Q^r_{\max}(R)$, $Q^\ell_{\max}(R)$	maximal right (left) ring of quotients for R
$Q^r(R)$, $Q^l(R)$	Martindale right (left) ring of quotients
$Q^s(R)$	symmetric Martindale ring of quotients
$\mathrm{ann}_r(S)$, $\mathrm{ann}_\ell(S)$	right, left annihilators of the set S
$\mathrm{ann}^M(S)$	annihilator of S taken in M
$\varinjlim$	injective (or direct) limit
$\varprojlim$	projective (or inverse) limit
kG, $k[G]$	(semi)group ring of the (semi)group G over the ring k
$k[x_i : i \in I]$	polynomial ring over k with (commuting) variables $\{x_i : i \in I\}$
$k\langle x_i : i \in I\rangle$	free ring over k generated by $\{x_i : i \in I\}$
$k[[x_1, \ldots, x_n]]$	power series in the x_i 's over k

Partial List of Abbreviations

FC	*First Course in Noncommutative Rings*
RHS, LHS	right-hand side, left-hand side
ACC	ascending chain condition
DCC	descending chain condition
IBN	"Invariant Basis Number" property
PRIR, PRID	principal right ideal ring (domain)
PLIR, PLID	principal left ideal ring (domain)
FFR	finite free resolution
QF	quasi-Frobenius
PF	pseudo-Frobenius
PP	"principal implies projective"
PI	"polynomial identity" (ring, algebra)
CS	"closed submodules are summands"
QI	quasi-injective (module)
Obj	object(s) (of a category)
iff	if and only if
resp.	respectively
ker	kernel
coker	cokernel
im	image
f.cog.	finitely cogenerated
f.g.	finitely generated
f.p.	finitely presented
f.r.	finitely related
l.c.	linearly compact
pd	projective dimension
id	injective dimension
fd	flat dimension
wd	weak dimension (of a ring)
r.gl.dim	right global dimension (of a ring)
l.gl.dim	left global dimension (of a ring)

Chapter 1

Free Modules, Projective, and Injective Modules

An effective way to understand the behavior of a ring R is to study the various ways in which R acts on its left and right modules. Thus, the theory of modules can be expected to be an essential chapter in the theory of rings. Classically, modules were used in the study of representation theory (see Chapter 3 in *First Course*). With the advent of homological methods in the 1950s, the theory of modules has become much broader in scope. Nowadays, this theory is often pursued as an end in itself. Quite a few books have been written on the theory of modules alone.

This chapter and the next are entirely devoted to module theory, with emphasis on the *homological* viewpoint. In the three sections of this chapter, we give an introduction to the notions of *freeness, projectivity* and *injectivity* for (right) modules. Flatness and homological dimensions will be taken up in the next chapter. The material in these two chapters constitutes the backbone of the modern homological theory of modules.

Limitation of space has made it necessary for us to present only the basic facts and the most standard theorems on free, projective, and injective modules in this chapter. Nevertheless, we will be able to introduce the reader to a number of interesting results. Readers desiring further reading in these areas are encouraged to consult the monographs of Faith [76], Kasch [82], Anderson-Fuller [92], and Wisbauer [91].

Much of the material in this chapter will be needed in a fundamental way in the subsequent chapters. For instance, both projectives and injectives will play a role in the study of flat modules, and are vital for the theory of homological dimensions in the next chapter. The idea of essential extensions will prove to be indispensable (even essential!) in dealing with uniform dimensions and complements in Chapter 3, and the formation of the injective hull of a ring is crucial for the theory of rings of quotients to be developed in Chapters 4 and 5. Finally, projective and injective modules are exactly what we need in Chapter 7 in studying Morita's important theory of equivalences and dualities for categories of modules over rings. Given the key roles projective and injective modules play in this book, the reader will be well-advised to study this beginning chapter carefully. However, the three sections in this chapter are largely independent, and can be tackled "almost" in any order.

Thus, readers interested in a quick start on projective (resp. injective) modules can proceed directly to §2 (resp. §3), and return to §1 whenever they please.

§1. Free Modules

§1A. Invariant Basis Number (IBN)

For a given ring R, we write $\mathfrak{M}_R$ (resp. ${}_R\mathfrak{M}$) for the category of right (resp. left) R-modules. The notation M_R (resp. ${}_RN$) means that M (resp. N) is a given right (resp. left) R-module. We shall also indicate this sometimes by writing $M \in \mathfrak{M}_R$, although strictly speaking we should have written $M \in \text{Obj}(\mathfrak{M}_R)$ since M is an object in (and not a member of) $\mathfrak{M}_R$. Throughout this chapter, we work with right modules, and write homomorphisms on the left so that we use the usual left-hand rule for the composition of homomorphisms. It goes without saying that all results have analogues for left modules (for which the homomorphisms are written on the right).

We begin our discussion by treating free modules in §1. For any ring R, the module R_R is called the *right regular module.* A right module F_R is called *free* if it is isomorphic to a (possibly infinite) direct sum of copies of R_R. We write $R^{(I)}$ for the direct sum $\bigoplus_{i\in I} R_i$ where each R_i is a copy of R_R, and I is an arbitrary indexing set. The notation R^I will be reserved for the direct product $\prod_{i\in I} R_i$. If I is a *finite* set with n elements, then the direct sum and the direct product coincide; in this case we write R^n for $R^{(I)} = R^I$.

There are two more ways of describing a free module, with which we assume the reader is familiar. First, a module F_R is free iff it has a *basis*, i.e. a set $\{e_i : i \in I\} \subseteq F$ such that any element of F is a unique finite (right) linear combination of the e_i 's. Second, a module F_R with a subset $B = \{e_i : i \in I\}$ is free with B as a basis iff the following "universal property" holds: for any family of elements $\{m_i : i \in I\}$ in any $M \in \mathfrak{M}_R$, there is a unique R-homomorphism $f : F \to M$ with $f(e_i) = m_i$ for all $i \in I$. By convention, the zero module (0) is free with the empty set $\emptyset$ as basis.

As an example, note that free $\mathbb{Z}$-modules are just the free abelian groups. If R is a division ring, then all $M \in \mathfrak{M}_R$ are free and the usual facts from linear algebra on independent sets and generating sets in vector spaces are valid. However, over general rings, many of these facts may no longer hold. One fact that does hold over any ring R is the following.

(1.1) Generation Lemma. *Let $\{e_i : i \in I\}$ be a minimal generating set of $M \in \mathfrak{M}_R$ where the cardinality $|I|$ is infinite. Then M cannot be generated by fewer than $|I|$ elements.*

Proof. Consider any set $A = \{a_j : j \in J\} \subseteq M$ where $|J| < |I|$. Each a_j is in the span of a finite number of the e_i's. First assume $|J|$ is infinite. Then there exists a subset $I_0 \subseteq I$ with $|I_0| \leq |J| \cdot \aleph_0 = |J|$ such that each a_j is in the span

of $\{e_i : i \in I_0\}$. Since $|I_0| \le |J| < |I|$, we have

$$\mathrm{span}(A) \subseteq \mathrm{span}\{e_i : i \in I_0\} \subsetneq M,$$

as desired. If $|J|$ is finite, then span(A) is contained in the span of a finite number of the e_i's. Since $|I|$ is infinite, the latter span is again properly contained in M. □

Remark. As the reader can see, the preceding proof already works under the weaker hypothesis that (I is infinite and) no subset $\{e_i : i \in I_0\}$ of $\{e_i : i \in I\}$ with $|I_0| < |I|$ can generate M.

From this Lemma, we can check easily that "finitely generated free module" is synonymous with "R^n for some non-negative integer n". More importantly, the Generation Lemma has the following interesting consequence.

(1.2) Corollary. *If $R^{(I)} \cong R^{(J)}$ as right R-modules, where $R \neq (0)$ and I is infinite, then $|I| = |J|$. (The rank of $R^{(I)}$, taken to be the cardinal $|I|$, is therefore well-defined in this case.)*

If I, J are both *finite* sets, this Corollary may no longer hold, as we shall see below. This prompts the following definition.

(1.3) Definition. A ring R is said to have (right) IBN ("*Invariant Basis Number*") if, for any natural numbers n, m, $R^n \cong R^m$ (as right modules) implies that $n = m$. Note that this means that any two bases on a f.g.[2] free module F_R have the same (finite) number of elements. This common number is defined to be the *rank* of F.

Another shorthand occasionally used for "IBN" in the literature is "URP", for "Unique Rank Property". As aptly pointed out by D. Shapiro, "URP" has the advantage of being more pronounceable (it rhymes with "burp"). In this book, we shall follow the majority of ring theorists and use the more traditional (if unpronounceable) term "IBN".

Recalling that any homomorphism $R^m \to R^n$ can be expressed by an $n \times m$ matrix via the natural bases on R^m and R^n, we can recast the definition (1.3) above in matrix terms. Thus, the ring R fails to have (right) IBN iff there exist natural numbers $n \neq m$ and matrices A, B over R of sizes $m \times n$ and $n \times m$ respectively, such that $AB = I_m$ and $BA = I_n$. One advantage of this statement is that it involves neither right nor left modules. In particular, we see that "right IBN" is synonymous with "left IBN". From now on, therefore, we can speak of the IBN property without specifying "right" or "left".

The zero ring is a rather dull example of a ring not satisfying IBN. C. J. Everett, Jr. was perhaps the first one to call attention to the following type of interesting examples.

[2]Hereafter, we shall abbreviate "finitely generated" by "f.g."

(1.4) Example. *Let V be a free right module of infinite rank over a ring $k \neq (0)$, and let $R = \text{End}(V_k)$. Then, as right R-modules, $R^n \cong R^m$ for any natural numbers n, m.* For this, it suffices to show that $R \cong R^2$. Fix a k-isomorphism $\varepsilon : V \to V \oplus V$ and apply the functor $\text{Hom}_k(V, -)$ to this isomorphism. We get an abelian group isomorphism

$$\lambda : R \to \text{Hom}_k(V, V \oplus V) = R \oplus R.$$

We finish by showing that λ is a *right R-module homomorphism.* To see this, note that

$$\lambda(f) = (\pi_1 \circ \varepsilon \circ f, \pi_2 \circ \varepsilon \circ f) \quad (\forall f \in R),$$

where π_1, π_2 are the two projections of $V \oplus V$ onto V. For any $g \in R$, we have

$$\begin{aligned}\lambda(fg) &= (\pi_1 \circ \varepsilon \circ f \circ g, \pi_2 \circ \varepsilon \circ f \circ g)\\ &= (\pi_1 \circ \varepsilon \circ f, \pi_2 \circ \varepsilon \circ f)\, g\\ &= \lambda(f)\, g,\end{aligned}$$

as desired. An explicit basis $\{f_1, f_2\}$ on R_R can be constructed easily from this analysis. In fact, in the case when $V = e_1 k \oplus e_2 k \oplus \ldots$, we have essentially used the above method to construct such $\{f_1, f_2\}$ in *FC*–Exercise 3.14. In the notation of that exercise, we have also a pair $\{g_1, g_2\}$ with

$$g_1 f_1 = g_2 f_2 = 1, \quad g_1 f_2 = g_2 f_1 = 0, \quad \text{and} \quad f_1 g_1 + f_2 g_2 = 1.$$

This yields explicitly the matrix equations

$$(*) \qquad (f_1, f_2)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 1, \qquad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(f_1, f_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for checking the lack of IBN for R.

(1.5) Remark. Let $f : R \to S$ be a ring homomorphism. (This includes the assumption that $f(1) = 1$.) *If S has* IBN, *then R also has* IBN. In fact, if there exist matrix equations $AB = I_m$, $BA = I_n$ over R as in the paragraph following (1.3), with $n \neq m$, then we'll get similar equations over S by applying the homomorphism f, contradicting the IBN on S. Alternatively, we can also prove the desired result by applying the functor $- \otimes_R S$ to free right R-modules.

Now we are in a good position to name some classes of rings that have IBN.

(1.6) Examples.

(a) As we have mentioned before, *division rings have* IBN.

(b) *Local rings* $(R, \mathfrak{m})$ *have* IBN. This follows from (1.5) since we have a natural surjection from R onto the division ring $R/\mathfrak{m}$.

(c) *Nonzero commutative rings R have* IBN. In fact, if $\mathfrak{m}$ is any maximal ideal in R, then we have a natural surjection from R onto the field $R/\mathfrak{m}$.

(d) *Any ring R with a homomorphism into a nonzero commutative ring k has* IBN. For instance, we can take R to be the group ring kG over any group G. We can also take R to be any k-algebra generated by $\{x_i : i \in I\}$ with relations $\{\lambda_j(x)\}$ where $\lambda_j(x)$ are polynomials in the x_i 's with a zero constant term.

(e) *A nonzero finite ring R has* IBN. In fact, if $R^n \cong R^m$, then $|R|^n = |R|^m$, which implies that $n = m$.

(f) (Generalizing (e).) *A nonzero right artinian ring R has* IBN. To see this, we can use, for instance, the fact that any f.g. right R-module has a composition series (*FC*–(4.15)). Suppose R_R has composition length ℓ. If $R^n \cong R^m$, comparing composition lengths gives $n\ell = m\ell$, so $n = m$.

§1B. Stable Finiteness

In order to understand IBN more thoroughly, and to come up with more classes of rings with IBN, it is advantageous to consider other, somewhat stronger, conditions. We do this in the present subsection and the next ones.

First we introduce the important notion of stable finiteness. Recall that a ring S is *Dedekind-finite* (*FC*–p.4) if, for any $c, d \in S$, $cd = 1$ implies $dc = 1$. We say that a ring R is *stably finite* if the matrix rings $\mathbb{M}_n(R)$ are Dedekind-finite for all natural numbers n. The terminology here follows the usage of workers in operator algebras. The alternative term "weakly finite" is sometimes used by other authors, but we prefer the more traditional term "stably finite" here. The fact that the stably finite property is of interest was already noted many years ago in topology by H. Hopf, and in the theory of operator algebras by F. J. Murray and J. von Neumann.

It is convenient to have some alternative descriptions of stable finiteness, which we assemble as follows.

(1.7) Proposition. *The following properties of R are equivalent:*

(1) *R is stably finite.*
(2) *For any n,* $R^n \cong R^n \oplus N \Longrightarrow N = 0$ *(in* $\mathfrak{M}_R$*).*
(3) *For any n, any epimorphism* $R^n \to R^n$ *in* $\mathfrak{M}_R$ *is an isomorphism.*[3]

The easy proof of this Proposition is left as an exercise. (In fact, Exercise 8 of this section offers a somewhat more general statement on the characterization of Dedekind-finite modules.) Of course, we could have added to (1.7) also the left module analogues of (2) and (3).

The next proposition elucidates the relationship between stable finiteness and IBN.

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[3]In general, a module M_R is said to be *hopfian* if any epimorphism $M \to M$ is an isomorphism. Therefore, (3) is the condition that any f.g. free right R-module be hopfian.

(1.8) Proposition. *For any nonzero[4] ring R, stable finiteness implies* IBN, *but not conversely (in general).*

Proof. The first half is clear from the characterization (2) of stable finiteness in (1.7). To see the second half, consider the algebra R generated over a commutative ring $k \neq 0$ by x, y with a single relation $xy = 1$. One can check by a specialization argument that $yx \neq 1$ in R (cf. *FC*–p.4), so R is not Dedekind-finite, in particular not stably finite. On the other hand, R admits a k-algebra homomorphism f into k defined by $f(x) = f(y) = 1$, so R has IBN by (1.6)(d). (For a refinement of this result, see (1.22) below.) □

The preceding example shows, incidentally, that there is no analogue of (1.5) for stably finite rings; that is, if $g : R \to S$ is a ring homomorphism and S is stably finite, R need not be stably finite. In compensation, however, we have the following result, which was brought to my attention by G. Bergman.

(1.9) Proposition. *Let $g : R \to S$ be an embedding of the ring R into the ring S, not necessarily taking the identity e of R to the identity 1 of S. If S is stably finite, then so is R.*

Proof. Upon identifying R with $g(R)$, the identity e of R is an idempotent in S, with the complementary idempotent $f = 1 - e$ satisfying $Rf = fR = 0$. Let A, B be $n \times n$ matrices over R such that $AB = eI_n$. Then

$$(A + fI_n)(B + fI_n) = AB + f^2 I_n = (e + f)I_n = I_n.$$

If S is stably finite, this implies that

$$I_n = (B + fI_n)(A + fI_n) = BA + fI_n,$$

so we get $BA = eI_n$. This shows that R is stably finite. □

The flexibility gained by allowing $g(e) \neq 1$ in S is seen, in part, from the following consequence of (1.9).

(1.10) Corollary. *A direct product ring $S = \prod_{i \in I} R_i$ is stably finite iff each component ring R_i is.*

Proof. The "only if" part follows from the natural embedding of R_i in S. The "if" part is done by a routine "componentwise" argument. □

Another noteworthy consequence of (1.9) is the following.

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[4]It is best to exclude the zero ring here. Of course, the zero ring is stably finite, but does not have IBN.

(1.11) Corollary. *Let k be any division ring. Then any free k-ring $R = k\langle x_i : i \in I\rangle$ is stably finite.*

Proof. By *FC*–(14.25), R can be embedded in a division ring S (see also (9.25) below). Since S is stably finite, so is R by (1.9). □

The next result shows that the "stably finite" property is worth exploring primarily for noncommutative rings.

(1.12) Proposition (cf. *FC*–Exercise 20.9). *Any commutative ring R is stably finite.*

Proof. The best way to prove this is perhaps by using determinant theory. Let $C, D \in \mathbb{M}_n(R)$ be such that $CD = I_n$. Then $(\det C)(\det D) = 1$, so $\det C$ is a unit in R. From this, it follows that C is invertible with inverse $(\det C)^{-1} \cdot \text{adj}(C)$, where $\text{adj}(C)$ denotes the classical adjoint of C. In particular, $D = (\det C)^{-1} \cdot \text{adj}(C)$, and $DC = I_n$. □

As it turns out, many "reasonable" noncommutative rings satisfy the stably finite property. For instance, in *FC*–(20.13), we have shown that *any ring with "stable range 1" is stably finite.* This includes the class of all semilocal rings, i.e. rings R such that $R/\text{rad } R$ is semisimple. In particular, any right (resp. left) artinian ring is stably finite. Improving upon this, we have the following result.

(1.13) Proposition (cf. *FC*–Exercise 20.9). *Any right noetherian ring R is stably finite.*

To prove this, we first make the following observation on noetherian modules.

(1.14) Proposition. *Let $M \in \mathfrak{M}_R$ be a noetherian module. Then M is hopfian; that is, any epimorphism $\varphi : M \to M$ is an isomorphism.*

Proof. Suppose there exists a nonzero $x \in \ker \varphi$. Consider any integer $n \geq 1$ and choose $y \in M$ such that $x = \varphi^n(y)$. Then $\varphi^{n+1}(y) = \varphi(x) = 0$, so $y \in \ker \varphi^{n+1}$, but $\varphi^n(y) = x \neq 0$ implies that $y \notin \ker \varphi^n$. Thus, we have a strictly ascending chain of submodules:

$$\ker \varphi \subsetneq \ker \varphi^2 \subsetneq \cdots \subsetneq \ker \varphi^n \subsetneq \cdots,$$

contradicting the fact that M is noetherian. □

It follows from (1.14) that, *for a right noetherian ring R, any f.g. module M_R is hopfian.* Applying this to the free modules R^n (and recalling (1.7)), we deduce (1.13).

Proving that a certain class of rings has the stably finite property can sometimes be tricky. For instance, consider the class of group algebras kG, where k is any

field and G is any group. *Is kG always stably finite?* If k has characteristic 0, Kaplansky has shown that the answer is yes. An elegant proof of this appeared in Montgomery [69] (cf. also Herstein [71: p. 34]; it uses some C^*-algebra techniques. But if the characteristic of k is $p > 0$, the answer seems still unknown.

We should also point out that stable finiteness is a *stronger* property than Dedekind finiteness. For more details on this, see Exercise 18.

One good feature about stable finiteness is that there is a canonical procedure by which we can associate a stably finite ring $\bar{R}$ to any given ring R. The idea is that we "kill" all obstruction to stable finiteness in R, and pass to the largest stably finite homomorphic image of R. This universal construction was first successfully carried out by P. Malcolmson [80]. We shall now present Malcolmson's construction below.

Starting from any ring R, let $\mathfrak{A}$ be the ideal of R generated by all entries of matrices of the form $I - YX$, where X, Y are arbitrary square matrices (of any size) over R such that $XY = I$. Let $\bar{R} = R/\mathfrak{A}$, and write "bar" for the quotient map. Admittedly, this is a brute force construction. But now, whenever $XY = I$ over R, we are assured that $\bar{Y}\bar{X} = \bar{I}$. Thus, $\bar{R}$ has come a little closer to being stably finite. However, square matrix relation $X'Y' = \bar{I}$ over $\bar{R}$ might not lift to one over R, so we cannot yet conclude that $\bar{R}$ is stably finite. To get a stably finite ring, it seems we would need to repeat the construction. Fortunately, the following result of Malcolmson saves our day.

(1.15) Theorem. *For any ring R, the ring $\bar{R}$ constructed above is always stably finite. Moreover, any homomorphism from R to a stably finite ring factors uniquely through $\bar{R}$.*

Proof (following Cohn [85: p. 8][5]). The universal property of $\bar{R}$ (in the second part) is clear from its construction. To prove the first part, let $A,\ B \in \mathbb{M}_n(R)$ be such that $\bar{A}\bar{B} = \bar{I}$. Then $I - AB = \sum c_{ij}E_{ij}$ where $c_{ij} \in \mathfrak{A}$ and the E_{ij} 's are matrix units. Using the definition of $\mathfrak{A}$ on each c_{ij}, we can find an equation

$$(1.16) \qquad I - AB = \sum_{k=1}^{r} U_k(I - Y_kX_k)V_k,$$

where $X_k, Y_k \in \mathbb{M}_{m_k}(R)$ are such that $X_kY_k = I$, and U_k, V_k have sizes, respectively, $n \times m_k$ and $m_k \times n$. Let

$$X = \operatorname{diag}(X_1, \ldots, X_r), \qquad Y = \operatorname{diag}(Y_1, \ldots, Y_r),$$
$$U = (U_1, \ldots, U_r), \qquad V = (V_1, \ldots, V_r)^t.$$

(Here, V is supposed to mean the matrix with column blocks $V_1, \ldots, V_r$: we do not transpose the V_i 's.) We have then $XY = I$ and $I - AB = U(I - YX)V$. Without loss of generality, we may assume that $m := m_1 + \cdots + m_r \geq n$. After adding zero rows to U and zero columns to V, we may further assume that

[5]There is a small error in the proof in Cohn's book, which is corrected here.

$U, V \in \mathbb{M}_m(R)$, with $I_m - A'B' = U(I - YX)V$, where $A' = \text{diag}(A, I_{m-n})$ and $B' = \text{diag}(B, I_{m-n})$. Now let

$$C = A'X + U(I - YX), \qquad D = YB' + (I - YX)V,$$

where *all* matrices are $m \times m$. Since

$$X(I - YX) = 0 = (I - YX)Y, \quad \text{and} \quad (I - YX)^2 = I - YX,$$

we have

(1.17) $$CD = A'XYB' + U(I - YX)V = A'B' + (I - A'B') = I.$$

On the other hand, $CY = A'XY = A'$ and $XD = XYB' = B'$, so

(1.18) $$X(I - DC)Y = XY - (XD)(CY) = I_m - B'A'.$$

In view of (1.17), (1.18) implies that $I_m - B'A' \in \mathbb{M}_m(\mathfrak{A})$. In particular, $I_n - BA \in \mathbb{M}_n(\mathfrak{A})$, and so $\bar{B}\bar{A} = \bar{I}_n \in \mathbb{M}_n(\bar{R})$, as desired. □

It goes without saying that $\bar{R}$ may sometimes be the zero ring. The preceding proof leads to an explicit criterion for this to happen.

(1.19) Corollary. *For any ring R, we have $\bar{R} = 0$ iff there exist (for some m) $C, D \in \mathbb{M}_m(R)$, a row vector x of size $1 \times m$, and a column vector y of size $m \times 1$, such that $CD = I$ and $x(I - DC)y = 1$.*

Proof. If $\bar{R} = 0$, we can apply the proof of (1.15) to $A = B = 0$ (and, say, for $n = 1$) to come up with the matrices $X, Y, C, D \in \mathbb{M}_m(R)$ such that $CD = I$ and $X(I - DC)Y = \text{diag}(1, 0, \ldots, 0)$. Letting x be the first row of X, and y be the first column of Y, we have $x(I - DC)y = 1$. Conversely, if x, y, C, D exist with the given properties, then clearly the entries of $I - DC$ generate the unit ideal in R, so we have $\mathfrak{A} = R$ and $\bar{R} = 0$. □

§1C. The Rank Condition

In the study of vector spaces over fields (or more generally over division rings), we have encountered the following two very basic properties. For any n-dimensional vector space V:

(A) *Any generating set for V has cardinality $\geq n$.*

(B) *Any linearly independent set in V has cardinality $\leq n$.*

Over an arbitrary ring R, it is therefore natural to pursue the analogues of these properties, say, for free modules of finite rank over R. This leads us to the following definitions.

(1.20) Definition.

(1) We say that R satisfies the *rank condition* if, for any $n < \infty$, any set of R-module generators for $(R^n)_R$ has cardinality $\geq n$. Equivalently, if there is an

epimorphism of right free modules $\alpha : R^k \to R^n$, then $k \geq n$. (It will be seen that this is indeed a left-right symmetric condition.)

(2) We say that R satisfies the *strong rank condition* if, for any $n < \infty$, any set of linearly independent elements in $(R^n)_R$ has cardinality $\leq n$. Equivalently, if there is a monomorphism of right free modules $\beta : R^m \to R^n$, then $m \leq n$. (It will be seen that this condition does depend on working with right modules, so a more proper name should have been the "right strong rank condition". Since this is too long, we propose to suppress the word "right".)

Our terminology in (1) and (2) is justified by the following basic observation.

(1.21) Proposition. *If R satisfies the strong rank condition, then it satisfies the rank condition.*

Proof. The Proposition is possibly a bit surprising, since the formulations of the two conditions in terms of free modules seem to be "dual". Nevertheless, the Proposition is true! Assume R satisfies the strong rank condition, and consider an epimorphism $\alpha : R^k \to R^n$. Then α must split (by the universal property of the free module R^n), and we get a monomorphism $\beta : R^n \to R^k$ with $\alpha \circ \beta = I_{R^n}$. By the strong rank condition, we have $n \leq k$, as desired. □

We shall give an example later (see (1.31)) to show that the strong rank condition is indeed stronger than the rank condition, in general. In this subsection, we focus our attention on the rank condition. The following is an elementary (but useful) observation due to P. M. Cohn [66].

(1.22) Proposition. *For any nonzero ring R,*

$$\textit{stable finiteness} \implies \textit{rank condition} \implies \text{IBN}.$$

Proof. First assume R satisfies the rank condition. If $R^n \cong R^m$, then the rank condition gives $n \leq m$ and $m \leq n$, so $m = n$. Therefore, R has IBN. Now assume R does not satisfy the rank condition. Then there exists an epimorphism $\alpha : R^k \to R^n$ with $k < n < \infty$. But then

$$R^k \cong R^n \oplus \ker \alpha \cong R^k \oplus (R^{n-k} \oplus \ker \alpha),$$

where $R^{n-k} \oplus \ker \alpha \neq 0$. Therefore, by (1.7), R is not stably finite. □

It follows from (1.12), (1.13), and (1.22) that (nonzero) commutative rings and right noetherian rings both satisfy the rank condition. In general, however, neither of the implications in (1.22) is reversible. To see this for the first implication, we can exploit the following observation on the rank condition, in parallel to (1.5).

(1.23) Proposition. *Let $f : R \to S$ be a ring homomorphism. If S satisfies the rank condition, so does R.*

Proof. Let $\alpha : R^k \to R^n$ be an epimorphism in $\mathfrak{M}_R$. Tensoring this with ${}_RS$, we get an epimorphism $\alpha \otimes_R S : S^k \to S^n$, so $k \geq n$ by the rank condition on S. □

Now consider any commutative ring $k \neq 0$, and the k-algebra $R = k\langle x, y\rangle$ with the single relation $xy = 1$. As in the proof of (1.8), we have a ring homomorphism $R \to k$. Since k satisfies the rank condition, R also does by (1.23). But R is not Dedekind-finite, a fortiori not stably finite.

To construct a ring that has IBN but not the rank condition, we use the following matrix-theoretic characterization of the (negation of the) latter.

(1.24) Proposition. *A ring R fails to satisfy the rank condition iff, for some integers $n > k \geq 1$, there exist an $n \times k$ matrix A and a $k \times n$ matrix B (over R) such that $AB = I_n$.*

Proof. If such matrices A, B exist, then $\alpha : R^k \to R^n$ defined by left multiplication by A on the column vectors of R^k is an epimorphism, so the rank condition fails. Conversely, if the rank condition fails, we can find an epimorphism $\alpha : R^k \to R^n$ (in $\mathfrak{M}_R$) with $k < n$. Fixing a splitting $\beta : R^n \to R^k$ for α, the matrices A, B representing α and β have the required properties. □

Incidentally, the Proposition above explains why we need not use the term "right rank condition". From (1.24), it is clear that right rank condition and left rank condition would have been the same thing. (God bless matrices!)

With the aid of (1.24), the construction of a ring with IBN but not satisfying the rank condition proceeds as follows. Let R be the $\mathbb{Q}$-algebra with generators a, b, c, d subject to the relations

$$ac = 1, \quad bd = 1, \quad bc = ad = 0. \tag{1.25}$$

Then $\begin{pmatrix} a \\ b \end{pmatrix}(c, d) = I_2$, so R fails to satisfy the rank condition, by (1.24). Nevertheless, R has IBN. A proof for this, using Exercise 5 below, can be found in Cohn [66].

As it turns out, there is a very close relationship between the rank condition and stable finiteness. The following remarkable theorem is due to P. Malcolmson (and in a special case to K. Goodearl and D. Handelman). Here, for any ring R, $\bar{R}$ denotes the largest homomorphic image of R that is stably finite; see (1.15).

(1.26) Theorem. *For any ring, the following are equivalent:*

(1) *R satisfies the rank condition.*
(2) $\bar{R} \neq 0$.
(3) *R has a nonzero homomorphic image that is stably finite.*
(4) *R has a homomorphism into a nonzero stably finite ring.*
(5) *For any $m \geq 1$ and $C, D \in \mathbb{M}_m(R)$ with $CD = I$, we have $x(I - DC)y \neq 1$ for any row vector x of size $1 \times m$ and any column vector y of size $m \times 1$.*

Proof. By (1.19), (2) and (5) are equivalent, so it suffices to prove that

$$(2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (1) \Longrightarrow (5).$$

Here, (2) $\Longrightarrow$ (3) follows from the fact that $\bar{R}$ is stably finite, and (3) $\Longrightarrow$ (4) is trivial. (4) $\Longrightarrow$ (1) follows from (1.22) and (1.23). For (1) $\Longrightarrow$ (5), assume that, for some m, there exist matrices x, y, C, D of sizes as in (5) such that $CD = I$ and $x(I - DC)y = 1$. Then

$$\begin{pmatrix} C \\ x(I-DC) \end{pmatrix} (D,\ (I-DC)y) = \begin{pmatrix} CD & 0 \\ 0 & x(I-DC)y \end{pmatrix} = I_{m+1}\,,$$

where the two matrices on the LHS have sizes $(m+1) \times m$ and $m \times (m+1)$, respectively. Therefore, R cannot satisfy the rank condition by (1.24). □

(1.27) Corollary. *A simple ring R satisfies the rank condition iff it is stably finite.*

Proof. The "if" part follows from (1.22). (Recall that $R \neq 0$ is part of the definition of a simple ring.) Conversely, if R satisfies the rank condition, then $\bar{R} \neq 0$ by (1.26). But then the projection map $R \to \bar{R}$ must be an isomorphism, so $R \cong \bar{R}$ is stably finite. □

(1.28) Example. The simplest nontrivial example to illustrate the formation of the quotient ring $\bar{R}$ is perhaps the following. Let $R = \mathbb{Q}\langle x, y\rangle$ with a single relation $xy = 1$. Let $\mathfrak{A}$ be the ideal in R such that $\bar{R} = R/\mathfrak{A}$. Since R satisfies the rank condition (see the paragraph following the proof of (1.23)), we expect $\bar{R}$ to be nonzero. Let $\mathfrak{B}$ be the ideal in R generated by $1 - yx$. Then $\mathfrak{B} \subseteq \mathfrak{A}$ (since $xy = 1$). But $R/\mathfrak{B} \cong \mathbb{Q}[x, x^{-1}]$ which is commutative and hence stably finite. Therefore, we must have $\mathfrak{A} = \mathfrak{B}$, and so $\bar{R} = R/\mathfrak{A} \cong \mathbb{Q}[x, x^{-1}]$.

§1D. *The Strong Rank Condition*

In this closing subsection of §1, we shall investigate the strong rank condition for rings. Recall that a ring R satisfies the *strong rank condition* if, whenever there is a monomorphism $\beta : R^m \to R^n$ in $\mathfrak{M}_R$, then $m \leq n$; or equivalently, for any n, any set of more than n vectors must be linearly dependent in R^n.

It is possible to express this condition in terms of linear equations. Writing $R^n = \bigoplus_{i=1}^n e_i R$, consider m vectors $\{u_1, \ldots, u_m\} \subseteq R^n$, with, say, $u_j = \sum_{i=1}^n e_i a_{ij}$ $(a_{ij} \in R)$. An R-linear combination of the u_j's has the form

$$\sum_j u_j x_j = \sum_j \Big(\sum_i e_i a_{ij}\Big) x_j = \sum_i e_i \Big(\sum_j a_{ij} x_j\Big).$$

For this to be zero, the condition is that the scalars $\{x_1, \ldots, x_m\}$ be a solution to the system of linear equations:

(1.29) $$\left\{ \sum_{j=1}^m a_{ij} x_j = 0 :\ 1 \leq i \leq n \right\}.$$

Therefore, we have the following alternative description of the strong rank condition (see also Exercise 19).

(1.30) Proposition. *A ring R satisfies the strong rank condition iff any homogeneous system of n linear equations over R (as in (1.29)) with $m > n$ unknowns has a nontrivial solution over R.*

While the strong rank condition implies the rank condition by (1.21), the following example shows that the two conditions are not equivalent in general.

(1.31) Example. Let R be the free algebra $k\langle X\rangle$ generated over a field k by a set X with $|X| \geq 2$. Since we can map R to k by a ring homomorphism, (1.23) implies that R *satisfies the rank condition.* But if $x \neq y$ in X, then in the right regular module R_R, the elements $\{u_j = x^j y : 0 \leq j < \infty\}$ are right linearly independent. (If $\sum u_j f_j(X) = 0$, the only monomials beginning with $x^j y$ can only occur in the summand $u_j f_j(X)$, so each $f_j(X) = 0$.) Therefore, R_R contains a free submodule $\bigoplus_{j=0}^{\infty} u_j R$ of countably infinite rank. In particular, *R does not satisfy the strong rank condition.*

(1.32) Remark. The strong rank condition should be more appropriately called the *right* strong rank condition. In the case of domains (nonzero rings in which $xy = 0 \Longrightarrow x = 0$ or $y = 0$), we shall see in §10 (cf. Exercise (10.21)) that *R satisfies the right strong rank condition iff R is "right Ore"*. Since there exist right Ore domains that are not left Ore (see the second paragraph of §10C), we see that the right strong rank condition is, in general, not the same as the left strong rank condition. However, for convenience, we shall continue to write "strong rank condition" to refer to the *right* strong rank condition.

The example in (1.31) shows that if $f : R \to S$ is a ring homomorphism, the fact that S satisfies the strong rank condition may not imply the same for R. However, a partial result is available; see Exercise 20.

Since (for nonzero rings) stable finiteness implies the rank condition, it is natural to ask for the relationship between stable finiteness and the strong rank condition. As it turns out, there is none. To see this, we first make the following observation.

(1.33) Proposition. *A direct product $R = A \times B$ satisfies the strong rank condition iff one of A, B does.*

Proof. Suppose A satisfies the strong rank condition. Given a homogeneous equation of n linear equations over R with $m > n$ unknowns, we can solve the system by taking a nontrivial solution in A and a trivial solution in B. Therefore, R also satisfies the strong rank condition. The converse can be shown by a similar consideration of linear equations, and is left as an exercise. □

(1.34) Remark. *Stable finiteness and the strong rank condition are independent properties.* First, the free algebra $R = \mathbb{Q}\langle x, y\rangle$ is stably finite by (1.11), but does not satisfy the strong rank condition by (1.31). Second, let A be a ring satisfying the strong rank condition, and B be a ring that is not stably finite. Then $R = A \times B$ satisfies the strong rank condition by (1.33), but is not stably finite by (1.10). (This construction was shown to me by G. Bergman.)

We shall now end this subsection by finding some interesting classes of rings that satisfy the strong rank condition. The most basic result in this direction is the following.

(1.35) Theorem. *Any right noetherian ring $R \neq 0$ satisfies the strong rank condition.*

Since right artinian rings are always right noetherian (*FC*–(4.15)), the conclusion of the theorem holds also over a nonzero right artinian ring. A direct verification for this case can be given quite easily by a composition length argument on f.g. free modules. In the right noetherian case, however, we cannot use the length function. Hence, we must exploit the available finiteness condition in a somewhat more subtle way.

(1.36) Lemma. *Let A, B be right modules over a ring R, where $B \neq 0$. If $A \oplus B$ can be embedded in A, then A is not a noetherian module.*

Proof. The hypothesis means that A has a submodule $A_1 \oplus B_1$, where $A_1 \cong A$ and $B_1 \cong B$. It also implies that $A \oplus B$ can be embedded in A_1, so A_1 in turn contains a submodule $A_2 \oplus B_2$, where $A_2 \cong A$ and $B_2 \cong B$. Iterating this process, we get an infinite direct sum $B_1 \oplus B_2 \oplus \dots$ in A, where each $B_i \cong B \neq 0$. In particular, it is clear that A cannot be a noetherian module. □

Proof of (1.35). Let $R \neq 0$ be a right noetherian ring. Then, for any n, $A = (R^n)_R$ is a noetherian module (*FC*–(1.21)). By (1.36), $A \oplus B$ cannot be embedded in A for any $B \neq 0$. In particular, for any $m > n$, $R^m = A \oplus R^{m-n}$ cannot be embedded in $A = R^n$. □

(1.37) Remark. After studying the theory of uniform dimensions in §6, we can make the following observation. The proof of (1.36) shows that, if A is a right R module of *finite* uniform dimension (i.e., not containing an infinite direct sum of nonzero submodules), then $A \oplus B$ cannot be embedded in A, for any $B \neq 0$. The argument for (1.35), therefore, yields a sharper result: *If $R_R \neq 0$ has finite uniform dimension, then R satisfies the strong rank condition.* (Of course, we need to use the fact that u.dim$(R^n)_R = n$(u. dim R_R).) This gives a large stock of examples of rings satisfying the strong rank condition.

(1.38) Corollary (to (1.35)). *Any commutative ring $R \neq 0$ satisfies the strong rank condition.*

Proof. Consider a system of n linear equations (1.29) in $m > n$ unknowns, where $a_{ij} \in R$. The subring R_0 generated over $\mathbb{Z} \cdot 1$ by the a_{ij}'s is a (nonzero) noetherian ring, by the Hilbert Basis Theorem. By (1.35), the system (1.29) has a nontrivial solution in R_0, so it also has a nontrivial solution in R. □

If $R \neq 0$ is a *commutative* ring, any two elements $a, b \in R_R$ are linearly dependent, because of the relation $a \cdot b - b \cdot a = 0$. This means that no R^m ($m > 1$) can be embedded in R^1. The conclusion that no R^m can be embedded in R^n for $m > n$ (ascertained in (1.38)) does not seem to be as well known as it should be. For fields, of course, this lies in the very foundation of the subject of linear algebra. But the usual methods of proof (e.g., Gauss-Jordan elimination for solving linear equations) do not work well over a commutative ring, due to the possible lack of units. Because of this, we deem it of interest to give another proof of (1.38), using the properties of the exterior algebra of a module over a commutative ring. This proof, adapted from Bourbaki's *Algèbre*, has the advantage of avoiding the reduction to the noetherian case. In particular, in this proof, the Hilbert Basis Theorem is not required.

For the duration of this proof, R shall denote a nonzero commutative ring. If M is any right R-module, the exterior algebra

(1.39) $$\Lambda(M) = \bigoplus_{i \geq 0} \Lambda^i(M) \qquad (\Lambda^0(M) = R, \ \ \Lambda^1(M) = M)$$

has the property that, for any right R-module N, the R-linear mappings from $\Lambda^r(M)$ to N correspond naturally to the multilinear alternating mappings from M^r to N. We shall use the following exterior algebra-theoretic characterization for linear dependence of vectors in the free module $M = R^n$.

(1.40) Theorem. *Let $u_1, \ldots, u_m \in M = R^n$. Then $u_1, \ldots, u_m$ are linearly dependent in R^n iff there exists a nonzero element $\alpha \in R$ such that $(u_1 \wedge \cdots \wedge u_m)\alpha = 0$ in $\Lambda^m(M)$.*

Proof. For the "only if" part, take an equation $\sum u_i \alpha_i = 0$, where the α_i's are not all zero in R. By skew-symmetry, we may assume that $\alpha_1 \neq 0$. Then $u_1 \alpha_1 = -\sum_{i \geq 2} u_i \alpha_i$ and hence

$$(u_1 \wedge u_2 \wedge \cdots \wedge u_m)\alpha_1 = -\textstyle\sum_{i \geq 2} u_i \alpha_i \wedge u_2 \wedge \cdots \wedge u_m = 0.$$

For the "if" part, we induct on m, the case $m = 1$ being clear. Suppose $(u_1 \wedge \cdots \wedge u_m)\alpha = 0$, where $\alpha \neq 0$. We may assume that $(u_2 \wedge \cdots \wedge u_m)\alpha \neq 0$, for otherwise $u_2, \ldots, u_m$ are already linearly dependent. Since $\Lambda^{m-1}(M)$ is a free module, there exists a linear map $f : \Lambda^{m-1}(M) \to R$ such that

$$f(u_2\alpha \wedge u_3 \wedge \cdots \wedge u_m) \neq 0.$$

This f corresponds to a multilinear alternating map $F : M^{m-1} \to R$ such that

(1.41) $$F(u_2\alpha,\ u_3,\ \ldots\ ,\ u_m) \neq 0.$$

Now construct a map $G :\ M^m \to M$ by

$$G(v_1, \ldots, v_m) = \sum_{i=1}^{m} (-1)^i v_i F(v_1, \ldots, \hat{v}_i, \ldots, v_m) \in M,$$

where $v_i \in M$. (As usual, the hat means "omission".) The map G is easily checked to be multilinear and alternating. (If two of the v_i 's are equal, there are only two terms left in the summation, one being the negative of the other.) Therefore, G corresponds to a linear map $g : \Lambda^m(M) \to M$. From $u_1 \wedge u_2\alpha \wedge \cdots \wedge u_m = 0$, we have then

$$\begin{aligned} 0 &= g(u_1 \wedge u_2\alpha \wedge u_3 \wedge \cdots \wedge u_m) \\ &= G(u_1,\ u_2\alpha,\ u_3,\ \ldots\ ,\ u_m) \\ &= -u_1 F(u_2\alpha,\ u_3,\ \ldots\ ,\ u_m) + u, \end{aligned}$$

where u is a right linear combination of $u_2\alpha, u_3, \ldots, u_m$. Recalling (1.41), we see that $u_1, u_2, \ldots, u_m$ are linearly dependent. □

It follows from (1.40) that, if $u_1, \ldots, u_m$ are linearly independent in $M = R^n$, then

$$u_1 \wedge \cdots \wedge u_m \neq 0 \quad \text{in} \quad \Lambda^m(M).$$

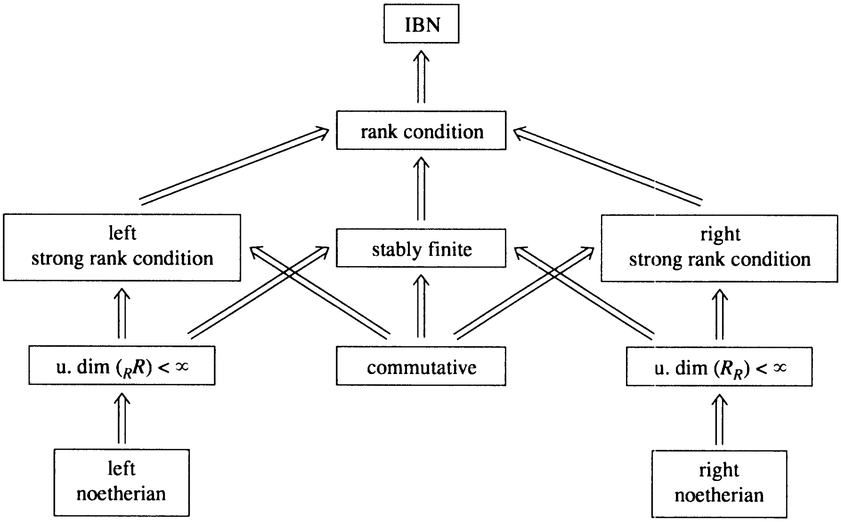
On the other hand, we know that $\Lambda^r(M) = 0$ for $r > n =$ rank M. Therefore, we must have $m \leq n$. This completes the alternative proof for (1.38). □

Another proof of (1.38) can be found in McCoy [48: pp. 159–160]. The main tool used in this proof is the *McCoy rank* of a matrix over a commutative ring, defined via the annihilators of its various minors. The Bourbaki proof we presented above, though couched in the language of exterior algebras, is in fact rather akin to McCoy's proof. Yet another proof of (1.38), using the fact that any commutative noetherian ring $R \neq 0$ has a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ embeds in R_R, can be found in Auslander-Buchsbaum [74: pp. 355–358].

§1E. Synopsis

The key notions discussed in §1 and some of their main interrelations can be summarized in the following chart (where we assume the ring R in question is

nonzero). All of the implications here are irreversible.



The fact that u.dim$(R_R) < \infty$ (resp. u.dim$({}_RR) < \infty$) implies stable finiteness can be deduced easily from Exercise (6.1) below.

Exercises for §1

1. Using (1.24), give a matrix-theoretic proof for "stable finiteness $\Rightarrow$ rank condition" (for nonzero rings).

2. A student gave the following argument to show that any algebra A over a field k has IBN. "Suppose A is generated over k by $\{x_i : i \in I\}$ with certain relations. Let $\bar{A}$ be the quotient of A obtained by introducing the further relations $x_ix_j - x_jx_i = 0\ (\forall\, i, j)$. Then A has a natural surjection onto $\bar{A}$. Since the commutative ring $\bar{A}$ has IBN, it follows from (1.5) that A has IBN." Is this argument valid?

3. Let R be the ring constructed in Example (1.4). Show that, for any integers n, m, $\mathbb{M}_n(R)$ and $\mathbb{M}_m(R)$ are isomorphic as rings.

4. Does every simple ring have IBN? A much harder optional question: does every domain have IBN? (See the discussion after (9.16).)

5. Suppose the ring R admits an additive group homomorphism T into an abelian group $(A, +)$ such that $T(cd) = T(dc)$ for all $c, d \in R$. (Such a T is called a *trace map*.) If $T(1)$ has infinite additive order in A, show that R must have IBN.

6. A module M_R is said to be *cohopfian* if every R-monomorphism $\varphi : M \longrightarrow M$ is an isomorphism. Dualize the argument in the proof of (1.14) to show that, if M_R is an artinian module, then M is cohopfian.

7. A ring that is Dedekind-finite is also known as von Neumann-finite. Is every von Neumann regular ring von Neumann-finite?

8. A module M_R is said to be *Dedekind-finite* if $M \cong M \oplus N$ (for some R-module N) implies that $N = 0$. Consider the following conditions:

   (A) M is Dedekind-finite.
   (B) The ring $E := \text{End}(M_R)$ is Dedekind-finite.
   (C) M is hopfian; that is, any R-epimorphism $M \to M$ is an isomorphism.

   Show that (C)$\Longrightarrow$(A)$\Longleftrightarrow$(B), and that (C)$\Longleftrightarrow$(A) if any R-epimorphism $M \to M$ splits (e.g., if M is a projective module). (Thus, the ring R is Dedekind-finite iff the module R_R is Dedekind-finite. And, applying the preceding to f.g. free modules, we also completely recover (1.7).) Give an example to show that, in general, $(A) \not\Rightarrow (C)$.

9. Show that a ring R is not Dedekind-finite iff there exists an idempotent $e \neq 1$ in R such that $eR \cong R$ in $\mathfrak{M}_R$.

10. (Vasconcelos, Strooker) We have shown in (1.12) that a commutative ring R is stably finite. More generally, show that any f.g. module ${}_RM$ over a commutative ring R is hopfian. (In particular, ${}_RM$ is Dedekind-finite in the sense of Exercise 8.) Is ${}_RM$ also cohopfian?

11. (Jacobson, Klein) Let R be a ring for which there exists a positive integer n such that $c^n = 0$ for any nilpotent element $c \in R$. Show that R is Dedekind-finite.

12. For any ring R, we can embed R into $S = \mathbb{M}_n(R)$ by sending $r \in R$ to $\text{diag}(r, \ldots, r)$. Therefore, S may be viewed as an (R, R)-bimodule. Show that $S_R \cong R_R^{n^2}$ and ${}_RS \cong ({}_RR)^{n^2}$, with the matrix units $\{E_{ij} : 1 \leq i, j \leq n\}$ as basis.

13. (Montgomery [83]) Let I be an ideal of a ring R contained in rad R (the Jacobson radical of R). Show that R is stably finite iff R/I is.

   For the following exercises ((14) *to* (17)), *let* "P" *denote any one of the properties:* IBN, *the rank condition, stable finiteness.*

14. Let $S = \mathbb{M}_n(R)$, where $n \geq 1$. Show that R satisfies the property "P" iff S does.

15. (Small) Let $S = R[[x]]$ (power series ring in one variable x over R). Show that R satisfies the property "P" iff S does. (**Hint.** For the "only if" part, note that the ideal $I = (x) \subseteq S$ is contained in rad S, with $S/I \cong R$. Then use (1.5), (1.23), and apply Exercise 13.)

16. (Small) Let $S = R[x]$. Show that R satisfies the property "P" iff S does. (**Hint.** In the case when "P" is stable finiteness, view $R[x]$ as a subring of $R[[x]]$. Can you also do it *without* using the power series ring?)

17. (Cohn [66]) Let $R = \varinjlim R_i$ (direct limit of a direct system of rings $\{R_i : i \in I\}$). Show that if each R_i satisfies the property "P", so does R.

18. Construct a ring R such that R is Dedekind-finite but $\mathbb{M}_2(R)$ is not Dedekind-finite. (In particular, R is not stably finite.) (**Hint.** Following a construction of Shepherdson [51], let R be the k-algebra generated over a field k by $\{s, t, u, v; w, x, y, z\}$ with relations dictated by the matrix equation $AB = I_2$, where $A = \begin{pmatrix} s & u \\ t & v \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Show that R is a domain, but that $BA \neq I_2$ in $\mathbb{M}_2(R)$. Thus, $\mathbb{M}_2(R)$ is not Dedekind-finite. Using similar methods, Cohn [66] has constructed (for any $n \geq 1$) a ring R for which $\mathbb{M}_\ell(R)$ is Dedekind-finite for all $\ell \leq n$, but $\mathbb{M}_{n+1}(R)$ is not Dedekind-finite. See also Montgomery [83] for results on finiteness questions for tensor products of algebras.)

19. Show that a ring $R \neq 0$ satisfies the strong rank condition iff, for any right R-module M generated by n elements, any $n + 1$ elements in M are linearly dependent.

20. Let $f : R \to S$ be a ring homomorphism such that S becomes a flat left R-module under f (i.e., the functor $- \otimes_R S$ is exact on $\mathfrak{M}_R$). Show that if S satisfies the (right) strong rank condition, so does R. Using this, give another proof for the "if" part of (1.33).

21. Supply a proof for the "only if" part of (1.33).

22. Let "P" be the strong rank condition. Redo Exercises 14, 16, and prove the "if" part of Exercise 15 for this "P".

23. If R satisfies the strong rank condition, does the same hold for $R[[x]]$? (This is a more challenging exercise. The answer, in general, is "no". A counterexample where R is, in fact, a domain can be found in (10.31). I don't know if it is easier to find a counterexample where R is allowed to have 0-divisors.)

24. Let R be a ring that satisfies the strong rank condition, and let $\beta : R^{(I)} \to R^{(J)}$ be a monomorphism from the free (right) module $R^{(I)}$ to the free module $R^{(J)}$, where I, J are (possibly infinite) sets. Show that $|I| \leq |J|$.

25. Let $R \neq 0$ be a commutative ring such that any ideal in R is free as an R-module. Show that R is a PID. (For a noncommutative version of this, see Exercise (10.25).)

26. Let R be any ring such that any right ideal in R is free as a right R-module. Show that any submodule of a free right R-module is free. (**Hint.** Look ahead at Kaplansky's Theorem (2.24).)

27. Let R be a ring and $\mathfrak{B} \subseteq R$ be an ideal that is free as a left R-module with a basis $\{b_j : j \in J\}$. For any free left R-module A with a basis $\{a_i : i \in I\}$, show that $\mathfrak{B}A$ is a free left R-module with a basis $\{b_j a_i : j \in J,\ i \in I\}$.

28. Let R and $\mathfrak{B}$ be as in Exercise 27, and let $\mathfrak{A} \supseteq \mathfrak{B}$ be a left ideal in R that is free as a left R-module. Show that

(1) for each $i \geq 0$, $\mathfrak{B}^i\mathfrak{A}/\mathfrak{B}^{i+1}\mathfrak{A}$ and $\mathfrak{B}^i/\mathfrak{B}^{i+1}$ are both free left $R/\mathfrak{B}$-modules;
(2) there is a long exact sequence of left $R/\mathfrak{B}$-modules:

$$\cdots \to \frac{\mathfrak{B}^2}{\mathfrak{B}^3} \to \frac{\mathfrak{B}\mathfrak{A}}{\mathfrak{B}^2\mathfrak{A}} \to \frac{\mathfrak{B}}{\mathfrak{B}^2} \to \frac{\mathfrak{A}}{\mathfrak{B}\mathfrak{A}} \to \frac{R}{\mathfrak{B}} \to \frac{R}{\mathfrak{A}} \to 0,$$

where all modules except $R/\mathfrak{A}$ are free over $R/\mathfrak{B}$. (Such a sequence is called a *free resolution* for the $R/\mathfrak{B}$-module $R/\mathfrak{A}$.)

29. Let G be a free group on a set of generators $\{x_i : i \in I\}$ and let R be the group ring kG, where k is a commutative ring. Show that, as a left R-module, the augmentation ideal $\mathfrak{A}$ (the kernel of the augmentation map $\varepsilon : R \to k$ defined by $\varepsilon(\sum_{z\in G} a_z z) = \sum_z a_z$) is R-free with basis $\{x_i - 1 : i \in I\}$. (In particular, if $|I| \geq 2$ and $k \neq 0$, R does not satisfy the left strong rank condition, although it does satisfy the rank condition.)

30. Let G and k be as in the preceding exercise, and let H be a subgroup of G. It is known that H is also a free group, say, on a set of generators $\{y_j : j \in J\}$. Let G/H be the coset space $\{gH : g \in G\}$ viewed as a left G-set, and let $k[G/H]$ be the permutation kG-module with k-basis G/H. Let $\alpha : kG \to k[G/H]$ be the kG-module homomorphism induced by the natural G-map $G \to G/H$. Show that, as a left kG-module, $\mathfrak{B} := \ker(\alpha)$ is free with basis $\{y_j - 1 : j \in J\}$. (This generalizes the last exercise, which corresponds to the case $H = G$.)

31. Let k be any commutative ring, and E be any (multiplicative) group. Fix a presentation of E by generators and relations, say,

$$1 \to H \to G \to E \to 1,$$

where G (and hence H) is free. (Here, H is the normal subgroup of G generated by the "relations".) Let $\mathfrak{A} := \ker(\varepsilon)$ and $\mathfrak{B} := \ker(\alpha)$ be as in the last two exercises. Show that k, viewed as a left kE-module with the trivial E-action, has the following free resolution

$$\cdots \to \frac{\mathfrak{B}^2}{\mathfrak{B}^3} \to \frac{\mathfrak{B}\mathfrak{A}}{\mathfrak{B}^2\mathfrak{A}} \to \frac{\mathfrak{B}}{\mathfrak{B}^2} \to \frac{\mathfrak{A}}{\mathfrak{B}\mathfrak{A}} \to \frac{R}{\mathfrak{B}} \to k \to 0$$

in the category of left kE-modules. (For $k = \mathbb{Z}$, this is known as the *Gruenberg resolution* of the trivial $\mathbb{Z}E$-module $\mathbb{Z}$. This free resolution is of basic importance in the cohomology theory of groups.)

32. (Bass) Show that any nonzero submodule of a free module F_R contains a copy of a nonzero principal right ideal aR.

33. (Bass) Let F_R be a free R-module on a basis $\{e_1, \ldots, e_n\}$, $\alpha = e_1a_1 + \cdots + e_na_n \in F$ ($a_i \in R$), and $A = \sum_i Ra_i$. Let f be an idempotent in R.

Show that the following are equivalent: (1) $A = Rf$; (2) $\alpha \cdot R$ is a direct summand of F isomorphic to fR with $\alpha \leftrightarrow f$.

34. ("Unimodular Column Lemma") Let $F = \bigoplus_{i=1}^{n} e_i R$ and $\alpha = \sum_i e_i a_i \in F$ be as in Exercise 33.
(1) Show that $\sum_{i=1}^{n} Ra_i = R$ iff $\alpha \cdot R$ is a direct summand of F free on $\{\alpha\}$.
(2) In case $\sum_{i=1}^{n} Ra_i = R$, show that a direct complement of $\alpha \cdot R$ in F is free of rank $n-1$ iff there exists a matrix $(a_{ij}) \in \mathrm{GL}_n(R)$ with $a_{i1} = a_i$ for all i.

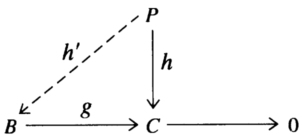
35. Let R be a ring with IBN such that any direct summand of R_R^n is free (for a fixed n). Show that $\sum_{i=1}^{n} Ra_i = R$ iff the column vector $(a_1, \ldots, a_n)^t$ can be completed to a matrix in $\mathrm{GL}_n(R)$.

§2. Projective Modules

§2A. *Basic Definitions and Examples*

This first subsection is a leisurely introduction to the basic facts on projective modules. Most of our readers have probably encountered projective modules in a graduate algebra course. To make the present exposition self-contained, however, we recall the definition here. A right R-module P is said to be *projective* (or *R-projective*) if, for any epimorphism of right R-modules, say, $g : B \to C$, and any R-homomorphism $h : P \to C$, there exists an R-homomorphism $h' : P \to B$ such that $h = g \circ h'$.

(2.1)



We refer to this property informally by saying that any $h : P \to C$ can be "lifted" (along g) to a homomorphism $h' : P \to B$. If P is free, this lifting is always possible, by an easy application of the universal property of a free module. Therefore, *a free module is always projective.* In general, however, such a lifting may not be possible. For instance, over $R = \mathbb{Z}$, if g is the unique epimorphism from $B = \mathbb{Z}/4\mathbb{Z}$ to $C = \mathbb{Z}/2\mathbb{Z}$, then the identity map h from $P = \mathbb{Z}/2\mathbb{Z}$ to C clearly cannot be lifted (along g) to a homomorphism $P \to B$. Therefore, $\mathbb{Z}/2\mathbb{Z}$ is not $\mathbb{Z}$-projective.

For any given R-module P_R, the functor $\mathrm{Hom}_R(P, -)$ from $\mathfrak{M}_R$ to the category of abelian groups is *left exact*, in the sense that, for any short exact sequence

(2.2) $$0 \to A \to B \to C \to 0$$

in $\mathfrak{M}_R$, we have a corresponding exact sequence of abelian groups:

(2.3) $$0 \to \mathrm{Hom}_R(P, A) \xrightarrow{f_*} \mathrm{Hom}_R(P, B) \xrightarrow{g_*} \mathrm{Hom}_R(P, C).$$

In this functorial language, we see that P_R *is projective iff* $\text{Hom}_R(P, -)$ *is exact*, which means that, for any short exact sequence (2.2), we have a short exact sequence

$$(2.4) \qquad 0 \to \text{Hom}_R(P, A) \xrightarrow{f_*} \text{Hom}_R(P, B) \xrightarrow{g_*} \text{Hom}_R(P, C) \to 0.$$

(2.5) Proposition. *A direct sum* $P = \bigoplus_i P_i$ *of right* R*-modules is projective iff each summand* P_i *is projective.*

Proof. The functor $\text{Hom}_R(P, -)$ is naturally equivalent to the direct product of the functors $\text{Hom}_R(P_i, -)$. Therefore, $\text{Hom}_R(P, -)$ is exact iff each functor $\text{Hom}_R(P_i, -)$ is exact. □

From this, we can easily deduce the following.

(2.6) Corollary. P_R *is projective iff it is (isomorphic to) a direct summand of a free module, iff any epimorphism* $B_R \to P_R$ *splits in* $\mathfrak{M}_R$.

(2.7) Corollary (Eilenberg's Trick). *If* P_R *is projective, then there exists a free module* F_R *such that* $P \oplus F \cong F$.

Proof. Fix an R-module Q_R such that $E = P \oplus Q$ is free. Then for the free module $F = E \oplus E \oplus \ldots$, we have

$$P \oplus F \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong F,$$

as desired. □

Note that if $P \neq 0$, then $E \neq 0$ also, so the free module F obtained above cannot be f.g., by (1.1). This fact, of course, severely limits the use of (2.7).

Coming back to Proposition 2.5, we should also note that in general, the direct product of projective modules need not be projective. The following example illustrating this is attributed to R. Baer.

(2.8) Example. *The direct product* $M = \mathbb{Z} \times \mathbb{Z} \times \cdots$ *is not a projective* $\mathbb{Z}$*-module.*

The proofs of this fact available in the literature mostly depend on the result that subgroups of free abelian groups are free (proved in (2.27) below). In particular, if M is $\mathbb{Z}$-projective, then it is free, and so is any of its subgroups. To complete the proof of (2.8), it suffices therefore to produce a *non-free* subgroup of M. For such a proof, see, for instance, Rotman [79: p. 122]. Instead of following this approach, however, we shall present below a proof of (2.8) avoiding any use of the fact that subgroups of free abelian groups are free. Our proof depends on the following lemma which is of independent interest (and will also be useful in solving the extra credit Exercise 8′ below).

(2.8)′ Lemma. *Let* $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \subset M$. *Then* $\mathrm{Hom}_{\mathbb{Z}}(M/P, \mathbb{Z}) = 0$.

Proof. Let $f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be such that $f(P) = 0$. Consider the following two subgroups of M:

$$A = \{(2a_1,\ 2^2a_2,\ \ldots,\ 2^na_n,\ \ldots) : \ a_i \in \mathbb{Z}\},$$
$$B = \{(3b_1,\ 3^2b_2,\ \ldots,\ 3^nb_n,\ \ldots) : \ b_i \in \mathbb{Z}\},$$

whose sum is clearly M. An element of A has the form

$$(2a_1,\ \ldots,\ 2^{n-1}a_{n-1},\ 0,\ 0,\ \ldots) + 2^n \cdot (\text{an element of } M).$$

Since $f(P) = 0$, we have $f(A) \subseteq 2^n\mathbb{Z}$ for every n, so $f(A) = 0$. Similarly, we have $f(B) = 0$, and so $f(M) = f(A) + f(B) = 0$. □

To prove (2.8), assume M is $\mathbb{Z}$-projective. We have $M \subseteq F$ for a suitable free abelian group F with basis $\{e_i : i \in I\}$. Since $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ is countable, we can decompose I into a disjoint union $I_1 \cup I_2$ such that I_1 is countable and P is contained in the span F_1 of $\{e_i : i \in I_1\}$ (cf. proof of (1.1)). Note that $M \not\subseteq F_1$, since F_1 is countable but M is not. Taking a projection of F into $e_i\mathbb{Z}$ for a suitable $i \in I_2$, we come up with a homomorphism $f : F \to \mathbb{Z}$ with $f(M) \neq 0$ but $f(F_1) = 0$ (and hence $f(P) = 0$), contradicting (2.8)′.

I thank I. Emmanouil, P. Farbman, and D. Shapiro for their help in formulating the preceding proof of (2.8). A sketch of another proof of (2.8) using cardinality arguments is given in Exercise 6 below.

§2B. Dual Basis Lemma and Invertible Modules

The main result in this subsection is the following basic characterization of a projective module P in terms of its (first) dual $P^* := \mathrm{Hom}_R(P, R)$.

(2.9) Dual Basis Lemma. *A right R-module P is projective iff there exist a family of elements $\{a_i : i \in I\} \subseteq P$ and linear functionals $\{f_i : i \in I\} \subseteq P^*$ such that, for any $a \in P$, $f_i(a) = 0$ for almost all i, and $a = \sum_i a_i f_i(a)$.*

Proof. Suppose the a_i's and f_i's exist as specified. Consider the epimorphism g from the free module $F = \bigoplus e_i R$ to P defined by $g(e_i) = a_i$ for all $i \in I$. The map $h : P \to F$ defined by $h(a) = \sum e_i f_i(a)$ is clearly an R-homomorphism splitting g. This implies that P is isomorphic to a direct summand of F; hence P is projective. Conversely, assume P is projective and fix an epimorphism g from a suitable free module $F = \bigoplus e_i R$ onto P. By the second part of (2.6), g admits a splitting $h : P \to F$, which may be expressed in the form

$$h(a) = \sum e_i f_i(a) \qquad (\forall a \in P).$$

Here, the f_i 's are easily checked to be R-linear (i.e. $f_i \in P^*$), and $f_i(a) = 0$ for almost all i. Applying g to the above equation, we see that

$$a = gh(a) = \sum a_i f_i(a),$$

where $a_i := g(e_i) \in P$. □

For convenience, we shall loosely refer to $\{a_i, f_i\}$ above as "a pair of dual bases" for the (projective) module P. Of course, the a_i's only form a generating set, not necessarily a basis, for P.

Note that for any right R-module P, the first dual $P^* = \mathrm{Hom}_R(P, R)$ is a *left* R-module by the action defined by $(rf)(a) = rf(a)$, where $r \in R$, $f \in P^*$, and $a \in P$. Following the convention that homomorphisms are written opposite scalars, we therefore write the linear functionals on P^* on the right. These linear functionals constitute the double dual P^{**} which, like P itself, is a right R-module. There is a well-known canonical R-homomorphism $\varepsilon : P \to P^{**}$ defined by $\varepsilon(a) = \hat{a}$ (for $a \in P$), where $f\hat{a} = f(a)$ for any $f \in P^*$.

(2.10) Corollary. *For any projective right R-module P, the natural map ε from P to P^{**} is a monomorphism.*

Proof. If $a \in \ker(\varepsilon)$, then $0 = f\hat{a} = f(a)$ for all $f \in P^*$. From the equation $a = \sum_i a_i f_i(a)$ in the Dual Basis Lemma, it follows that $a = 0$. □

(2.11) Remark. The proof of (2.9) also shows that P is f.g. projective iff there exist $\{a_i, f_i : 1 \le i \le n\}$ as in (2.9) such that $a = \sum_{i=1}^{n} a_i f_i(a)$ for every $a \in P$. In this case, it can be shown that *the f_i's also generate P^**. Moreover, the map $\varepsilon : P \to P^{**}$ defined above is an *isomorphism* of right R-modules. For more details, see Exercise 7.

(2.12A) Example. Let e be an idempotent in R. Then $R = eR \oplus (1-e)R$, so $P := eR$ is a projective right R-module. In the Dual Basis Lemma, we can choose $I = \{1\}$, $a_1 = e \in P$, and $f_1 : P \to R$ to be the inclusion map. To check that these choices work, simply note that if $a = er \in P$ $(r \in R)$, then

$$a_1 f_1(a) = ea = eer = er = a.$$

We note, incidentally, that *if $e \neq 0, 1$, and R is Dedekind-finite, then P_R cannot be free.* For, if P were free, then $P \cong R^n$ for some $n \ge 1$, and we would have

$$R \cong R \oplus (R^{n-1} \oplus (1-e)R),$$

contradicting Dedekind-finiteness. (If R is *not* assumed to be Dedekind-finite, eR may indeed be free: see Exercise (1.9).)

(2.12B) Example. Sometimes, an R-module P may be of the form Re in a somewhat non-obvious way. The following is an example. Let A be a ring and M_A be a right A-module with a decomposition $X \oplus Y$ where X is A-free of rank 1, say with a basis $\{x\}$. For $R = \mathrm{End}(M_A)$ (operating on the left of M), we'll show that *${}_RM$ is a projective left R-module.* To do this, let $e \in R$ be the projection of M onto X with respect to the decomposition $M = X \oplus Y$, and consider the map $\varphi : R \longrightarrow {}_RM$ given by $\varphi(g) = g(x)$ $(\forall g \in R)$. Clearly,

this is a left R-module homomorphism. *We claim that* (1) φ *is onto, and* (2) $\ker(\varphi) = R(1-e)$. Once we have proved these, it will follow that

$$M \cong R/\ker(\varphi) = R/R(1-e) \cong Re,$$

and so (by (2.12A)) ${}_RM$ is a projective R-module. To see (1), take any $z \in M$. There exists an A-homomorphism $h_0 : X \longrightarrow M$ given by $h_0(x) = z$, and h_0 can be extended to an $h \in \mathrm{Hom}_A(M, M) = R$ by the rule $h(Y) = 0$. Then $\varphi(h) = h(x) = h_0(x) = z$. To prove (2), note first that $\varphi(1-e) = (1-e)x = x - e(x) = 0$, so $R(1-e) \subseteq \ker(\varphi)$. Conversely, if $g \in \ker(\varphi)$, then $ge(x) = g(x) = \varphi(g) = 0$ and $ge(Y) = 0$ imply that $ge = 0$. Therefore, $g = g - ge = g(1-e) \in R(1-e)$, as claimed.

For a concrete example, let M be any f.g. additive abelian group of rank ≥ 1, viewed as a $\mathbb{Z}$-module. Then M has the decomposition $X \oplus Y$ above, and it follows that M is a left projective module over $\mathrm{End}_{\mathbb{Z}}(M)$. The case when M has rank 0 can be handled similarly; for more details, see Exercise (3.37).

(2.12C) Example. Let $R = \prod_{j\in I} R_j$, where R_j are arbitrary rings. Each ideal R_j is projective as a right R-module, so the ideal $P_R = \bigoplus_{j\in I} R_j$ is also projective. To see how the Dual Basis Lemma works here, let a_i be the "i^{th} unit vector" in P, and let $f_i \in P^*$ be defined by $f_i((r_j)) = (0, \ldots, r_i, 0, \ldots)$, with r_i kept in the ith position. Then, for any $r = (r_j) \in P$, $f_i(r)$ is zero for almost all i, and we have $r = \sum_i a_i f_i(r)$ by a direct calculation. Note that, as an ideal in R, P is idempotent ($P^2 = P$). However, if infinitely many of the rings R_j are nonzero, P is not f.g. as a right ideal.

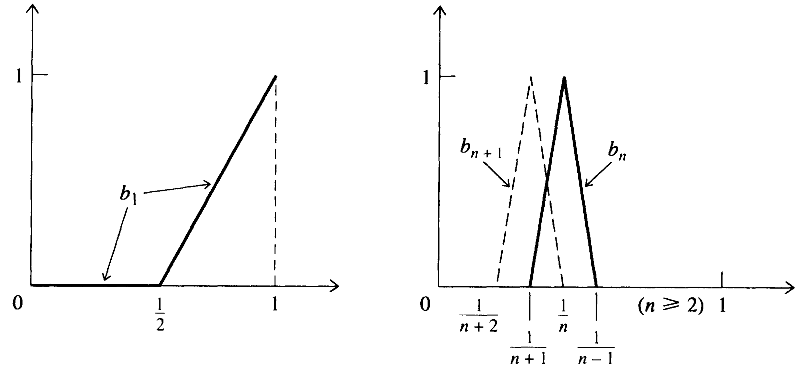
(2.12D) Example (Kaplansky). This is another elegant example illustrating the use of a countably infinite family of a_i 's and f_i 's in the Dual Basis Lemma (2.9). We take R here to be the (commutative) ring of real-valued continuous functions on $[0, 1]$, with pointwise addition and multiplication for functions. Let

$$P = \{f \in R : \ f \text{ vanishes on } [0, \varepsilon] \text{ for some } \varepsilon = \varepsilon(f) \in (0, 1)\}$$

which is easily seen to be an ideal of R. We shall check that P is *R-projective* by explicitly constructing a family $\{a_n, f_n : n = 1, 2, \ldots\}$ as in (2.9). For any $a \in R$, write

$$\mathrm{supp}(a) = \{x \in [0, 1] : \ a(x) \neq 0\}.$$

We define $b_n \in P$ $(n \geq 1)$ by the following graphs:



where $\text{supp}(b_n) = \left(\frac{1}{n+1}, \frac{1}{n-1}\right) \cap [0, 1]$. For any $x > 0$, we have $b_n(x) \neq 0$ for at most two n's, and it is easy to see that $\sum_{n=1}^{\infty} b_n(x) = 1$. (The b_n's come close to forming a "partition of unity"; however, $\sum_{n=1}^{\infty} b_n \notin R$ since it is not continuous at 0.) Now define $f_n : R \to R$ by multiplication by $a_n := \sqrt{b_n} \in P$, where $\left(\sqrt{b_n}\right)(x) = \sqrt{b_n(x)}$ for all $x \in [0, 1]$. By restricting to the ideal P, we may view f_n as an element of P^*. For any function $a \in P$, we have $f_n(a) = aa_n = 0$ whenever n is so big that a vanishes on $[0, 1/(n-1)]$. Finally, we have $a = \sum_{n=1}^{\infty} a_n f_n(a)$ since both sides are 0 at the origin, and for $x > 0$:

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_n f_n(a)\right)(x) &= \sum_{n=1}^{\infty} b_n(x)^{1/2} b_n(x)^{1/2} a(x) \\ &= \sum_{n=1}^{\infty} b_n(x) a(x) \\ &= a(x).\end{aligned}$$

Three things are worth pointing out about the projective ideal P. First, since $f_n(a) = a_n a \in P$, the formula $a = \sum a_n f_n(a)$ above shows that P is an idempotent ideal, i.e. $P = P^2$. Second, the projective ideal P is not free, since any $a \in P$ has a nonzero annihilator in R. Third, P is not a f.g. ideal since the functions in any f.g. subideal of P must vanish on $[0, 1/m]$ for some m, while P clearly contains functions not vanishing identically there.

A large number of examples of f.g. projective modules over a commutative ring R can be obtained by considering a commutative ring extension $S \supseteq R$.[6] Since R is commutative, we can afford to be somewhat sloppy about the term "R-module", noting that a right R-module may be viewed as a left R-module, and vice versa.

[6]A certain amount of what follows can be done for any pair of rings $R \subseteq S$, without assuming commutativity. However, in order to simplify the exposition and to focus on the main ideas, we work with commutative rings here.

Now consider the R-submodules of S. If P, Q are such submodules, we make the following two definitions:

$$PQ = \left\{\sum p_i q_i : \ p_i \in P, \ q_i \in Q\right\},$$
$$P^{-1} = \{s \in S : \ sP \subseteq R\}.$$

Clearly, these are also R-submodules of S. Of course, the definition of P^{-1} here depends on the choice of the extension $S \supseteq R$.

(2.13) Lemma. *For any R-submodule $P \subseteq S$, the following are equivalent:*

(1) *There exists an R-submodule Q of S such that $PQ = R$.*
(2) $PP^{-1} = R$.

If (1) *(or* (2)*) holds, we say that P is an invertible R-submodule of S.*

Proof. (2) $\Longrightarrow$ (1) is trivial, so we need only prove (1) $\Longrightarrow$ (2). Given $PQ = R$, we have clearly $Q \subseteq P^{-1}$. Thus, $R \supseteq PP^{-1} \supseteq PQ = R$, which implies that $PP^{-1} = R$. □

(2.14) Theorem. *Let P be an invertible R-submodule of S. Then*

(1) *P is a f.g. projective R-module.*
(2) *For any R-submodule $M \subseteq S$, the natural map $\alpha : P \otimes_R M \to PM$ is an R-module isomorphism.*
(3) *$P^* \cong P^{-1}$ as R-modules.*
(4) *P_R is free iff $P = sR$ for some $s \in S$ (necessarily a unit of S).*

Proof. (1) Let $Q = P^{-1}$. Since $PQ = R$, there exists an equation $1 = \sum_{i=1}^n p_i q_i$, where $p_i \in P$, $q_i \in Q$. Define $f_i \in P^*$ by $f_i(p) = pq_i$ ($\forall p \in P$). Then, for any $p \in P$, $p = \sum pp_iq_i = \sum p_i f_i(p)$. By (2.9), P is a projective R-module generated by $p_1, \ldots, p_n$.

(2) Recall that $P \otimes_R M$ has a natural R-module structure via the action

$$(p \otimes m)r = (pr) \otimes m = p \otimes (mr).$$

The map α is defined by $\alpha(\sum a_i \otimes m_i) = \sum a_i m_i$, and it is clearly an R-module epimorphism. To show that α is also injective, we use the notations set up in the proof of (1). Since $P \otimes_R M = \sum p_i \otimes M$, an arbitrary element of $P \otimes_R M$ may be expressed in the form $z = \sum_i p_i \otimes m_i$. Assume $\alpha(z) = \sum_i p_i m_i = 0$. Then

$$z = \sum_i \left(p_i \sum_j p_j q_j\right) \otimes m_i = \sum_j p_j \otimes \sum_i (p_i q_j) m_i.$$

But in S, $\sum_i (p_i q_j) m_i = q_j \sum_i p_i m_i = 0$, so $z = 0$.

(3) Define $\beta : Q \to P^*$ by $\beta(q)(p) = pq \in R$ ($\forall p \in P, q \in Q$). If $\beta(q) = 0$, then $q \in qR = qPQ = 0$, so β is injective. Using again the notations in (1), we have $f_i = \beta(q_i)$. By (2.11), $P^* = \sum Rf_i = \beta\left(\sum Rq_i\right)$, so β is also surjective.

(4) First assume $P = sR$ where $s \in S$. Then $R = PQ = sQ$, so $sq = 1$ for some $q \in Q$. This shows that $s \in \mathrm{U}(S)$. In particular, P_R is free on the basis $\{s\}$. Conversely, assume P_R is free. Then $P \cong R^n$ for some $n < \infty$, and so $Q \cong P^* \cong R^n$ for the same n. But then

$$R = PQ \cong P \otimes_R Q \cong (R^n) \otimes_R (R^n) \cong R^{n^2}.$$

If $R \neq 0$, we must have $n = 1$ and so $P = sR$ for some $s \in S$. The latter is, of course, also true if $R = 0$. □

The theorem we proved above enables us to construct many examples of nonfree projective modules. Continuing in the spirit of Kaplansky's example (2.12D), we first construct an explicit invertible module over a ring of continuous functions.

Example. Let S be the (commutative) ring of real-valued continuous functions $f(x)$ on $[0, \pi]$ (with pointwise addition and multiplication as before). Let

$$R = \{f(x) \in S : \ f(0) = f(\pi)\},$$
$$P = \{f(x) \in S : \ f(0) = -f(\pi)\}.$$

An easy check shows that R is a subring of S, and that $R \cdot P \subseteq P$, $P^2 \subseteq R$. We claim that P is an *invertible R-module with $P \oplus P \cong R \oplus R$, but P_R is not free*. To see this, the crucial observation is that $\sin x, \cos x \in P$, so that $1 = \sin^2 x + \cos^2 x \in P^2$. This yields $P^2 = R$, so P is indeed invertible (and "self-dual"). Letting

$$A = \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix},$$

we can define R-homomorphisms

$$\varphi : P \oplus P \to R \oplus R, \qquad \psi : R \oplus R \to P \oplus P \ \text{ by}$$
$$\varphi\begin{pmatrix} f \\ g \end{pmatrix} = A\begin{pmatrix} f \\ g \end{pmatrix} \quad \text{and} \quad \psi\begin{pmatrix} h \\ k \end{pmatrix} = A\begin{pmatrix} h \\ k \end{pmatrix}.$$

Since $A^2 = I$, φ and ψ are mutually inverse isomorphisms. Therefore, $P \oplus P \cong R \oplus R$ (and the proof above shows that $(\sin x, \cos x)$, $(\cos x, -\sin x)$ give a basis for $P \oplus P$). To see that P is *nonfree*, assume the contrary for the moment, so that, by (2.14)(4), $P = f(x)R$ for some unit $f(x)$ of S. Being a unit, $f(x)$ is nowhere zero on $[0, \pi]$. On the other hand, $f(0)$ and $f(\pi)$ have opposite signs, since $f(x) \in P$. This obviously contradicts the Intermediate Value Theorem in calculus. Therefore, P_R cannot be free.

Many other examples of invertible modules over commutative domains will be constructed in the next subsection. Here, to conclude §2B, we would like to construct a class of examples (called "Schanuel modules") over a fairly general commutative ring R. I thank H. Lenstra heartily for his invaluable help in formulating the ideas in (2.15). In particular, the nice proof of (2.15B) below is his.

(2.15) Example. Let $R \subseteq S$ be commutative rings, and let $g \in S$ be such that $g^2, g^3 \in R$. This implies (easily) that $g^n \in R$ for all $n \geq 2$, but g itself may or may not be in R. For $r \in R$, let $P_r = (1 + rg,\ g^2)$ be the R-submodule of S generated by the indicated elements.[7] For $r, s \in R$:

$$P_r P_s = (1+rg,\ g^2)(1+sg,\ g^2) = (1+(r+s)g+rsg^2,\ g^2+rg^3,\ g^2+sg^3,\ g^4)$$

contains $g^2(1 + (r + s)g + rsg^2) - rsg^4 = g^2 + (r + s)g^3$, so it also contains rg^3, sg^3 and $g^2, 1 + (r + s)g$. Therefore,

$$P_r P_s = (1 + (r + s)g,\ g^2,\ rg^3,\ sg^3).$$

From $g^3(1 + (r + s)g) - (r + s)g^2g^2 = g^3$, we see further that

$$\text{(2.15A)} \qquad P_r P_s = (1 + (r + s)g,\ g^2) = P_{r+s}.$$

In particular, $P_r P_{-r} = P_0 = (1,\ g^2) = R$, so $\{P_r : r \in R\}$ is a family of invertible (hence projective) R-submodules of S, with $P_r^* = P_r^{-1} = P_{-r}$. The criterion for P_r to be free turns out to be the following:

$$\text{(2.15B)} \qquad P_r \text{ is } R\text{-free iff } u(1 + rg) \in R \text{ for some } u \in \mathrm{U}(R[g]).$$

In fact, if P_r is R-free, by (2.14)(4) we have $P_{-r} = uR$ for some $u \in \mathrm{U}(S)$. Since

$$u \in P_{-r} = (1 - rg,\ g^2) \subseteq R[g] \quad \text{and} \quad u^{-1} \in P_r = (1 + rg,\ g^2) \subseteq R[g],$$

we have $u \in \mathrm{U}(R[g])$ and $u(1 + rg) \in uP_r = R$. Conversely, suppose $u(1 + rg) \in R$ for some $u \in \mathrm{U}(R[g])$. Since $ug^2 \in R[g]g^2 \subseteq R$, we have $uP_r = (u(1 + rg),\ ug^2) \subseteq R$ so $P_r \subseteq u^{-1}R$. *We finish by showing that* $P_r = u^{-1}R$. Let $J = \{c \in R : cg \in R\}$. This is an ideal of $R[g]$ (called the "conductor" of the pair $R \subseteq R[g]$). Clearly, $Rg^2 \subseteq J$. Also, for any $c \in J$:

$$c = c(1 - r^2g^2 + r^2g^2) = [c(1 - rg)](1 + rg) + [cr^2]g^2 \in P_r,$$

so $J \subseteq P_r$. Let $t = u(1 + rg) \in R$. Since $(1 + rg)(1 - rg) \in 1 + J$, we have $\overline{1 + rg} \in \mathrm{U}(R[g]/J)$, and hence $\bar{t} \in \mathrm{U}(R[g]/J)$. Using the fact that $R \subseteq R[g]$ is an integral extension, we see that $\bar{t} \in \mathrm{U}(R/J)$. Therefore,

$$P_r/J = (1 + rg) \cdot (R/J) = u^{-1}t \cdot (R/J) = u^{-1} \cdot (R/J).$$

This implies that $u^{-1}R \subseteq P_r$, as desired.

To further simplify the criterion for P_r to be free, we can impose an additional hypothesis.

(2.15C) Proposition. *In the preceding example, assume that*

$$\text{(2.15D)} \qquad \textit{For } u \in \mathrm{U}(R[g]),\ u(1 + rg) \in R \implies u \in R.$$

[7]The construction of $P_1 = (1 + g,\ g^2)$ was first given by S. Schanuel. It is, therefore, reasonable to call the P_r 's Schanuel modules. The direct calculation checking $P_r P_s = P_{r+s}$ in (2.15A) appears to be new.

Then P_r is R-free iff $r \in J$ (the conductor defined above).

Proof. If $r \in J$, then $P_r \subseteq R$ (since $rg \in R$), and it contains

$$(1 - rg)(1 + rg) + r^2 g^2 = 1,$$

so $P_r = R$. Conversely, assume P_r is R-free. By (2.15B), there exists $u \in \mathrm{U}(R[g])$ such that $u(1+rg) \in R$. By (2.15D), this implies that $u \in R$. Since $R \subseteq R[g]$ is an integral extension, we have as before $u \in \mathrm{U}(R)$. Therefore, $1 + rg \in u^{-1}R = R$, and so $r \in J$. □

Though a bit technical, (2.15D) is not at all an unreasonable hypothesis. For instance, if $\mathrm{U}(R[g]) \subseteq R$, (2.15D) certainly holds. For a specific example, look at the subring $R = k[z^2, z^3]$ of $S = k[z]$, where k is a field. (R is the coordinate ring of the "cusp" $x^3 = y^2$ over k.) Since $\mathrm{U}(S) = k^* \subseteq \mathrm{U}(R)$, (2.15D) holds. Choosing g to be z, we see easily that the conductor J for $R \subseteq R[g] = S$ is the R-ideal (z^2, z^3). It follows from (2.15C) that, for $r \in R$, *the projective R-module P_r is free iff $r \in (z^2, z^3)$.*

For a less obvious example, fix commutative rings $A \subseteq F$ with $\mathrm{Nil}(F) = 0$, and an element $b \in F$ such that $b^2,\ b^3 \in A$. Let $R = A[x] \subseteq S = F[x]$, where x is an indeterminate. For $g := bx \in S$, we have clearly $g^2 = b^2x^2 \in R,\ g^3 = b^3x^3 \in R$. *We claim that* (2.15D) *always holds for the pair $R \subseteq S$.* In fact, let $r \in R$ and $u \in \mathrm{U}(S)$ be such that $u(1 + rg) \in R$. Since $\mathrm{Nil}(F) = 0$, we see easily that $\mathrm{U}(S) = \mathrm{U}(F)$, so $u \in F$. From

$$u(1 + rg) = u + urbx \in A[x],$$

we have $u \in A \subseteq R$, so (2.15D) holds. If

$$J_0 = \{a \in A : ab \in A\}$$

is the conductor for $A \subseteq A[b]$, the conductor J for $R \subseteq R[g]$ is easily seen to be $J_0[x]$. It follows from (2.15C) that, for $r \in R$, *the projective R-module P_r is free iff $r \in J_0[x]$.*

The preceding considerations will have nice applications to the computation of Picard groups in (2.23) below.

§2C. Invertible Fractional Ideals

In this subsection, we shall specialize the study of invertible R-submodules in a ring extension $S \supseteq R$ by taking S to be the total ring of quotients of R. Throughout this subsection, R continues to denote a commutative ring.

An element $r \in R$ is called a *non 0-divisor* if $ra = 0$ implies $a = 0$ for any $a \in R$. Such an element r is also said to be *regular* in R. The set C_R of regular elements in R is clearly multiplicatively closed, and $1 \in C_R$. The localization of R at C_R is a commutative ring K containing R, with the property that any regular element of K is invertible in K. In commutative ring theory, K is referred to

as the *total ring of quotients* of R, and is denoted by $Q(R)$. Of course, if R is a commutative domain, $Q(R)$ is just the quotient field of R.

As in §2B, it will be useful to look at the R-submodules of $K = Q(R)$: we call them *fractional ideals*. Note that a fractional ideal contained in R is simply an ideal of R; it is sometimes called an *integral ideal* for emphasis.

(2.16) Lemma. *For any fractional ideal $\mathfrak{A} \subseteq K$ such that $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$, we have a K-isomorphism* $\operatorname{Hom}_R(\mathfrak{A}, K) \cong K$.

Proof. Define $\lambda : K \to \operatorname{Hom}_R(\mathfrak{A}, K)$ by $\lambda(k)(a) = ka$, where $k \in K$. The hypothesis $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$ implies that λ is an *injective* K-homomorphism. Fix an element $b \in \mathfrak{A} \cap \mathcal{C}_R$. For any $f \in \operatorname{Hom}_R(\mathfrak{A}, K)$ and any $a \in \mathfrak{A}$, pick $r \in \mathcal{C}_R$ such that $ra \in R$. Then

$$rbf(a) = f(rba) = (ra)f(b).$$

Since $b, r \in \mathcal{C}_R$, this gives $f(a) = f(b)b^{-1} \cdot a$. Therefore, $f = \lambda(f(b)b^{-1})$, so λ is a K-isomorphism. □

For fractional ideals $\mathfrak{A}, \mathfrak{B} \subseteq K$, we define

$$\mathfrak{A}\mathfrak{B} = \left\{\sum a_i b_i : a_i \in \mathfrak{A},\ b_i \in \mathfrak{B}\right\},$$

$$\mathfrak{B} : \mathfrak{A} = \{k \in K : k\mathfrak{A} \subseteq \mathfrak{B}\}, \quad \text{and}$$

$$\mathfrak{A}^{-1} = \{k \in K : k\mathfrak{A} \subseteq R\} = R : \mathfrak{A}.$$

These are easily checked to be fractional ideals also. Note that if $\mathfrak{A} \subseteq R$, then $\mathfrak{A}^{-1} \supseteq R$. On the other hand, if $1 \in \mathfrak{A}$, then $\mathfrak{A}^{-1} \subseteq R$.

Consider the R-module $\operatorname{Hom}_R(\mathfrak{A}, \mathfrak{B})$, which consists of those R-homomorphisms in $\operatorname{Hom}_R(\mathfrak{A}, K)$ sending $\mathfrak{A}$ into $\mathfrak{B}$. Assuming that $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$, and interpreting $\operatorname{Hom}_R(\mathfrak{A}, K)$ as in (2.16), we see that there is a natural identification

(2.16′) $$\operatorname{Hom}_R(\mathfrak{A}, \mathfrak{B}) \cong \mathfrak{B} : \mathfrak{A} \quad (\text{whenever } \mathfrak{A} \cap \mathcal{C}_R \neq \emptyset).$$

In particular, for such $\mathfrak{A}$, the (first) dual $\mathfrak{A}^* = \operatorname{Hom}_R(\mathfrak{A}, R)$ may be identified with $R : \mathfrak{A} = \mathfrak{A}^{-1}$.

(2.17) Theorem. *For any fractional ideal $\mathfrak{A} \subseteq K = Q(R)$, the following are equivalent:*

(1) *$\mathfrak{A}$ is invertible in the sense of* §2B *(i.e., $\mathfrak{A}\mathfrak{A}^{-1} = R$).*
(2) *$\mathfrak{A}_R$ is projective, and $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$.*

If (1) *(or* (2)*) holds, $\mathfrak{A}_R$ must be f.g., and it is free iff $\mathfrak{A} = sR$ for some $s \in K$ (necessarily a unit of K).*

Proof. (1) $\Longrightarrow$ (2). By (2.14)(1), $\mathfrak{A}_R$ is projective. Take an equation $\sum p_i q_i = 1$, where $p_i \in \mathfrak{A}$, $q_i \in \mathfrak{A}^{-1}$. For a suitable common denominator $r \in \mathcal{C}_R$, we can

write $p_i = a_i r^{-1}$, $q_i = b_i r^{-1}$, where $a_i, b_i \in R$. Then $a_i = rp_i \in \mathfrak{A}$ and $r^2 = \sum a_i b_i \in \mathfrak{A} \cap \mathcal{C}_R$.

(2) $\Longrightarrow$ (1). By (2.9), there exist $a_i \in \mathfrak{A}$ and $f_i \in \mathfrak{A}^*$ ($i \in I$) such that $a = \sum a_i f_i(a)$ for any $a \in \mathfrak{A}$. By what we said in the paragraph preceding the theorem, each f_i is multiplication by some $b_i \in \mathfrak{A}^{-1}$. Fix an element $r \in \mathfrak{A} \cap \mathcal{C}_R$. Since $f_i(r) = b_i r$ are almost all zero, we see that the b_i's are almost all zero. After dropping the unnecessary indices, we may therefore assume that I is *finite*. From

$$r = \sum a_i f_i(r) = \sum a_i b_i r,$$

we have $1 = \sum a_i b_i \in \mathfrak{A}\mathfrak{A}^{-1}$. Hence $\mathfrak{A}\mathfrak{A}^{-1} = R$.

The last statement in (2.17) is just a repeat of parts of (2.14). □

(2.18) Corollary. *If $R \neq K = Q(R)$, then K is not a projective R-module.*

Proof. It suffices to show that K_R is not f.g. If it is, say,

$$K = \sum_{i=1}^{n} R \cdot c_i d_i^{-1} \qquad (c_i \in R,\ d_i \in \mathcal{C}_R),$$

then, for $d = d_1 \cdots d_n \in \mathcal{C}_R \subseteq \mathrm{U}(K)$, we have $K = dK \subseteq R$, a contradiction. □

In the following, we give some explicit examples of invertible and noninvertible ideals in commutative domains.

(2.19A) Example. Let k be a field. Then, in the commutative polynomial ring $R = k[x_1, \ldots, x_n]$ with $n \geq 2$ variables, *the ideal $\mathfrak{A} = (x_1, \ldots, x_n)$ is not invertible, and hence not projective.* In fact, it turns out that $\mathfrak{A}^{-1} = R$, so $\mathfrak{A}\mathfrak{A}^{-1} = \mathfrak{A} \neq R$. To see this, assume instead that there is some $f/g \in \mathfrak{A}^{-1}$, where $f, g \in R$ are relatively prime to each other, and $g \notin k$. Then $(f/g) \cdot x_i = h_i \in R$. If $x_1 | g$, then x_1 also divides $gh_2 = x_2 f$, so $x_1 | f$, a contradiction. Therefore, $x_1 \nmid g$, and $x_1 f = gh_1$ implies that $x_1 | h_1$. But then $f/g = h_1/x_1 \in R$, again a contradiction.

(2.19B) Example. Let k be a field in which $-1 \notin k^2$, and R be the coordinate ring of the "circle" over k; that is, $R = k[x, y]$ with the relation $x^2 + y^2 = 1$.[8] Let us compute $\mathfrak{A}^{-1}$ for the prime ideal $\mathfrak{A} = (1 - y, x)$. For $z := x/(1 - y)$ in the quotient field $k(x, y)$ of R, we have

$$z(1 - y) = x \in R \quad \text{and} \quad zx = x^2/(1 - y) = (1 - y^2)/(1 - y) = 1 + y \in R,$$

so $z \in \mathfrak{A}^{-1}$. Letting $\mathfrak{B} = R + Rz \subseteq \mathfrak{A}^{-1}$, we see that $\mathfrak{B}\mathfrak{A}$ contains

$$1 \cdot (1 - y) + zx = (1 - y) + (1 + y) = 2.$$

[8] It is easy to verify that R is a domain.

Since $-1 \notin k^2$, we have $2 \neq 0$ in k, so $\mathfrak{B}\mathfrak{A} = R$. This shows that $\mathfrak{A}$ *is invertible*, with $\mathfrak{A}^{-1} = \mathfrak{B}$. We claim that $\mathfrak{A}$ *is not principal* (so $\mathfrak{A}$ is a projective R-module that is not free). To see this, we represent R as $k[y] \oplus xk[y]$, and use the field norm N for the quadratic extension $k(x, y)/k(y)$. For $f, g \in k[y]$, we have

$$\begin{aligned} N(f + xg) &= (f + xg)(f - xg) = f^2 - x^2g^2 \\ &= f^2 + y^2g^2 - g^2 \in k[y]. \end{aligned} \tag{1}$$

If $\mathfrak{A}$ is principal, say, $\mathfrak{A} = (f + xg)$, then we'll have

$$1 - y = (f + xg)\alpha, \quad \text{and} \quad x = (f + xg)\beta$$

for some $\alpha, \beta \in R$. Taking norms and subtracting, we get

$$[N(\alpha) - N(\beta)]N(f + xg) = (1 - y)^2 + x^2 = 2(1 - y) \quad \text{in } k[y]. \tag{2}$$

From (1) above, however, it is clear that the highest degree term in $N(f + xg)$ can come only from y^2g^2, or f^2, or both. From the hypothesis that $-1 \notin k^2$, we see that $N(f + xg)$ has (even) degree ≥ 2. This clearly contradicts the equation (2). Therefore, $\mathfrak{A}$ is not principal. Nevertheless, $\mathfrak{A}^2$ is principal, since

$$\begin{aligned} \mathfrak{A}^2 &= (x^2,\ x(1 - y),\ (1 - y)^2) \\ &= (x^2 + (1 - y)^2,\ x(1 - y),\ (1 - y)^2) \\ &= (2(1 - y),\ x(1 - y),\ (1 - y)^2) \\ &= (1 - y). \end{aligned}$$

(2.19C) Example. Let $R = \mathbb{Z}[\theta]$ where $\theta = \sqrt{5}$. Note that R is a proper subring of the ring $S = \mathbb{Z}[\tau]$ of algebraic integers in the number field $K = \mathbb{Q}(\theta)$, where τ is the "golden ratio" $(\theta + 1)/2 = 2\cos 36°$. Let us compute $\mathfrak{A}^{-1}$ for the R-ideal $\mathfrak{A} = (2,\ 1 + \theta)$. Clearly, an element in $\mathfrak{A}^{-1}$ must have the form $(a + b\theta)/2$ where $a, b \in \mathbb{Z}$. For $(a + b\theta)/2$ to be in $\mathfrak{A}^{-1}$, the condition is that

$$(1 + \theta)(a + b\theta)/2 = [(a + 5b) + (a + b)\theta]/2 \in R,$$

which amounts to $a \equiv b \pmod 2$. This shows that $\mathfrak{A}^{-1} = S$. However, from

$$\mathfrak{A} = 2R + 2R\tau = 2S$$

and the fact that S is a ring, we have

$$\mathfrak{A}\mathfrak{A}^{-1} = 2S \cdot S = 2S = \mathfrak{A} \subsetneq R.$$

Hence, $\mathfrak{A}$ is not invertible. Similarly, we can show that S is *not* invertible, with $S^{-1} = \mathfrak{A}$.

(2.19D) Example. Let $R = \mathbb{Z}[\theta]$ where $\theta = \sqrt{-5}$. Here, R is the full ring of algebraic integers in $K = \mathbb{Q}(\theta)$. Let $\tau = (\theta + 1)/2$, and $\mathfrak{A} = (2,\ 1 + \theta) \subseteq R$. Just as in (2.19C), we can show that $\mathfrak{A}^{-1}$ is given by $S := \mathbb{Z} + \mathbb{Z}\tau$. However, contrary to

the case in (2.19C), S is only an R-module, *not a ring*. From $S = R + R\,\tau = \frac{1}{2}\mathfrak{A}$, we have now

$$\mathfrak{A}\mathfrak{A}^{-1} = \frac{1}{2}\mathfrak{A}^2 = \frac{1}{2}\left(4,\ 2(1+\theta),\ 1+2\theta+\theta^2\right)$$
$$= (2,\ 1+\theta,\ \theta-2) = R.$$

Therefore, $\mathfrak{A}$ is invertible. The equation $\mathfrak{A}^2 = 2R$ obtained above can be used to show that $\mathfrak{A}$ *is not principal* (and hence not free). For, if $\mathfrak{A} = (x + y\theta)R$ $(x, y \in \mathbb{Z})$, then we have $2 = \alpha(x + y\theta)^2$ for some $\alpha \in R$. Letting N be the field norm from K to $\mathbb{Q}$, we get

$$4 = N(\alpha)N(x + y\theta)^2 = N(\alpha)(x^2 + 5y^2)^2.$$

This forces (x, y) to be $(\pm 1, 0)$, which is clearly impossible.

(2.19E) Example. Let $i = \sqrt{-1}$ and consider the domain

$$R = \mathbb{Z}[6i] = \mathbb{Z} + 6\mathbb{Z}\cdot i \subset \mathbb{Z}[i] \subset K = \mathbb{Q}(i).$$

It is well known that $\mathbb{Z}[i]$ is a PID, but we shall show that, for the subring R, there exist nonprincipal invertible (fractional) ideals. In fact, for the fractional ideal $\mathfrak{A} = (3,\ 1+2i) \subset K$, an easy calculation shows that $\mathfrak{A}^2 = (3,\ 2i)$ and $\mathfrak{A}^4 = R$, so $\mathfrak{A}$ is invertible, with $\mathfrak{A}^{-1} = \mathfrak{A}^3$. *We claim that* $\mathfrak{A}^2$ (and hence $\mathfrak{A}$) *is not principal.* To see this, assume, instead, that $\mathfrak{A}^2$ is principal. Then $3\mathfrak{A}^2 = (9,\ 6i) \subset R$ is a principal ideal, say, $3\mathfrak{A}^2 = (x + 6yi)R$ for some $x,\ y \in \mathbb{Z}$. This gives

$$9R = 9\mathfrak{A}^4 = (x + 6yi)^2 R,$$

so $9 = \alpha(x + 6yi)^2$ for some $\alpha \in R$. Taking the field norm from K to $\mathbb{Q}$ gives $9^2 = N(\alpha)(x^2 + 36y^2)^2$, so $y = 0$ and $x \in \{\pm 1, \pm 3\}$. But then $(9,\ 6i) = (x + 6yi)R$ is either R or $3R$, both of which are easily seen to be impossible. (What can we say about $\mathfrak{A}$ if we define it as a fractional ideal, instead, over the rings $\mathbb{Z}[ki]$, where $1 \le k \le 5$?)

§2D. *The Picard Group of a Commutative Ring*

In this subsection, we shall give an introduction to the notion of the Picard group. The material here is a natural continuation of that of §2C. The principal object of study will be the class of f.g. projective modules of rank 1.

Throughout this subsection, R shall denote a commutative ring. By an R-module, we shall mean a right R-module, although it may also be viewed naturally as a left R-module. If $P,\ Q$ are R-modules, so is $P \otimes_R Q$. If $P,\ Q$ are both projective, then they are direct summands of suitable free modules, and so is $P \otimes_R Q$. Therefore, $P \otimes_R Q$ is also projective. This fact is a special feature of the commutative case, and will eventually enable us to define the Picard group.

Let P be a f.g. projective R-module. For any prime ideal $\mathfrak{p} \subset R$, the localization $P_\mathfrak{p} := P \otimes_R R_\mathfrak{p}$ is a f.g. projective $R_\mathfrak{p}$-module. Since $R_\mathfrak{p}$ is a local ring, $P_\mathfrak{p}$ must actually be free (*FC*–(19.29)), say, $P_\mathfrak{p} \cong R_\mathfrak{p}^{n_\mathfrak{p}}$. Thus, we get a function $\mathfrak{p} \mapsto n_\mathfrak{p}$

from Spec R (the prime spectrum of R) to $\mathbb{Z}$. The case when this is a constant function is especially important. We shall say that P *has rank* n (rk $P = n$ for short) if all $n_{\mathfrak{p}} = n$; that is, $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^n$ for all prime ideals $\mathfrak{p}$. The verification of the following fact is straightforward, and will be left to the reader:

Fact. *Let P, Q be f.g. projective R-modules of rank n and rank m respectively, and let $P^* = \mathrm{Hom}_R(P, R)$. Then $\mathrm{rk}(P^*) = n$ and $\mathrm{rk}(P \otimes_R Q) = nm$.*

For many commutative rings R, *every* f.g. projective R-module P has constant rank. In fact, this is the case precisely for the class of commutative rings with no idempotents other than 0, 1. We shall not pause to prove this fact here (see Exercise 22); suffice it for us to keep in mind that, if R is a commutative domain with quotient field K, then every f.g. projective R-module P has constant rank, equal to $\dim_K(P \otimes_R K)$. This is easily seen by localizing P first to $R_{\mathfrak{p}}$ (for $\mathfrak{p} \in$ Spec R), and then to its quotient field K.

Next we make the following basic observation.

(2.20) Lemma. *Let $R \subseteq S$ be commutative rings. Then any invertible R-module $P \subseteq S$ is f.g. projective of rank 1.*

Proof. Say $PQ = R$, where Q is a suitable R-submodule of S. For any $\mathfrak{p} \in$ Spec R, localization gives $P_{\mathfrak{p}} Q_{\mathfrak{p}} = R_{\mathfrak{p}} \subseteq S_{\mathfrak{p}}$. Therefore, $P_{\mathfrak{p}}$ is a f.g. projective $R_{\mathfrak{p}}$-module. Since $R_{\mathfrak{p}}$ is local, $P_{\mathfrak{p}}$ is free and therefore isomorphic to $R_{\mathfrak{p}}$ by (2.14)(4). This shows that $\mathrm{rk}(P) = 1$. □

Now let Pic(R) be the set of isomorphism classes of f.g. projective R-modules of rank 1. We shall denote the isomorphism class of a module P by $[P]$. From the fact on ranks stated earlier, it follows that Pic(R) has the structure of an abelian semigroup, by the operation $[P][Q] = [P \otimes_R Q]$. Obviously, $[R]$ serves as the identity for Pic(R).

For any $[P] \in$ Pic(R), we have a natural map $f : P \otimes_R P^* \to R$ in $\mathfrak{M}_R$. Using the fact that $(P^*)_{\mathfrak{p}} \cong (P_{\mathfrak{p}})^*$ (the proof of which is left as an exercise), we see that the localization $f_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in$ Spec R. It follows that f is an isomorphism, and so

$$[P][P^*] = [P \otimes_R P^*] = [R] \in \mathrm{Pic}(R).$$

Therefore, Pic(R) is an abelian group, with $[P]^{-1} = [P^*]$ for any $[P] \in$ Pic(R). We say that Pic(R) is the *Picard group* of the commutative ring R.

If $f : R \to K$ is a homomorphism of commutative rings, we may view K as an R-module via f. For any $[P] \in$ Pic(R), it is easy to check that $[P \otimes_R K] \in$ Pic(K), and that $[P] \mapsto [P \otimes_R K]$ defines a group homomorphism $f_* : \mathrm{Pic}(R) \to \mathrm{Pic}(K)$, with the property that $(fg)_* = f_* g_*$ and $(\mathrm{id})_* = \mathrm{id}$. With these definitions, we see that "Pic" *is a covariant functor from the category of commutative rings to the category of abelian groups.* (See Exercise 13.)

Now let $K = Q(R)$ (the localization of R at the multiplicative set $\mathcal{C}_R$ of all regular elements), and let $f : R \to K$ be the inclusion map. By (2.20), every invertible fractional ideal in K is f.g. projective of rank 1 over R, but in general, a f.g. projective module of rank 1 over R need not be isomorphic to an invertible fractional ideal in K. To account for this discrepancy, we proceed as follows. Let $\mathbb{I}_R$ be the set of invertible fractional ideals in K. Clearly, $\mathbb{I}_R$ is an abelian group under multiplication, with identity R. For any unit $s \in \mathrm{U}(K)$, $\pi(s) := sR$ is in $\mathbb{I}_R$ since $(sR)(s^{-1}R) = R$. On the other hand, for any $\mathfrak{A} \in \mathbb{I}_R$, $\alpha(\mathfrak{A}) := [\mathfrak{A}]$ is in $\mathrm{Pic}(R)$. By (2.14)(2), we have $\alpha(\mathfrak{A}\mathfrak{B}) = \alpha(\mathfrak{A})\alpha(\mathfrak{B})$ for $\mathfrak{A}, \mathfrak{B} \in \mathbb{I}_R$ so $\alpha : \mathbb{I}_R \to \mathrm{Pic}(R)$ is a group homomorphism. We are now in a position to prove the following important theorem.

(2.21) Theorem. *For $f : R \to K = Q(R)$, we have a five-term exact sequence*

$$1 \to \mathrm{U}(R) \xrightarrow{f} \mathrm{U}(K) \xrightarrow{\pi} \mathbb{I}_R \xrightarrow{\alpha} \mathrm{Pic}(R) \xrightarrow{f_*} \mathrm{Pic}(K).$$

Proof. It is straightforward to see that this is a 0-sequence. (For $f_*\alpha = 0$, use the fact that $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$ for any $\mathfrak{A} \in \mathbb{I}_R$.) For $s \in \mathrm{U}(K)$, if $\pi(s) = sR = R$, we have clearly $s \in \mathrm{U}(R)$. For $\mathfrak{A} \in \mathbb{I}_R$, if $\alpha(\mathfrak{A}) = [\mathfrak{A}] = [R]$, then $\mathfrak{A} \in \mathrm{im}(\pi)$ by the last statement of (2.17). Finally, let $[P] \in \mathrm{Pic}(R)$ be such that $f_*[P] = [K]$. Then $P \otimes_R K \cong K$ as K-modules. Since P can be embedded in a free R-module, the localization map $P \to P \otimes_R K$ is injective. Composing this with $P \otimes_R K \cong K$, we can embed P (as an R-module) in K. There exist $p \in P$ and $r \in \mathcal{C}_R$ such that $pr^{-1} = 1 \in K$, so $p = r \in P \cap \mathcal{C}_R$ and $P \cap \mathcal{C}_R \neq \emptyset$. By (2.17), $P \in \mathbb{I}_R$, so $[P] = \alpha(P) \in \mathrm{im}(\alpha)$. □

Let $\mathbb{P}_R = \pi(\mathrm{U}(K))$ be the subgroup of $\mathbb{I}_R$ consisting of the principal invertible fractional ideals. Using (2.21), we see easily that $\mathrm{coker}(\pi) = \mathbb{I}_R/\mathbb{P}_R$ is (essentially) the group of isomorphism classes of invertible fractional ideals. In view of this, $\mathbb{I}_R/\mathbb{P}_R$ is called the *ideal class group* of R. (Note that any $\mathfrak{A} \in \mathbb{I}_R$ is isomorphic to an invertible ideal in R, since $\mathfrak{A}_R$ is f.g.) From (2.21), we deduce immediately the following.

(2.21)′ Corollary. *Define the relative Picard group* $\mathrm{Pic}(K/R)$ *to be* $\ker(f_*)$ *in* (2.21). *Then* $\mathrm{Pic}(K/R) \cong \mathbb{I}_R/\mathbb{P}_R$. *In particular, if f_* is the trivial homomorphism (e.g., in the case when R is a commutative domain and K is its quotient field), we have* $\mathrm{Pic}(R) \cong \mathbb{I}_R/\mathbb{P}_R$. *In this case, P is a f.g. projective R-module of rank 1 iff P is isomorphic to an invertible ideal of R.*

In general, however, f_* may not be trivial. In this case, $\mathrm{Pic}(K/R) \cong \mathbb{I}_R/\mathbb{P}_R$ is only a *proper* subgroup of $\mathrm{Pic}(R)$, so there are (f.g.) rank 1 projectives over R that are *not* isomorphic to any invertible ideals. An easy example of this nature, shown to me by H. Lenstra, is given in (A) below.

(2.22) Examples.

(A) We start by describing a very useful ring-theoretic construction called the "trivial extension". Let S be any ring and M be any (S, S)-bimodule. This means that M is at the same time a left and a right S-module, with the property that $(sm)s' = s(ms')$ for all $s, s' \in S$ and all $m \in M$. Given such a bimodule M, we form $R := S \oplus M$, and define a multiplication on R by the rule

$$(s, m)(s', m') = (ss', sm' + ms').$$

It is routine to verify that R is a ring with identity $(1, 0)$, having $S = S \oplus (0)$ as a subring, and with $M = (0) \oplus M$ as an ideal of square zero, such that $R/M \cong S$ as rings. The (S, S)-bimodule structure on M is recovered by the ideal structure on M. The ring R constructed in this way is called the "trivial extension" of M by S. It is basically the only ring we can construct out of $S \oplus M$ with all of the aforementioned properties.[9] In the special case when the ring S is *commutative*, we could just take M to be, say, a right S-module, since we can define the left S-structure on M to be identical to its right S-structure. The resulting trivial extension $R = S \oplus M$ will then be also a commutative ring.

To apply this to our setting, we start with any commutative ring S, and take M to be any S-module such that any nonunit in S kills some nonzero element in M. (Such an S-module always exists; for instance, take $M = \bigoplus\{S/aS : a \in S \setminus \mathrm{U}(S)\}$.) We then form the (commutative) trivial extension $R = S \oplus M$. It is easy to check that

$$\mathcal{C}_R = \{(s, m) : \ s \in \mathrm{U}(S), \ m \in M\} = \mathrm{U}(R).$$

Thus, R coincides with its total ring of quotients, and the ideal class group $\mathbb{I}_R/\mathbb{P}_R$ is trivial. To compute $\mathrm{Pic}(R)$, consider the homomorphisms

$$\mathrm{Pic}(S) \xrightarrow{i_*} \mathrm{Pic}(R) \xrightarrow{j_*} \mathrm{Pic}(S)$$

induced by the inclusion $i : S \to R$ and the projection $j : R \to R/M = S$. Since $ji = \mathrm{Id}_S$, $\mathrm{Pic}(R) \cong \mathrm{Pic}(S) \oplus \ker(j_*)$. But by *FC*–(19.27), $\ker(j_*) = 0$, since $M^2 = 0$. Therefore, $\mathrm{Pic}(R) \cong \mathrm{Pic}(S)$, which, of course, may not be trivial. Summarizing, *the (f.g.) rank* 1 *projectives over* R *all "come" from those over* S, *but none of the nonfree ones is isomorphic to invertible ideals of* R, *since all invertible ideals are principal.* (For an explicit construction, let $S = \mathbb{Z}[\sqrt{-5}]$ and M be the quotient S-module $\mathbb{Q}[\sqrt{-5}]/S$, on which every nonunit of S acts as a 0-divisor. By (B) below, $\mathrm{Pic}(R) \cong \mathrm{Pic}(S) \cong \mathbb{Z}/2\mathbb{Z}$.)

(B) For the commutative domains R in (2.19B), (2.19D), and (2.19E), we have constructed invertible fractional ideals that represent, respectively, elements of order 2, 2, and 4 in the ideal class group $\mathbb{I}_R/\mathbb{P}_R \cong \mathrm{Pic}(R)$. As a matter of fact,

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[9] We can also think of R, if we like, as the subring of the triangular ring $\begin{pmatrix} S & M \\ 0 & S \end{pmatrix}$ consisting of matrices $\begin{pmatrix} s & m \\ 0 & s \end{pmatrix}$ where $s \in S$ and $m \in M$.

Pic(R) turns out to be the group $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/4\mathbb{Z}$ respectively in these three examples, although we shall not go into the details of the computation here.

(C) If R is a *local* ring, $\text{Pic}(R) = \{1\}$ by *FC*–(19.29). (See also Exercise 12.)

(D) Let R be a *semilocal* ring, with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Then the Jacobson radical of R is $\text{rad } R = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$, and by the Chinese Remainder Theorem,

$$\bar{R} := R/\text{rad } R \cong R/\mathfrak{m}_1 \times \cdots \times R/\mathfrak{m}_n .$$

Since $\text{Pic}(A \times B) \cong \text{Pic}(A) \times \text{Pic}(B)$, Example (C) above gives $\text{Pic}(\bar{R}) = \{1\}$. Applying *FC*–(19.27) again, we see that $\text{Pic}(R) = \{1\}$. In particular, this applies to all (commutative) artinian rings.

(E) Let R be any noetherian ring, or more generally, a (commutative) ring with ACC on its annihilator ideals.[10] We will show in (8.31)(2) that $K := Q(R)$ is a semilocal ring. Assuming this result, we have $\text{Pic}(K) = \{1\}$ by (D) above, and the "good" case in (2.21)$'$ applies. It follows that $\text{Pic}(R) \cong \mathbb{I}_R/\mathbb{P}_R$; that is, any f.g. rank 1 projective over R is isomorphic to an invertible ideal of R.

(F) *If* R *is a* UFD, $\text{Pic}(R) = \{1\}$. To see this, let us verify that any invertible ideal $\mathfrak{A} \subseteq R$ is principal. Start with an equation $1 = \sum_{i=1}^{n} b_i a_i$, where the a_i 's generate $\mathfrak{A}$, and $b_i \in \mathfrak{A}^{-1}$. Write $b_i = c_i/d_i$, where $c_i, d_i \in R$ have no common prime divisor. Since $b_i a_j \in R$, we have $d_i | c_i a_j$, and hence $d_i | a_j$ for any i, j. Let $d = \text{lcm}\{d_1, \ldots, d_n\}$. Then $d | a_j$ for every j, so $\mathfrak{A} \subseteq R \cdot d$. On the other hand,

$$d = \sum_{i=1}^{n} \frac{d}{d_i} c_i a_i \in \sum_{i=1}^{n} R a_i = \mathfrak{A} ,$$

so we have $\mathfrak{A} = R \cdot d$, as desired. In particular, if k is any field, and $R = k[x_1, \ldots, x_n]$, then any f.g. rank 1 projective R-module is free. Confirming "Serre's Conjecture," A. Suslin and D. Quillen have independently shown that, in fact, any f.g. projective R-module is free. For a detailed exposition of this, see [Lam: 78].

In commutative algebra, a *Dedekind ring* (or a Dedekind domain) is defined to be a commutative domain R whose nonzero ideals are all invertible (or projective). By (2.17), it follows that such an R must be a *noetherian* domain. Among all (commutative) noetherian domains, Dedekind domains may be characterized as those that are integrally closed of Krull dimension ≤ 1. Alternatively, Dedekind rings may also be characterized as commutative domains in which every ideal is a finite product of prime ideals. These results can be found in most standard treatises in commutative algebra, so they will not be repeated here.

In algebraic geometry, Dedekind rings arise as the coordinate rings of smooth affine curves. For instance, the ring considered in Example (2.19B) is such a

[10] An annihilator ideal (in R) is one of the form $\text{ann}^R(X)$, where X is any subset of R. Rings satisfying chain conditions on annihilator ideals will be considered in more detail in §6E.

Dedekind ring. In number theory, Dedekind rings arise as (full) rings of algebraic integers in number fields (= finite field extensions of $\mathbb{Q}$). For instance, the ring considered in Example (2.19D) is such a Dedekind ring. If R is a Dedekind ring of this type, a basic theorem in number theory states that $\text{Pic}(R)$ is a *finite* (abelian) group; the cardinality h_R of this group is called the *class number* of R (or of its quotient field). Using this famous "finiteness of class number" theorem, it can further be shown that if R is *any* commutative domain whose field of quotients is a number field, then $\text{Pic}(R)$ is a finite group. These groups $\text{Pic}(R)$ are among the best known and most computed invariants in the theory of algebraic numbers. In the case when R is the full ring of algebraic integers in a quadratic number field $\mathbb{Q}(\sqrt{d})$, the structure of $\text{Pic}(R)$ can even be determined (by experts!) on a programmable hand-held calculator, if $|d|$ is within a reasonable range. However, to treat any of these results in detail would take us too far afield.

In general, not much can be said about the structure of the Picard group of a Dedekind domain. In fact, L. Claborn has shown that, for any given (finite or infinite) abelian group G, there always exists a Dedekind domain R with $\text{Pic}(R) \cong G$!

Using the notion of the Picard group, we can also streamline some of our earlier results presented in (2.15). As in (2.15), let $R \subseteq S$ be commutative rings and let $g \in S$ be such that $g^2, g^3 \in R$. For $r \in R$, let $P_r = (1 + rg, g^2)$ be the R-submodule of S generated by $1 + rg$ and g^2. Since P_r is an invertible R-submodule, $[P_r] \in \text{Pic}(R)$. The relation $P_r P_s = P_{r+s}$ proved earlier shows that $r \mapsto [P_r]$ defines a group homomorphism $\pi : R \to \text{Pic}(R)$. By (2.15B), the kernel of π is given by the following subgroup of R:

(2.23A) $$\tilde{J} := \{r \in R : u(1 + rg) \in R \text{ for some } u \in \text{U}(R[g])\}.$$

Therefore, π induces an injective homomorphism

(2.23B) $$\bar{\pi} : R/\tilde{J} \longrightarrow \text{Pic}(R).$$

Note that $\tilde{J}$ contains the conductor ideal $J = \{r \in R : rg \in R\}$. In the good case when the technical condition (2.15)(D) is satisfied (i.e., for $u \in \text{U}(R[g])$, $u(1+rg) \in R \Longrightarrow u \in R$), then $\tilde{J} = J$ by (2.15)(C), and $\bar{\pi}$ *gives an embedding of* R/J *into* $\text{Pic}(R)$.

While the preceding results were based on the ad hoc computations in (2.15), more precise results can be obtained by using the tools of algebraic K-theory. The following remarks are due to H. Lenstra, R. Swan, and R. Wiegand. From the "conductor square":

$$\begin{array}{ccc} R & \longrightarrow & R[g] \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R[g]/J \end{array}$$

there results a Mayer-Vietoris sequence (see Bass [68: p. 482]):

$$\begin{array}{ccccc} U(R) & \longrightarrow & U(R[g]) \oplus U(R/J) & \longrightarrow & U(R[g]/J) \\ & & & \swarrow & \\ \operatorname{Pic}(R) & \longrightarrow & \operatorname{Pic}(R[g]) \oplus \operatorname{Pic}(R/J) & \longrightarrow & \operatorname{Pic}(R[g]/J). \end{array}$$

After "unhooking" this exact sequence and identifying $\operatorname{Pic}(R[g]/J)$ with $\operatorname{Pic}(R/J)$ (omitting all details), one obtains a short exact sequence:

(2.23C) $$0 \longrightarrow R/\tilde{J} \xrightarrow{\bar{\pi}} \operatorname{Pic}(R) \longrightarrow \operatorname{Pic}(R[g]) \longrightarrow 0,$$

where $\tilde{J}$ and $\bar{\pi}$ are as defined before. *In particular, this enables us to identify* $R/\tilde{J}$ *with the relative Picard group* $\operatorname{Pic}(R[g]/R)$.

For an explicit example, consider $R = k[z^2, z^3] \subset S = k[z]$, where k is a field. (R is the coordinate ring of the degenerate elliptic curve $y^2 = x^3$.) For $g = z$, we have observed before that (2.15D) is satisfied, and so $\tilde{J} = J = (z^2, z^3)$. Since

$$\operatorname{Pic}(R[g]) = \operatorname{Pic}(k[z]) = \{1\},$$

the short exact sequence (2.23C) yields:

$$\operatorname{Pic}(k[z^2, z^3]) \cong k[z^2, z^3]/(z^2, z^3) \cong k.$$

This is a well-known fact in algebraic geometry, usually proved by using more sophisticated machinery.

There is also a useful application of this circle of ideas to the study of the Picard group of a polynomial extension. Let $R = A[x]$, where A is a commutative ring. Consider the natural ring homomorphisms:

$$A \xrightarrow{i} R = A[x] \xrightarrow{j} A,$$

where i is the inclusion and $j(h(x)) = h(0)$. We have the induced homomorphisms

$$\operatorname{Pic}(A) \xrightarrow{i_*} \operatorname{Pic}(A[x]) \xrightarrow{j_*} \operatorname{Pic}(A)$$

whose composition is the identity. Following standard notation in algebraic K-theory, we write $\operatorname{NPic}(A) := \ker(j_*)$. Since i_* is a split monomorphism from the above, we have

$$\operatorname{Pic}(A[x]) \cong \operatorname{Pic}(A) \oplus \operatorname{NPic}(A).$$

Thus, $\operatorname{NPic}(A)$ may also be interpreted as the cokernel of i_*. Assuming that $\operatorname{Pic}(A)$ is known, the computation of $\operatorname{Pic}(A[x])$ boils down to that of $\operatorname{NPic}(A)$.

Now let $A \subseteq F$ be (commutative) rings with $\operatorname{Nil}(F) = 0$, and let $b \in F$ be such that $b^2, b^3 \in A$. Form the polynomial rings

$$R = A[x] \subseteq S = F[x],$$

and let $g = bx \in S$, with $g^2, g^3 \in R$. For any $r \in R$, we can then form the Schanuel module $P_r = (1 + rg, g^2)$ which defines an element $[P_r] \in \operatorname{Pic}(R)$. Putting together our earlier results on Schanuel modules, we have the following.

(2.23) Theorem. *Let $J_0 = \{a \in A : ab \in A\}$ be the conductor ideal for the pair $A \subseteq A[b]$. Then $r \mapsto [P_r]$ defines an injective group homomorphism $\bar{\pi}$: $(A/J_0)[x] \to \mathrm{NPic}(A)$.*

Proof. As we have observed at the end of §2B, the subgroup $\tilde{J}$ defined in (2.23A) is given by the ideal $J_0[x]$. Therefore, (2.23B) gives an embedding $\bar{\pi} : (A/J_0)[x] \to \mathrm{Pic}(R)$. We now finish by proving that $[P_r] \in \mathrm{NPic}(A)$ for any $r \in R$. To this end, first note that

$$g^2 = (1+rg)(g^2 - rg^3) + r^2g^4 \in (1+rg)xR + g^2xR = P_rx.$$

This shows that $P_r = (1+rg)A \oplus P_rx$ (as A-modules), and so

$$P_r \otimes_R (R/xR) \cong P_r/P_rx \cong (1+rg)A \cong A,$$

whence $[P_r] \in \mathrm{NPic}(A)$. □

To explain the significance of (2.23), we shall record an application of it to the study of "p-seminormal rings". For simplicity, *we specialize now to the case where A is a commutative domain, and F is its quotient field.* The following definition should not be surprising to the reader in view of what we have done so far. Let p be either a prime number, or 0. We say that A is *p-seminormal* if, for any element $b \in F$:

$$b^2 \in A, \quad b^3 \in A, \quad pb \in A \implies b \in A.$$

If A is 0-seminormal, we simply say that it is *seminormal.* (The defining property here is that $b^2 \in A, b^3 \in A \implies b \in A$.)

The notion of p-seminormality is intended to be a generalization of the more familiar classical notion of normality. Note that if A is normal (i.e., integrally closed), then A is seminormal, and if A is seminormal, then A is p-seminormal for any prime p.

With all this terminology in place, we can now derive the following consequence of (2.23).

(2.23)′ Corollary. *Let p be a prime number, or 0. If the group $\mathrm{NPic}(A)$ has no p-torsion, then the domain A must be p-seminormal.*[11] *If A is not seminormal and $\mathbb{Q} \subseteq A$, then $\mathrm{NPic}(A)$ contains, in fact, an infinite-dimensional vector space over $\mathbb{Q}$.*

Proof. Assume A is *not* p-seminormal, and fix an element $b \in F \backslash A$ such that b^2, b^3, $pb \in A$. The conductor ideal $J_0 = \{a \in A : ab \in A\}$ contains then the element $p \cdot 1$, but is different from A, and, by (2.23), $\mathrm{NPic}(A)$ contains a copy of $(A/J_0)[x]$. If $p > 0$, $\mathrm{NPic}(A)$ then contains an $\mathbb{F}_p$-vector space of infinite dimension; in particular, it has p-torsion. If $p = 0$, $\mathrm{NPic}(A)$ is an infinite group; in particular $\mathrm{NPic}(A) \neq 0$. If, in the latter case, $\mathbb{Q} \subseteq A$, then $\mathbb{Q} \cap J_0 = 0$, and

[11] **Convention**: In the case $p = 0$, "no 0-torsion" is to be interpreted as "equal to 0".

$(A/J_0)[x]$ is an infinite-dimensional $\mathbb{Q}$-vector space contained in $\mathrm{NPic}(A)$. □

Remarkably, the converse of the first statement in (2.23)′ is also true; that is, *the domain* A *is* p*-seminormal iff* $\mathrm{NPic}(A)$ *has no* p*-torsion*! The proof of this, using basic techniques of commutative algebra and algebraic K-theory, can be found in Swan [80]. The case when $p = 0$ appeared earlier in Brewer-Costa [79]. In the above, we have managed to prove half of the theorem of Brewer, Costa, and Swan without invoking commutative algebra or algebraic K-theory, and using the direct calculations in (2.15) instead.

§2E. Hereditary and Semihereditary Rings

In §§2C–2D, we have chosen to give a fairly detailed account of invertible fractional ideals and the Picard group since this material is truly basic to the theory of projective modules. Our coverage of invertible ideals and Dedekind domains in §2D was also intended as a motivation for what is to come in this subsection. Returning now to general (not necessarily commutative) rings, we introduce the following important definition.

Definition. A ring R is said to be *right* (resp. *left*) *hereditary*[12] if every right (resp. left) ideal of R is projective as a right (resp. left) R-module. If R is both right and left hereditary, we say that R is *hereditary.*

According to this definition, the hereditary rings among commutative domains are precisely the Dedekind rings. Before giving further examples of right hereditary rings, let us first prove the following major result on submodules of free modules over such rings.

(2.24) Kaplansky's Theorem. *Let R be a right hereditary ring. Then any submodule P of a free right R-module $F = \bigoplus_{\alpha\in I} e_\alpha R$ is isomorphic to a direct sum of right ideals of R; in particular, P is a projective module.*

Proof. We fix a well-ordering "$<$" on the indexing set I. For any $\alpha \in I$, let F_α (resp. G_α) be the span of the e_β's with $\beta \leq \alpha$ (resp. $\beta < \alpha$). Then each $a \in P \cap F_\alpha$ has a unique decomposition $a = b + e_\alpha r$ with $b \in G_\alpha$ and $r \in R$. The mapping $\varphi : a \mapsto r$ maps $P \cap F_\alpha$ onto a right ideal $\mathfrak{A}_\alpha$ with kernel $P \cap G_\alpha$. Since $(\mathfrak{A}_\alpha)_R$ is projective, φ splits, so we have

$$P \cap F_\alpha = (P \cap G_\alpha) \oplus A_\alpha$$

for some submodule A_α of $P \cap F_\alpha$ isomorphic to $\mathfrak{A}_\alpha$. We finish off by showing that $P = \bigoplus_{\alpha\in I} A_\alpha$. First suppose we have $a_1 + \cdots + a_n = 0$, where $a_i \in A_{\alpha_i}$. We may assume that $\alpha_1 < \cdots < \alpha_n$ in the ordering of I. Then $a_1, \ldots, a_{n-1} \in G_{\alpha_n}$

[12]We have briefly encountered this notion in an example in *FC*–§25 (p. 378).

and $a_n \in A_{\alpha_n}$. Since $G_{\alpha_n} \cap A_{\alpha_n} = 0$, we have $a_n = 0$ and hence all $a_i = 0$ by induction. Finally, we need to show that $P = \sum_{\alpha \in I} A_\alpha$. If this does not hold, there would exist a smallest β such that $P \cap F_\beta$ contains an element, say, a, not belonging to $\sum_\alpha A_\alpha$. Write $a = b + c$, where $b \in P \cap G_\beta$ and $c \in A_\beta$. The element b lies in $P \cap F_\gamma$ for some $\gamma < \beta$. By the minimal choice of β, we must have $b \in \sum_\alpha A_\alpha$. But then $a = b + c \in \sum_\alpha A_\alpha$, a contradiction. □

(2.25) Corollary. *Over a right hereditary ring R, an R-module P_R is projective iff it is embeddable into a free right R-module.*

(2.26) Corollary. *A ring R is right hereditary iff submodules of projective right R-modules are projective.*

(2.27) Corollary. *If R is a* PRID *(principal right ideal domain), then any submodule of a free right R-module is free.*

(Recall that a PRID is a domain R in which any right ideal $\mathfrak{A} \subseteq R$ is principal.[13] Since R is a domain, $\mathfrak{A}$ is in fact free, and hence projective; in particular, R is an example of a right hereditary domain.)

(2.28) Definition. A ring R is said to be *right (resp. left) semihereditary* if every f.g. right (resp. left) ideal of R is projective as a right (resp. left) R-module. If R is both right and left semihereditary, we say that R is *semihereditary*. A commutative semihereditary domain is called a *Prüfer domain* (just as a commutative hereditary domain is called a Dedekind domain).

For right semihereditary rings, we have the following easy analogue of (2.24), due to F. Albrecht.

(2.29) Theorem. *Let R be a right semihereditary ring. Then any f.g. submodule P of a free (or projective) right R-module F is isomorphic to a finite direct sum of f.g. right ideals of R; in particular, P is a projective module.*

Proof. It suffices to deal with the case when F is free, and we may assume that $F = \bigoplus_{i=1}^n e_i R$. We proceed by induction on n, the case $n = 1$ being covered by the definition. Arguing as in the proof of (2.24), we have a direct sum decomposition $P = (P \cap F_1) \oplus A$ where $F_1 = \bigoplus_{i=1}^{n-1} e_i R$, and A is a submodule of P isomorphic to a f.g. right ideal of R. As a direct summand of P, $P \cap F_1$ is also f.g. We are done by invoking our inductive hypothesis on $P \cap F_1 \subseteq F_1$. □

(2.30) Corollary. *A ring R is right semihereditary iff f.g. submodules of projective right R-modules are projective.*

[13] Similarly, a PLID (principal left ideal domain) is a domain in which any left ideal is principal.

A right module M over a commutative domain R is said to be *torsionfree* if, for $m \in M$ and $r \in R$, $mr = 0 \implies m = 0$ or $r = 0$. Clearly, a free module is torsionfree, so a projective module is also torsionfree. Conversely, a torsionfree module need not be projective. For instance, if $R \neq K$, the quotient field of R, then K_R is torsionfree, but not projective according to (2.18). A more interesting question to ask would be: *when is a f.g. torsionfree R-module projective*?

Note that any f.g. torsionfree module M over a commutative domain R can be embedded into a free module R^n. In fact, localizing M at $S = R \backslash \{0\}$, we have an inclusion

$$M \subseteq S^{-1}M = M \otimes_R K = K^n \quad \text{(for some } n\text{)},$$

after making appropriate identifications. Since M is f.g., there exists $r \in S$ such that $Mr \subseteq R^n$, so we can embed M into R^n by right multiplication by r. This observation leads quickly to the following consequence of (2.30).

(2.31) Corollary. *A commutative domain R is a Prüfer domain iff every f.g. torsionfree R-module is projective.*

Let us now give some examples of right hereditary (and semihereditary) rings.

(2.32) Examples.

(a) *Every semisimple ring R is hereditary.* In fact, over R, all left or right modules are projective.

(b) As we have already mentioned, *any* PRID *is right hereditary.* A nice example is $D[x]$, the polynomial ring in one variable over a division ring D. Another example is Hurwitz' ring of integral quaternions, mentioned in *FC*–(1.1). (A detailed proof for the fact that this latter ring is a PRID can be found in Herstein's *Topics in Algebra*, p. 375.)

(c) Let R be a Dedekind ring, and $S = \mathbb{M}_n(R)$. In Chapter 7, we will show that there is a natural equivalence between the two module categories $\mathfrak{M}_S$ and $\mathfrak{M}_R$. Under this natural equivalence, projective right S-modules M correspond to projective right R-modules P, and submodules of M correspond to submodules of P. Since submodules of P are projective in $\mathfrak{M}_R$ (by (2.26)), we see that submodules of M are projective in $\mathfrak{M}_S$. It follows again from (2.26) that S is a right hereditary ring. Similarly, if R is a Prüfer domain, then $S = \mathbb{M}_n(R)$ is a semihereditary ring.

(d) Consider any von Neumann regular ring, that is, a ring R in which every element $a \in R$ can be written in the form axa for some $x \in R$ (depending on a). Here, $e = ax \in aR$ is an idempotent, and $a = ea \in eR$ implies that $aR = eR$. More generally, any f.g. right ideal $\mathfrak{A}$ of R can be expressed in the form eR for a suitable idempotent e (see *FC*–(4.23)). Since $R = eR \oplus (1-e)R$, $\mathfrak{A}_R$ is projective. Therefore, *R is a right (and left) semihereditary ring.* From (2.29), it follows that a f.g. right R-module P is projective iff P is isomorphic to a finite direct sum of principal right ideals.

(e) *In a von Neumann regular ring* R, *any countably generated right ideal* $\mathfrak{B}$ *is projective.* In fact, write $\mathfrak{B}$ as the ascending union of a chain $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$, where each $\mathfrak{A}_n$ is a f.g. right ideal. By what we said in (d) above, $(\mathfrak{A}_n)_R$ is a direct summand in R, so we can write $\mathfrak{A}_{n+1} = \mathfrak{A}_n \oplus \mathfrak{B}_n$ for a suitable right ideal $\mathfrak{B}_n$. Since $(\mathfrak{A}_{n+1})_R$ is projective, $(\mathfrak{B}_n)_R$ is also projective. By (2.5), it follows that $\mathfrak{B} = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \cdots$ is projective. In particular, *if* R *is a countable von Neumann regular ring, then* R *is right (and left) hereditary.*

(f) "Hereditary orders" occur frequently in the theory of integral representations. In fact, if R is a Dedekind domain with quotient field K, then any maximal R-order $\mathcal{O}$ in a finite-dimensional separable K-algebra A is always hereditary. (See Reiner's book *Maximal Orders* for the details.) For instance, let $R = \mathbb{Z}$, $K = \mathbb{Q}$, and A be the division algebra of all rational quaternions. Then, Hurwitz' ring of integral quaternions, $\mathcal{O}$, is a maximal order in A. Here, $\mathcal{O}$ is not just hereditary; it is in fact a PRID (and a PLID), as we have mentioned in (b) above.

(g) It is easy to see that the direct product of any two right (semi) hereditary rings is also right (semi) hereditary.

(h) The Weyl algebras $A_n(k)$ over a field k of characteristic zero are simple domains (cf. *FC*–(3.17)). The first Weyl algebra $A_1(k)$ turns out to be hereditary; for a proof of this, see McConnell-Robson [87: p. 250].

(i) Let R be a free algebra generated over a field by $\{x_i : i \in I\}$. It is known that R is a *right free ideal ring* (or "right fir") in the sense that R has IBN, and any right ideal of R is free. (See Cohn [85: p. 106].) In particular, R is right (and of course also left) hereditary.

(j) For *commutative* domains R, (2.17) implies that if R is hereditary, then R is noetherian. Taking $|I| \geq 2$ in (i), however, we see that a hereditary domain need not be right noetherian or left noetherian. For a necessary and sufficient condition for a right hereditary ring to be right noetherian, see (7.58).

(k) A commutative domain R is said to be a *valuation domain* if, for every nonzero element x in its quotient field, either x or x^{-1} is in R. For such a domain R, it is easy to see that the ideals of R form a chain. It follows that (R is local and) every f.g. ideal of R is principal; thus, R is a semihereditary domain.

§2F. Chase Small Examples[14]

The first book in which the notion of a right hereditary ring appeared was probably Cartan-Eilenberg [56]. At the time when this classic was written, it was not known whether "right hereditary" and "left hereditary" were equivalent properties. One year later, I. Kaplansky constructed the first example of a right hereditary ring that is not a left hereditary ring. Another, considerably easier, example was later constructed by L. Small. We shall present this Small example below: it shows not

[14]Due to lack of space, we won't pursue large ones.

only that "right hereditary" is different from "left hereditary", but also that "left hereditary" is different from "left semihereditary".

(2.33) Small's Example. The "triangular ring" $T = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right noetherian but not left noetherian, by *FC*–(1.22). We will show that it has the following three additional properties:

(a) *T is right hereditary.*
(b) *T is not left hereditary.*
(c) *T is left semihereditary.*

The proofs are based upon the knowledge of the structure of the left and right ideals of T, as obtained in *FC*–(1.17). We begin with (a). According to *FC*–(1.17), the right ideals of T are of the following types:

$$M_1 = \begin{pmatrix} n\mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \qquad (n \neq 0),$$

$$M_2 = \begin{pmatrix} n\mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix} \qquad (n \neq 0),$$

$$M_V = \left\{ \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} : \begin{pmatrix} p \\ q \end{pmatrix} \in V \;\; (\text{a } \mathbb{Q}\text{-subspace of } \mathbb{Q} \oplus \mathbb{Q}) \right\}.$$

We need to show that these are *projective* as right T-modules. Since

$$\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} r & p \\ 0 & q \end{pmatrix} = \begin{pmatrix} nr & np \\ 0 & nq \end{pmatrix},$$

we have $M_1 = nT$, and this is free of rank 1. On the other hand,

$$(*) \qquad M_1 = M_2 \oplus \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{pmatrix} = M_2 \oplus M_{V_0} \qquad \left(V_0 = (0) \oplus \mathbb{Q} \right),$$

so M_2 and M_{V_0} are also T-projective. Lastly, M_V as a T-module is obtained by "pulling back" the $\mathbb{Q}$-module $V_\mathbb{Q}$ along the ring homomorphism $T \to \mathbb{Q}$ sending $\begin{pmatrix} r & p \\ 0 & q \end{pmatrix}$ to q, since

$$\begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} \begin{pmatrix} r' & p' \\ 0 & q' \end{pmatrix} = \begin{pmatrix} 0 & pq' \\ 0 & qq' \end{pmatrix}.$$

As a $\mathbb{Q}$-module, $V_\mathbb{Q}$ is isomorphic to either (0), $\mathbb{Q}$, or $\mathbb{Q} \oplus \mathbb{Q}$. In the case when $V \cong \mathbb{Q} \cong V_0$, we already know from $(*)$ that $M_V \cong M_{V_0}$ is T-projective. (Alternatively, M_{V_0} is just eT for $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$!) Therefore, when $V \cong \mathbb{Q} \oplus \mathbb{Q}$, $M_V \cong M_{V_0} \oplus M_{V_0}$ is T-projective as well. This proves (a).

For (b) and (c), recall from FC–(1.17) that the *left* ideals of T are of the following types:

$$N_1 = \begin{pmatrix} n\mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \qquad (n \neq 0),$$

$$N_2 = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix},$$

$$N_G = \left\{ \begin{pmatrix} r & p \\ 0 & 0 \end{pmatrix} : (r, p) \in G \ \big(\text{a subgroup of } \mathbb{Z} \oplus \mathbb{Q}\big) \right\}.$$

As before, we see easily that $N_1 = T \cdot n$ is free of rank 1, and $N_2 = T \cdot e$ (for $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$) is T-projective. Lastly, notice that N_G as a T-module is obtained by "pulling back" the $\mathbb{Z}$-module ${}_{\mathbb{Z}}G$ along the ring homomorphism $\varphi : T \to \mathbb{Z}$ sending $\begin{pmatrix} r & p \\ 0 & q \end{pmatrix}$ to r. *Take such an* N_G *which is f.g. as a left* T*-ideal.* Then G is f.g. as a subgroup of $\mathbb{Z} \oplus \mathbb{Q}$, and is thus isomorphic to (0), $\mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z}$. For $G_0 = \mathbb{Z} \oplus (0)$, $N_{G_0} = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix} = T \cdot e'$ for $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so N_{G_0} is T-projective. It follows that, in general, N_G is isomorphic to (0), N_{G_0}, or $N_{G_0} \oplus N_{G_0}$, and is therefore T-projective as well. This proves (c).

Finally, take G to be $(0) \oplus \mathbb{Q}$, which is *not* f.g. over $\mathbb{Z}$. Then N_G is not f.g. as a left ideal in T. If N_G is T-projective, then, reducing modulo the kernel I of the homomorphism $\varphi : T \to \mathbb{Z}$, $N_G / I \cdot N_G \cong G \cong \mathbb{Q}$ would be $\mathbb{Z}$-projective. This contradicts (2.18), so we have proved (b).[15]

By a slight variation of the preceding arguments, we can show that the properties (a), (b), and (c) already hold for any triangular ring $T = \begin{pmatrix} R & K \\ 0 & K \end{pmatrix}$, where R is any Dedekind domain not equal to its quotient field K. (For a further generalization, see Exercise (5.23).)

(2.34) Chase's Example. Let S be a von Neumann regular ring with an ideal I such that, as a submodule of S_S, I is not a direct summand. (For instance, any commutative nonsemisimple von Neumann regular ring S has such an ideal I.) Let $R = S/I$, which is also a von Neumann regular ring. As a right S-module, R is not projective (for otherwise $S \twoheadrightarrow S/I$ splits in $\mathfrak{M}_S$). Viewing R as an (R, S)-bimodule, we can form the triangular ring $T = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$. We claim that *$T$ is left semihereditary but not right semihereditary.*

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[15]Actually, given the property (a) and the fact that T is right noetherian, (c) is not an accident. Small has shown in general that *any right noetherian right hereditary ring is left semihereditary.* For a proof of this result, see (7.65).

To see the latter, simply look at the principal right ideal

$$M = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T \subseteq T.$$

This is obtained by pulling back R_S along the natural surjection $T \twoheadrightarrow S$. Since R_S is not projective, it follows as in the previous example that M_T is not projective. Therefore, T *is not right semihereditary.*

Next, consider a finitely generated left ideal N of T. By FC–(1.17), N has the form $V \oplus \mathfrak{A}$, where $\mathfrak{A}$ is a left ideal of S, and V is a left R-submodule of $R \oplus R$ containing $0 \oplus \bar{\mathfrak{A}}$, where $\bar{\mathfrak{A}}$ denotes the image of $\mathfrak{A}$ in R. (Here we follow the notation in FC–(1.17) and think of T as $R \oplus R \oplus S$.) If $\begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix}$ $(1 \leq i \leq n)$ are left ideal generators of N, we see easily that

$$\mathfrak{A} = \sum Sc_i \quad \text{and} \quad V = \sum R \cdot (a_i, b_i) + \sum R \cdot (0, \bar{c}_i).$$

The former implies that $\mathfrak{A} = S \cdot e$ for some idempotent $e \in S$, since S is von Neumann regular. Let $f = 1 - e$ and define

$$U = \{(a,\, b\bar{f}) :\ (a, b) \in V\}.$$

This is a left R-submodule of $R \oplus R$, and hence a left ideal of T. Also, since

$$(a,\, b\bar{f}) = (a,\, b) - (0,\, b\bar{e}),$$

we have $U \subseteq V$. Writing $E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in N$, we have

$$T \cdot E = \left\{ \begin{pmatrix} 0 & b\bar{e} \\ 0 & se \end{pmatrix} :\ b \in R,\ s \in S \right\} \subseteq N.$$

From the matrix decomposition

$$\begin{pmatrix} a & b \\ 0 & se \end{pmatrix} = \begin{pmatrix} a & b\bar{f} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b\bar{e} \\ 0 & se \end{pmatrix},$$

we see immediately that $N = U \oplus T \cdot E$. Here, $T \cdot E$ is projective as a left T-module, since E is an idempotent. The last step will be to show that ${}_TU$ is also projective. Now ${}_TU$ is obtained as the pullback of ${}_RU$ along the natural surjection $T \twoheadrightarrow R$. The pullback of ${}_RR$ is $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} = T \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which is T-projective. Therefore, the pullback of any projective left R-module is also T-projective. Now recall that ${}_RV$ is f.g., which implies that ${}_RU \subseteq R \oplus R$ is f.g. Since $R = \bar{S}$ is von Neumann regular, (2.32)(d) and (2.29) imply that ${}_RU$ is R-projective. It follows that ${}_TU$ is T-projective, and therefore so is ${}_TN$. This completes the proof that T is left semihereditary.

§2G. Hereditary Artinian Rings

In the study of the structure of right hereditary rings, the class of such rings that are *right artinian* plays a rather significant role. It turns out that there is a simple way

to recognize whether a right artinian ring R is right hereditary: one only needs to check whether rad R, the Jacobson radical of R, is projective as a right R-module. This result is given in (2.35) below. In this result, we shall use the notion of a *primitive idempotent* (introduced in *FC*–(21.8)) and some basic facts about such idempotents. However, if the reader so prefers, he or she can ignore the primitive idempotents used in the proof and simply work with *all* idempotents. In this case, one would bypass the condition (4) below, and prove the equivalence of the other conditions by using the cycle of implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (5) $\Longrightarrow$ (1).

(2.35) Theorem. *For a right artinian ring R with J = rad R, the following conditions are equivalent:*

(1) *R is right hereditary.*
(2) *J is projective as a right R-module.*
(3) *eJ is projective as a right R-module, for any idempotent $e \in R$.*
(4) *eJ is projective as a right R-module, for any primitive idempotent $e \in R$.*
(5) *Any maximal right ideal $\mathfrak{m}$ of R is projective as a right R-module.*

Proof. (1) $\Longrightarrow$ (2) is a tautology.

(2) $\Longrightarrow$ (3) follows from the right ideal decomposition $J = eJ \oplus (1-e)J$ for any idempotent $e \in R$.

(3) $\Longrightarrow$ (4) is a tautology.

For the two remaining implications, we shall use Schanuel's Lemma, which will be proved a bit later in §5. As a matter of fact, the following arguments turn out to provide a very good illustration of how Schanuel's Lemma is applied in practice. We hope this would, at least in part, justify our use of a result whose proof is not yet given. This point aside, it does seem more natural to cover (2.35) in this section, rather than in §5.

(4) $\Longrightarrow$ (5). Let $\bar{R} = R/J$, and consider any maximal right ideal $\mathfrak{m}$ of R. The simple right R-module $R/\mathfrak{m}$ is isomorphic to $\bar{e}\bar{R}$ for some primitive idempotent $\bar{e}$ of $\bar{R}$. Since J is nil, we can lift $\bar{e}$ to an idempotent $e \in R$ (*FC*–(21.28)). Then, e is a primitive idempotent of R, by *FC*–(21.18). We have the following two short exact sequences:

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow R/\mathfrak{m} \longrightarrow 0,$$
$$0 \longrightarrow eJ \longrightarrow eR \longrightarrow eR/eJ \longrightarrow 0,$$

where $R/\mathfrak{m} \cong \bar{e}\bar{R} \cong eR/eJ$. Since R_R and $(eR)_R$ are both projective, Schanuel's Lemma (5.1) implies that

$$\mathfrak{m} \oplus eR \cong eJ \oplus R, \quad \text{as right } R\text{-modules.}$$

By (4), the RHS is projective, so it follows that $\mathfrak{m}_R$ is projective as well.

(5) $\Longrightarrow$ (1). To show that any right ideal $I \subseteq R$ is projective, we apply an induction on $n = \text{length}(R/I)$. (Recall that any f.g. right module over the right artinian

ring R has finite length.) If $n = 0$, $I = R$ is, of course, projective. If $n > 0$, choose a right ideal $I' \supset I$ such that I'/I is a *simple* right R-module, say, isomorphic to $R/\mathfrak{m}$, where $\mathfrak{m}$ is a suitable maximal right ideal. Then length$(R/I') = n - 1$, so by the inductive hypothesis, *I'_R is projective.* Applying Schanuel's Lemma to the short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & I' & \longrightarrow & I'/I & \longrightarrow & 0, \\ & & & & & & \wr\| & & \\ 0 & \longrightarrow & \mathfrak{m} & \longrightarrow & R & \longrightarrow & R/\mathfrak{m} & \longrightarrow & 0, \end{array}$$

we get an isomorphism $I \oplus R \cong I' \oplus \mathfrak{m}$. Using (5), we see that $I' \oplus \mathfrak{m}$ is projective, so it follows that I_R is projective as well. □

We note in passing that (1) $\Longleftrightarrow$ (2) in the theorem actually holds more generally for any *semiprimary* ring; see Auslander's Theorem in (5.62).

As an application of (2.35), we shall give some classical examples of hereditary rings below.

(2.36) Example. *Let k be any division ring, and R be the ring of $n \times n$ upper triangular matrices over k. Then R is a right (and also left) hereditary ring.*

We have verified this before in *FC*–§25 (p.378), where we used an ad hoc argument to show that any submodule of a f.g. projective right R-module is projective. Here, we shall directly apply our newly proved criterion (2.35). For the (artinian) ring R in question, $J = \text{rad } R$ consists of all matrices in R with a zero diagonal (*FC*–p. 60). *We want to show that J_R is projective.* Consider the surjection $\varphi : R_R \to J_R$ defined by *left* multiplication by the matrix $E_{12}+E_{23}+\cdots+E_{n-1,n}$, where $\{E_{ij}\}$ denote the matrix units. The kernel of φ consists of all matrices in R with all rows zero except perhaps the first. This is just the right ideal eR where e is the idempotent $E_{11} \in R$. From the short exact sequence

(2.37) $$0 \longrightarrow eR \longrightarrow R \xrightarrow{\varphi} J \longrightarrow 0,$$

we see immediately that $J_R \cong R/eR \cong (1 - e)R$, which is a projective right R-module.

There is also another, perhaps more direct, method by which we can prove the projectivity of J_R. Note first that J decomposes into $J_1 \oplus \cdots \oplus J_{n-1}$, where J_i consists of all matrices of J with all rows zero except perhaps the ith. Each J_i is easily seen to be a right ideal of R, so it suffices to show that $(J_i)_R$ is projective. Now, R itself decomposes into $P_1 \oplus \cdots \oplus P_n$, where P_i consists of all matrices of R with all rows zero except perhaps the ith. Again, each P_i is a right ideal of R, so $(P_i)_R$ is projective. Now we finish by noting that $J_i \cong P_{i+1}$ as right R-modules (both being isomorphic to the module of row vectors

(2.38) $$\{(0, \ldots, 0, b_{i+1}, \ldots, b_n)\}$$

on which R acts by right multiplication.) Alternatively, we can get an explicit isomorphism $J_i \to P_{i+1}$ by left multiplication by $E_{i+1,i}$.

A closely related example is the subring $R' \subset R$ consisting of matrices (a_{ij}) in R with $a_{ij} = 0$ for $2 \leq i < j$. The Jacobson radical $J' = \text{rad } R'$ is just $J \cap R'$, and similar arguments as before will show that $(J')_{R'}$ is projective. Therefore, R' is also a right (and left) hereditary ring.

In both of the preceding examples, we could have replaced the division ring k by any semisimple ring. All arguments carry over to this case without any essential change.

Of course, not all right artinian rings are right hereditary. For instance, if $(R, \mathfrak{m})$ is a local algebra of finite dimension $n > 1$ over a field k, and $\mathfrak{m} \neq 0$, then $\mathfrak{m}$ cannot be projective as a right R-module. In fact, if $\mathfrak{m}_R$ is projective, then it must be free by *FC*–(19.29), and hence $\dim_k \mathfrak{m} \neq 0$ is a multiple of n, a contradiction. This shows that R is neither right hereditary nor left hereditary.

§2H. *Trace Ideals*

We shall close §2 with a short introduction to the *trace ideal* of a (right) module. This notion will prove to be useful in the development of the Morita Theory of equivalence of module categories in Chapter 7.

(2.39) Definition. For any right module P_R, we define $\text{tr}(P)$ to be $\sum \text{im}(f)$, where f ranges over $P^* = \text{Hom}_R(P, R)$.

(2.40) Proposition. *For $T = \text{tr}(P)$ defined above, we have:*

(1) *T is an ideal in R (henceforth called the trace ideal of P).*
(2) *If P_R is projective, then $PT = P$, $T^2 = T$, and $\text{ann}(P) = \text{ann}_r(T)$.*

Proof. (1) Clearly, T is a right ideal. For any $f \in P^*$, $a \in P$, and $s \in R$, we have $s(f(a)) = (sf)(a)$, where $sf \in P^*$ (recalling that P^* is a left R-module). This shows that $RT \subseteq T$, so T is an ideal. For (2), fix a pair of dual bases $\{a_i, f_i\}$ $(i \in I)$ on P as in (2.9). For any $a \in P$, $a = \sum a_i f_i(a)$ shows that $PT = P$. For any $f \in P^*$,

$$f(a) = f\left(\sum a_i f_i(a)\right) = \sum f(a_i) f_i(a)$$

shows that $T^2 = T$. Next, for any $s \in \text{ann}(P)$, we have $f(a)s = f(as) = 0$, so $\text{ann}(P) \subseteq \text{ann}_r(T)$. Finally, for any $s \in \text{ann}_r(T)$ and $a \in P$ as above, $as = \left(\sum a_i f_i(a)\right)s = 0$ shows that $\text{ann}_r(T) \subseteq \text{ann}(P)$. □

(2.41) Remark and Example. Assume that P_R is f.g. and projective. Then we could have used the dual bases $\{a_i, f_i\}$ with a *finite* indexing set I. In this case, $T = \text{tr}(P)$ *is generated as an ideal by the finite set* $\{f_j(a_i)\}$. Indeed, for any $f \in P^*$, we have $f = \sum_j s_j f_j$ for suitable $s_i \in R$ (cf. (2.11)), so for any

$a \in P$:

(2.42)
$$f(a) = \sum_j s_j f_j(a) = \sum_{i,j} s_j f_j(a_i) f_i(a),$$

proving our claim. For instance, consider the case $P = eR$, where e is an idempotent. Recalling the choice of the dual bases in (2.12A), we conclude from the preceding that $T = \text{tr}(P) = ReR$. Here the last equation in (2.40)(2) is clear since $\text{ann}(P) = \{s \in R : eRs = 0\}$. Note that in this case the dual of P is just the left ideal Re.

In the commutative case, much more can be said about $\text{tr}(P)$ for f.g. projective modules P, thanks to the following easy (but important) observation on idempotent ideals.

(2.43) Lemma. *Let T be any idempotent ideal in a commutative ring R. If T is f.g., then $T = eR$ for a suitable idempotent $e \in R$.*

Proof. Let $T = \sum_{i=1}^n Rx_i$. Then $T = T \cdot \sum Rx_i = \sum Tx_i$. Writing each x_j as a T-linear combination of $x_1, \ldots, x_n$, the usual determinant argument yields an element $e \in T$ such that $(1-e)T = 0$. In particular, $0 = (1-e)e = e - e^2$, so e is an idempotent. Finally, for any $t \in T$, $(1-e)t = 0$ implies that $t = et \in eR$, so $T = eR$. □

(2.44) Theorem. *Let P_R be a f.g. projective module over a commutative ring R. Let $T = \text{tr}(P)$ and $N = \text{ann}(P)$. Then there exists an idempotent $e \in R$ such that $T = eR$ and $N = (1-e)R$; in particular, $R = T \oplus N$. The projective module P is faithful iff $T = R$, and both conditions hold if R has no nontrivial idempotents and $P \neq 0$.*

Proof. We know from (2.41) that T is f.g. Since $T^2 = T$, it follows from (2.43) that $T = eR$ for some $e = e^2 \in R$. Finally, from (2.40)(2),

$$\text{ann}(P) = \text{ann}_r(T) = \text{ann}_r(eR) = (1-e)R.$$

The rest of the theorem is now clear.[16] □

Keeping the hypotheses and notations in (2.44), let $\{a_i, f_i\}$ $(1 \le i \le n)$ be a pair of dual bases for P. Instead of using the generators $\{f_j(a_i)\}$ for T as in the argument of (2.43), we can also construct an idempotent generator for T explicitly as follows. Let $M \in \mathbb{M}_n(R)$ be defined by $M_{ij} = f_i(a_j)$. Then for any $a \in P$,

(2.45)
$$f_i(a) = f_i\left(\sum_j a_j f_j(a)\right) = \sum_j f_i(a_j) f_j(a)$$

---

[16] All we need for this argument is that T be f.g. Thus, (2.44) also holds if, instead of assuming that P is f.g., we assume that R is a (commutative) *noetherian* ring.

implies that $M \cdot (f_1(a), \dots, f_n(a))^t = (f_1(a), \dots, f_n(a))^t$. In particular, letting a be $a_1, \dots, a_n$, we see that $M^2 = M$. So far, we did not require R to be commutative, but for the rest of the construction we do. Define $e \in R$ by $\det(I - M) = 1 - e$. Since $(I - M)^2 = I - M$, we have $(1 - e)^2 = 1 - e$, and hence $e^2 = e$. *We claim that* $T = \mathrm{tr}(P)$ *is given by* eR. First, $e = 1 - \det(I - M)$ is clearly in T upon expanding $\det(I - M)$, so $eR \subseteq T$. For the reverse inclusion, note that

$$(I - M) \cdot (f_1(a), \dots, f_n(a))^t = 0$$

implies (by Cramer's Rule) that $(1 - e)f_i(a) = 0$ for all i, and all $a \in P$. Thus, $f_i(a) = ef_i(a) \in eR$. Since the $f_i(a)$'s generate T, we have $T \subseteq eR$, as desired.

(2.46) Remark. In the case when P is a f.g. projective *ideal* in the commutative ring R, there is a yet simpler construction for an $e_0 = e_0^2$ such that $T = e_0R$. Keeping the same dual bases $\{a_i, f_i\}$ as above, we simply define $e_0 = \sum f_i(a_i) = \mathrm{trace}(M) \in T$. For any $a \in P$ and $f \in P^*$,

$$\begin{aligned} f(a)e_0 &= \sum f(af_i(a_i)) = \sum f(f_i(aa_i)) \\ &= \sum f(a_i f_i(a)) = f\left(\sum a_i f_i(a)\right) = f(a). \end{aligned}$$

Letting $f = f_i$, $a = a_i$, and summing, we have, in particular, $e_0^2 = e_0$. The above equations also show that $T \subseteq e_0R$, so we have $T = e_0R$. (For more information on the element $\sum f_i(a_i)$, see Exercises 29, 30, and 31.)

(2.47) Remark. If the projective module P_R is *not* f.g., the equation $\mathrm{tr}(P) + \mathrm{ann}(P) = R$ need not hold. For instance, take $R = \mathbb{Z} \times \mathbb{Z} \times \cdots$, and its ideal $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$. Then P_R is projective by (2.5). Using the fact that $P = P^2$, we see easily that $\mathrm{tr}(P) = P$. On the other hand, $\mathrm{ann}(P) = (0)$, so $\mathrm{tr}(P) + \mathrm{ann}(P) = P \neq R$.

(2.48) Remark. If R is not a commutative ring, the equation $\mathrm{tr}(P) + \mathrm{ann}(P) = R$ need not hold for a f.g. projective module P_R. For a simple example, take R to be the ring of upper triangular $n \times n$ matrices over a ring $k \neq 0$, and take $P = eR$ where e is the matrix unit E_{11}. Then $P = \sum_j E_{1j}k$ is an ideal of R, and by (2.41), $\mathrm{tr}(P) = ReR = RP = P$. However, it is easy to see that $\mathrm{ann}(P_R) = 0$, so $\mathrm{tr}(P) + \mathrm{ann}(P_R) = P \neq R$ if $n > 1$.

To give a nontrivial application of trace ideals, we first prove the following lemma due to Müller and Azumaya.

(2.49) Lemma. *Let* $R \subseteq S$ *be rings. Then* R *is a direct summand of* S_R *if and only if* $\mathrm{tr}(S_R) = R$.

Proof. First suppose $S = R \oplus A$, where A is an R-submodule of S_R. Then $\mathrm{tr}(S_R) \supseteq \mathrm{tr}(R_R) = R$, so $\mathrm{tr}(S_R) = R$. Conversely, assume $\mathrm{tr}(S_R) = R$. Then

there exist $f_i \in (S_R)^*$ and $s_i \in S$ $(1 \leq i \leq n)$ such that $\sum f_i(s_i) = 1$. Let $\lambda : S \to R$ be defined by $\lambda(s) = \sum f_i(s_i s)$ for $s \in S$. Then, for $r \in R$:

$$\lambda(sr) = \sum f_i(s_i sr) = \sum f_i(s_i s)r = \lambda(s)r,$$

so $\lambda \in (S_R)^*$. Finally,

$$\lambda(r) = \sum f_i(s_i r) = \sum f_i(s_i)r = r$$

for any $r \in R$, so $\lambda : S \to R$ splits the inclusion map $R \hookrightarrow S$ in $\mathfrak{M}_R$. □

(2.50) Theorem. *Let $R \subseteq S$ be rings such that R is commutative and S is a f.g. projective right R-module. Then R is a direct summand of S_R.*

Proof. Clearly, $\text{ann}(S_R) = 0$, so the last part of (2.44) gives $\text{tr}(S_R) = R$. Now apply (2.49). □

(2.51) Remark. As in (2.44), the assumption in (2.50) that S_R be f.g. can be replaced by R being noetherian. But again, the assumption that R be *commutative* is essential in (2.50). For a counterexample in the general case, consider the R and $P = eR$ constructed in (2.48), taking k there to be a division ring (and $n \geq 2$). Take the extension ring S to be $\mathbb{M}_n(k)$. By breaking up S into its "row spaces", we see that $S_R \cong P \oplus \cdots \oplus P$ (n copies), so S_R is f.g. projective. However, this decomposition also shows that

$$\text{tr}(S_R) = \text{tr}(P) + \cdots + \text{tr}(P) = P \neq R \quad \text{(cf. (2.48))},$$

so R is *not* a direct summand of S_R by (2.49). This can also be seen directly as follows: If $S_R = R \oplus A$ for some R-submodule A of S_R, then the LHS has Krull-Schmidt decomposition $P \oplus \cdots \oplus P$, but the RHS has Krull-Schmidt decomposition $P \oplus P_2 \oplus \cdots \oplus P_n \oplus \cdots$, where $\dim_k P_i = n - i + 1 < n = \dim_k P$ (see (2.36), or *FC*–p. 377), a contradiction.

Later in the text, we shall return to the notion of trace ideals. In §18, we define a module P_R to be a *generator* if $\text{tr}(P) = R$ (several characterizations are given in (18.8)), and we define P_R to be a *progenerator* if P is a f.g. projective generator. The notion of progenerators will play a crucial role later in the development of the Morita theory of equivalence of module categories in §18.

Exercises for §2

1. Let S, R be rings and let ${}_SP_R$ be an (S, R)-bimodule such that P_R is a projective right R-module. Show that, for any projective S-module M_S, the tensor product $M \otimes_S P$ is a projective right R-module. In particular, if there is a given ring homomorphism $S \to R$, whereby we can view R as an (S, R)-bimodule, then for any projective S-module M_S, $M \otimes_S R$ is a projective right R-module. Deduce that, for any ideal $\mathfrak{A} \subseteq S$, if M_S is any projective S-module, then $M/M\mathfrak{A}$ is a projective right $S/\mathfrak{A}$-module.

2. Show that a principal right ideal aR in a ring R is projective as a right R-module iff $\text{ann}_r(a)$ (the right annihilator of a) is of the form eR where e is an idempotent of R.

3. (Ojanguren-Sridharan) Let a, b be two noncommuting elements in a division ring D, and let $R = D[x, y]$. Define a right R-homomorphism $\varphi : R^2 \to R$ by $\varphi(1, 0) = x + a$, $\varphi(0, 1) = y + b$, and let $P = \ker(\varphi)$. Show that:

   (1) P is a f.g. projective R-module with $P \oplus R \cong R^2$, and
   (2) P is isomorphic to a right ideal of R.

   (**Remark**: Ojanguren and Sridharan have also shown that P is nonfree. On the other hand, it is known that $P^n \cong R^n$ for any $n \geq 2$, at least when D has an infinite center.)

4. ([Lam: 76]) Let P be a projective right R-module that has R as a direct summand. If $P \oplus R^m \cong R^n$ where $n > m$, show that P^{m+1} is free.

5. Suppose R has IBN and f.g. projective right R-modules are free. Show that R satisfies the rank condition, and conclude that R is stably finite.

6. Give another proof for (2.8) using a cardinality argument (but again not using the fact that subgroups of free abelian groups are free). (**Hint.** Assume that M embeds in a free abelian group F and use the notations in the proof of (2.8). Then $M/M \cap F_1$ embeds in F_2. Get a contradiction by showing that $M \setminus F_1$ contains an element of the form $(2\varepsilon_1, 4\varepsilon_2, 8\varepsilon_3, \ldots)$, where $\varepsilon_i = \pm 1$.)

7. Let P be a f.g. projective right R-module, with a pair of dual bases

$$\{a_i, f_i : 1 \leq i \leq n\}.$$

   Recall that P^* is a left R-module, and that, for $a \in P$, $\hat{a} \in P^{**}$ is defined by $f\hat{a} = f(a)$, for every $f \in P^*$. Show that

   (1) $\{f_i, \hat{a}_i : 1 \leq i \leq n\}$ is a pair of dual bases for P^*;
   (2) P^* is a f.g. projective left R-module; and
   (3) the natural map $\varepsilon : P \to P^{**}$ defined by $\varepsilon(a) = \hat{a}$ (for every $a \in P$) is an isomorphism of right R-modules.

8. Give examples of (necessarily non-f.g.) projective right R-modules P, P_1 such that

   (1) the first dual P^* of P is *not* a projective left R-module, and
   (2) the natural embedding of P_1 into P_1^{**} is *not* an isomorphism.

   (**Hint.** For (1), take $R = \mathbb{Z}$, $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, and use (2.8). For (2), take $R = \mathbb{Q}$ and $P_1 = \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots$.)

8′. (Extra Credit) Let M be the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z} \times \cdots$ and let $e_1, e_2, \ldots \in M$ be the standard unit vectors.

(1) For any $f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, show that $f(e_i) = 0$ for almost all i.
(2) Using (1) and (2.8)′, show that, for the free $\mathbb{Z}$-module $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, the natural map $\varepsilon : P \to P^{**}$ is an isomorphism.

9. Let R be the ring $\mathbb{Z}[\theta]$ in Example (2.19D).

(1) Show that the ideal $\mathfrak{B} = (3, 1+\theta)$ is invertible, and compute $\mathfrak{B}^{-1}$ explicitly.
(2) Show that $\mathfrak{B}$ and the ideal $\mathfrak{A} = (2, 1+\theta)$ in (2.19D) represent the same element in $\mathrm{Pic}(R)$.
(3) Show that $\mathfrak{A} \oplus \mathfrak{A}$ is free of rank 2, and construct a basis for it explicitly.

10. Show that a Dedekind ring R has trivial Picard group iff R is a PID, iff R is a unique factorization domain.

11. Show that any semilocal Dedekind ring is a PID. (**Hint.** Use (2.22)(D).)

12. Let R be a commutative local ring. Since $\mathrm{Pic}(R) = \{1\}$ by (2.22)(C), we know that every invertible ideal $\mathfrak{A}$ of R is principal. Give a direct proof for this fact without using the notion of projective modules.

13. In the text, we have stated that "Pic" is a covariant functor from the category of commutative rings to the category of abelian groups. Supply the details for a full verification of this fact.

14. Show that the ideal P of the ring R in Example (2.12D) is the union of a strict ascending chain of principal ideals $A_1 \subsetneq A_2 \subsetneq \cdots$ in R. (In particular, R does not satisfy ACC on principal ideals.)

15. (1) Let R be a commutative ring, with $[P][Q] = 1$ in $\mathrm{Pic}(R)$. If P can be generated by two elements, show that $P \oplus Q \cong R^2$.
(2) For the Schanuel modules

$$P_r = (1 + rg,\ g^2), \qquad P_{-r} = (1 - rg,\ g^2) \qquad (r \in R)$$

introduced in (2.15), construct an explicit isomorphism $(\alpha, \beta) : R^2 \to P_r \oplus P_{-r}$.
(**Hint.** (1) Fix P' such that $P \oplus P' \cong R^2$. Take the second exterior power to show that $P' \cong Q$. (2) Define $\alpha : R^2 \to P_r$ by $\alpha(e_1) = 1 + rg$, $\alpha(e_2) = g^2$, and $\beta : R^2 \to P_{-r}$ by

$$\beta(e_1) = -r^4 g^2, \quad \beta(e_2) = (1 + r^2 g^2)(1 - rg).$$

To show that (α, β) is onto, check that $P_{-r} = (g^2,\ (1 + r^2 g^2)(1 - rg))$.)

16. (**Modified Projectivity Test**) Let $\mathfrak{B}$ be a class of objects in $\mathfrak{M}_R$ such that any module in $\mathfrak{M}_R$ can be embedded in some module in $\mathfrak{B}$. Show that, in testing whether a right module P is projective, it is sufficient to check the lifting condition in (2.1) in the case $B \in \mathfrak{B}$. (**Note.** In §3, we'll see that $\mathfrak{B}$ may be taken, for instance, to be the class of all *injective* right R-modules.)

17. Let P be a projective right module over a von Neumann regular ring R. Show that any f.g. submodule of P is a direct summand of P (and hence also a projective module).

18. Show that any f.g. projective right R-module P can be represented as $e(R^n)$, where $e : R^n \to R^n$ is left multiplication by some idempotent matrix $(a_{ij}) \in \mathbb{M}_n(R)$. With respect to this representation, show that $\text{tr}(P) = \sum Ra_{ij}R$, and deduce that $\mathbb{M}_n(\text{tr}(P)) = \mathbb{M}_n(R)\, e\, \mathbb{M}_n(R)$. (**Hint.** Note that $rE_{ij}eE_{k\ell}r' = ra_{jk}r'E_{i\ell}$.)

19. If, for any n, any idempotent in $\mathbb{M}_n(R)$ is conjugate to some diag$(1, \ldots, 1, 0, \ldots, 0)$, show that any f.g. projective right R-module is free. Show that the converse is also true if R has IBN.

20. For right modules A, B over a ring R, define

$$\sigma = \sigma_{A,B} : \; B \otimes_R A^* \longrightarrow \text{Hom}_R(A, B)$$

by $\sigma(b \otimes f)(a) = bf(a)$, where $b \in B$, $a \in A$, and $f \in A^* = \text{Hom}_R(A, R)$. (Recall that $A^* \in {}_R\mathfrak{M}$.) Show that, for any given $A \in \mathfrak{M}_R$, the following are equivalent:

(1) A is a f.g. projective module;
(2) $\sigma_{A,B} : B \otimes_R A^* \to \text{Hom}_R(A, B)$ is an isomorphism for all $B \in \mathfrak{M}_R$;
(3) $\sigma_{B,A} : A \otimes_R B^* \to \text{Hom}_R(B, A)$ is an isomorphism for all $B \in \mathfrak{M}_R$;
(4) $\sigma_{A,A} : A \otimes A^* \to \text{End}_R(A)$ is an epimorphism (resp. isomorphism).

For more information on $\sigma_{A,B}$, see Exercise (4.11).

In Exercises 21–31 *below, R denotes a commutative ring.*

21. (Bourbaki) Let P_R be a f.g. R-module. We say that P is *locally free* if the localization $P_\mathfrak{p}$ of P at any maximal (or prime) ideal $\mathfrak{p}$ is free over the local ring $R_\mathfrak{p}$. (It turns out that these P's are exactly the f.g. flat modules; see Exercise (4.15).) For such a locally free (f.g.) module P, define rk P : Spec $R \to \mathbb{Z}$ by

(rk P)($\mathfrak{p}$) = the (uniquely defined) rank of the free module $P_\mathfrak{p}$ over $R_\mathfrak{p}$.

Here, $\mathbb{Z}$ is given the discrete topology, and the prime spectrum Spec R is given the Zariski topology. (The Zariski closed sets are of the form $V(\mathfrak{A}) = \{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{A}\}$, where $\mathfrak{A}$ ranges over the ideals of R.) Show that the following are equivalent:

(1) P is a projective R-module.
(2) P is *finitely presented*; that is, there exists an exact sequence

$$R^m \to R^n \to P \to 0$$

for some integers m, n.
(3) rk P is a continuous function from Spec R to $\mathbb{Z}$.
(4) rk P is a "locally constant" function; i.e., for any $\mathfrak{p} \in$ Spec R, rk P is constant on a suitable neighborhood of $\mathfrak{p}$.

22. Keeping the notations in Exercise 21, show that a subset $S \subseteq \operatorname{Spec} R$ is clopen (closed and open) iff $S = V(eR)$ for some idempotent $e \in R$. Using this, show that the following statements are equivalent for any nonzero ring R:

    (1) R has no idempotents other than 0, 1.
    (2) Spec R is connected.
    (3) Every f.g. projective R-module has constant rank.

23. The *support* of an R-module P is defined to be

$$\operatorname{supp} P = \{\mathfrak{p} \in \operatorname{Spec} R : \; P_{\mathfrak{p}} \neq 0\}.$$

    For any f.g. $P \in \mathfrak{M}_R$, show that $\operatorname{supp} P = V(\operatorname{ann} P)$. Deduce that $P_{\mathfrak{p}} \neq 0$ for all primes $\mathfrak{p}$ iff $\operatorname{ann} P \subseteq \operatorname{Nil}(R)$.

24. For any f.g. projective R-module P, show that P is faithful iff the function $\operatorname{rk} P : \operatorname{Spec} R \to \mathbb{Z}$ is everywhere positive. (In particular, a f.g. projective module P_R of rank $n > 0$ is always faithful.)

25. Suppose $P,\ Q \in \mathfrak{M}_R$ are such that $P \otimes_R Q \cong R^n$ where $n > 0$. Show that P and Q must be faithful f.g. projective R-modules.

26. Deduce from Exercise 25 that $P \in \mathfrak{M}_R$ is f.g. projective of rank 1 iff there exists $Q \in \mathfrak{M}_R$ such that $P \otimes_R Q \cong R$. In this case, show that necessarily $Q \cong P^*$.

27. Show that a f.g. projective module P_R has rank 1 iff the natural map $\lambda : R \to \operatorname{End}_R(P)$ (defined by $\lambda(r)(p) = pr$) is an isomorphism of rings.

28. Let P be a f.g. projective R-module. Show that there is a natural way to define the trace of an R-endomorphism of P so that we get an R-module homomorphism tr: $\operatorname{End}_R(P) \to R$. The definition should be such that, in case $P = R^n$, we get back the usual trace map on $n \times n$ matrices, upon identifying $\operatorname{End}_R(P)$ with $\mathbb{M}_n(R)$. (**Hint.** Identify $\operatorname{End}_R(P)$ with $P \otimes_R P^*$ and show that $a \otimes f \mapsto f(a)$ gives a well-defined R-homomorphism $\alpha : P \otimes_R P^* \to R$.)

29. Let P be a f.g. projective R-module, and let $\{a_i, f_i\}$ $(1 \leq i \leq n)$ be a pair of dual bases as in (2.9). Show that $\tau(P) := \sum_i f_i(a_i) \in R$ is an invariant of P (not depending on the choice of $\{a_i, f_i\}$). (**Hint.** Show that $\tau(P) = \operatorname{tr}(\operatorname{Id}_P)$, where "tr" is the trace map in Exercise 28.) Is the same conclusion true if R is not commutative?

30. Recall that every idempotent (square) matrix defines a f.g. projective module (as in Exercise 18). Show that, if $e,\ e' \in \mathbb{M}_m(R)$ are idempotent matrices that define isomorphic projective modules, then $\operatorname{trace}(e) = \operatorname{trace}(e')$. (**Hint.** If e defines P, show that $\operatorname{trace}(e) = \tau(P)$ in the notation of Exercise 29.) Is the same conclusion true if R is not commutative?

31. Let P_R be a f.g. projective R-module of rank 1.

(a) For $a, b \in P$ and $f \in P^*$, show that $af(b) = bf(a) \in P$.
(b) Show that the following diagram is commutative:

$$\begin{array}{ccc} P \otimes P^* & \xrightarrow{\ \sigma\ } & \mathrm{End}_R(P) \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\lambda} & \\ & R & \end{array}$$

Here, $\sigma = \sigma_{P,P}$, λ, and α are defined, respectively, in Exercises 20, 27, and 28.
(c) Show that the trace map $\mathrm{tr}\colon \mathrm{End}_R(P) \to R$ defined in Exercise 28 is the same as λ^{-1}.
(d) Show that $\tau(P) = 1$ (in the notation of Exercise 29).

32. Let P be a projective module that is not f.g. Show that there is a split monomorphism $f : P \to \bigoplus_{i\in I} P$ for a suitable infinite indexing set I such that $f(P)$ is not contained in $\bigoplus_{i\in J} P$ for any finite subset $J \subseteq I$.

33. In a ring theory monograph, the following statement appeared: "If G is a finite group, every projective module over (the integral group ring) $\mathbb{Z}G$ is free." Give a counterexample!

The next two problems are suggested by R. Swan. Recall that, for a commutative ring R, $Q(R)$ denotes the total ring of quotients of R; that is, the localization of R at the multiplicative set of its non 0-divisors.

34. For any commutative ring S, show that there exists another commutative ring $R \supseteq S$ with the following properties: (1) $Q(R) = R$; (2) S is a "retract" of R (i.e., the inclusion map $S \to R$ is split by a ring homomorphism $R \to S$); (3) $\mathrm{Nil}(S) = \mathrm{Nil}(R)$; and (4) $\mathrm{U}(S) = \mathrm{U}(R)$.

35. In (2.22)(A), we have constructed a commutative ring R with $Q(R) = R$ and $\mathrm{Pic}(R) \neq \{1\}$. However, this ring R has nonzero nilpotent elements. Now use Exercise 34 to construct a *reduced* ring R with the same properties.

The last two problems below are from a paper of H. Bass.

36. For any right R-module P and $x \in P$, let $o_P(x) = \{f(x) : f \in P^*\}$, where, as usual, P^* denotes the left R-module $\mathrm{Hom}_R(P, R)$.

(1) If P' is any right R-module and $F = P \oplus P'$, show that $o_P(x) = o_F(x)$ for any $x \in P \subseteq F$.
(2) If P is a projective right R-module and $x \in P$, show that $o_P(x)$ is a f.g. left ideal of R, with $o_P(x) \neq 0$ if $x \neq 0$. Deduce that the natural map $P \to P^{**}$ is a monomorphism.

37. For any right R-module P and $x \in P$, let

$$o'_P(x) = \{y \in P : \forall\, f \in P^*,\ f(x) = 0 \Longrightarrow f(y) = 0\}.$$

(1) If P' is any projective R-module and $F = P \oplus P'$, show that $o'_P(x) = o'_F(x)$ for any $x \in P \subseteq F$.
(2) If P is a projective right R-module and $x \in P$, show that $o'_P(x) \cong o_P(x)^*$, where $o_P(x)$ is the left ideal associated with $x \in P$ in Exercise 36.
(3) Under the same hypothesis as in (2), show that $o'_P(x)$ is a direct summand of P iff $o_P(x)$ is a projective left R-module.

(**Comment.** Bass used (2) and (3) above to show that, over any left semihereditary ring R, any projective right R-module is a direct sum of f.g. (projective) R-modules, each isomorphic to the dual of a f.g. left ideal in R. In particular, if each f.g. left ideal in R is free, then any projective right R-module is free.)

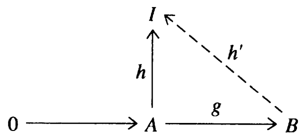
§3. Injective Modules

§3A. *Baer's Test for Injectivity*

In this subsection, we present the definition and the basic properties of injective modules. The definition of an injective module is formally "dual" to that of a projective module, in the sense that we just take the old definition and "turn all the arrows around." Indeed, some of the key properties of injective modules are dual versions of corresponding properties of projective modules. However, the process of dualization works only up to a point. In general, we are in no position to assume that properties of projectives always have analogues for injectives, or vice versa.

By definition, a right R-module I is said to be *injective* if, for any monomorphism $g : A \to B$ of right R-modules and any R-homomorphism $h : A \to I$, there exists an R-homomorphism $h' : B \to I$ such that $h = h' \circ g$:

(3.1)



We refer to this property informally by saying that any $h : A \to I$ can be "extended" to B, or to a homomorphism $h' : B \to I$.

The injectivity of I may also be expressed as a property of the contravariant functor $\mathrm{Hom}_R(-, I)$ from $\mathfrak{M}_R$ to abelian groups. In general, if

$$(3.2) \qquad 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

is a short exact sequence in $\mathfrak{M}_R$, then, for any $I \in \mathfrak{M}_R$, the induced sequence of abelian groups

$$(3.3) \qquad 0 \longrightarrow \mathrm{Hom}_R(C, I) \xrightarrow{f^*} \mathrm{Hom}_R(B, I) \xrightarrow{g^*} \mathrm{Hom}_R(A, I)$$

is also exact. In other words, $\mathrm{Hom}_R(-, I)$ is a left exact (contravariant) functor. For I_R to be injective, we require precisely that g^* be surjective; i.e., that (3.3) be

a short exact sequence when we add a zero term at the right end. We see, therefore, that I_R *is injective iff* $\operatorname{Hom}_R(-, I)$ *is an exact functor.*

(3.4) Proposition. (1) *A direct product* $I = \prod_\alpha I_\alpha$ *of right R-modules is injective iff each* I_α *is.* (2) *A right R-module I is injective iff any monomorphism* $I_R \to M_R$ *splits in* $\mathfrak{M}_R$.

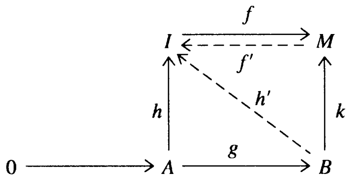
Proof. (1) follows from the natural equivalence of functors

$$\operatorname{Hom}_R(-, \prod_\alpha I_\alpha) \cong \prod_\alpha \operatorname{Hom}_R(-, I_\alpha).$$

For (2), the "only if" part follows by extending the identity map $I \to I$ to a map $M \to I$. For the "if" part, suppose we are given the maps h and g in (3.1). We form the "pushout":

$$M := \frac{I \oplus B}{\{(h(a), -g(a)) : a \in A\}},$$

and let $f : I \to M$, $k : B \to M$ be the obvious maps. Then we have a commutative diagram:



The map f is clearly injective. For, if $i \in \ker(f)$, then $(i, 0) = (h(a), -g(a))$ for some $a \in A$. The injectivity of g implies that $a = 0$, and so $i = h(a) = 0$. By assumption, there exists a splitting $f' : M \to I$ for the monomorphism f. Taking $h' = f'k : B \to I$, we have $h'g = f'kg = f'fh = h$, as desired. □

Remark. (1) above implies that a finite direct sum of injectives is injective. In general, however, an arbitrary direct sum of injectives need not be injective. See (3.46) below.

The next result, called the "Injective Producing Lemma", enables us to use known injective modules over one ring to produce injective modules over another. It will be one of our main tools for constructing examples of injective modules. We proceed, in general, as follows. Let S, R be rings, and let P be a *fixed* (R, S)-bimodule that is flat as a left R-module (cf. *FC*–(24.20), or (4.0) below). For any $M \in \mathfrak{M}_S$, let us write

$$\tilde{M} = \operatorname{Hom}_S(P_S, M_S).$$

Using the left R-action on P, we can make $\tilde{M}$ into a *right* R-module by:

$$(fr)(p) = f(rp), \quad \text{where } f \in \tilde{M},\ r \in R, \text{ and } p \in P.$$

(3.5) Injective Producing Lemma. *If M is an injective right S-module, then $\tilde{M}$ above is an injective right R-module.*

Proof. Our job is to prove the exactness of the functor $\text{Hom}_R(-, \tilde{M})$. For any $A \in \mathfrak{M}_R$, we have the following standard isomorphism:

$$\text{Hom}_R(A, \tilde{M}) = \text{Hom}_R(A, \text{Hom}_S(P, M)) \cong \text{Hom}_S(A \otimes_R P, M).$$

The fact that ${}_RP$ is flat means that the functor $- \otimes_R P$ is exact on $\mathfrak{M}_R$. The fact that M_S is injective means that $\text{Hom}_S(-, M)$ is exact on $\mathfrak{M}_S$. Combining these, we see from the isomorphism above that the functor $\text{Hom}_R(-, \tilde{M})$ is exact on $\mathfrak{M}_R$. Therefore, $\tilde{M}$ is an injective right R-module. □

Let us indicate some of the principal ways in which the above lemma will be used. For the first one, we take a ring homomorphism $\eta : R \to S$. We can view S as a left R-module via η, and choose P to be ${}_RS_S$. For this choice of P, we can identify the right R-module $\text{Hom}_S({}_RS_S, M_S)$ with M_R, where the right R-module structure on M is obtained by "pullback" along η. In this case, (3.5) gives the following.

(3.6A) Corollary. *Let $\eta : R \to S$ be a ring homomorphism such that S becomes a flat left R-module under η. Then, for any injective module M_S, the right R-module M_R (obtained by pullback along η) is also injective.*

For the second way of using (3.5), we take now a ring homomorphism $\varepsilon : S \to R$, and choose $P = {}_RR_S$, where R_S is obtained via ε. Here ${}_RR$ is projective and hence flat (*FC*–p. 365), so we have the following special case of (3.5).

(3.6B) Corollary. *Let $\varepsilon : S \to R$ be a ring homomorphism. Then for any injective module M_S, the right R-module* $\text{Hom}_S({}_RR_S, M_S)$ *is also injective.*

For the purposes of studying modules over a k-algebra R where k is a field, we note that any left R-module P may be viewed as an (R, k)-bimodule, and any k-vector space is certainly k-injective. Thus, we have the following special case of (3.5):

(3.6C) Corollary. *Let R be a k-algebra where k is a field. Let ${}_RP$ be a fixed projective left R-module, viewed as an (R, k)-bimodule in the natural way (e.g., $P = R$). Then, for any k-vector space M,* $\text{Hom}_k(P, M)$ *is an injective right R-module.*

If we recall what happens with projective modules, (3.5) is not surprising at all. In fact, if S, R are rings and ${}_SP_R$ is such that P_R is projective, then, for any projective S-module M_S, $M \otimes_S P$ is a projective right R-module according to Exercise (2.1). This may be called the *Projective Producing Lemma*, and (3.5) is just its injective analogue.

While we are on this theme, let us recall another piece of information on projective modules. In Exercise 2.16, we have noted that to check that a module P_R is projective, it is sufficient to check the lifting condition in (2.1) for B belonging to a family $\mathfrak{B}$ with the property that any right R-module embeds into a module in $\mathfrak{B}$. We called this the *Modified Projectivity Test.* The proof is a completely routine diagram argument. The same kind of argument also yields a *Modified Injectivity Test*, as follows. Let $\mathfrak{B}$ be a family of right R-modules such that any right R-module is an epimorphic image of a module in $\mathfrak{B}$. Then, to test that an $I \in \mathfrak{M}_R$ is injective, it is sufficient to check the extendibility condition in (3.1) for B belonging to $\mathfrak{B}$. For instance, we can take $\mathfrak{B}$ to be the family of all free right R-modules.

As it turns out, we can even do better than that. The following remarkable criterion for injectivity, due to R. Baer, says that it will be sufficient to test the extendibility condition in (3.1) with B chosen to be the right regular module, R_R. This result does not seem to have an analogue for the case of projective modules.

(3.7) Baer's Criterion (or Baer's Test). *A right R-module I is injective iff, for any right ideal $\mathfrak{A}$ of R, any R-homomorphism $f : \mathfrak{A} \to I$ can be extended to $f' : R \to I$.*

(**Note**. A homomorphism $f' : R \to I$ is uniquely determined by specifying the image $f'(1) \in I$. Therefore, to extend f to *some* f' means exactly to find an element $i \in I$ such that $f(r) = ir$ for every $r \in \mathfrak{A}$.)

Proof. ("If" part) We test the injectivity of I by considering the diagram (3.1), where h and g are given. To simplify the notation, we think of A as a submodule of B. By a simple application of Zorn's Lemma, we can find some $h_0 : A_0 \to I$ where $A \subseteq A_0 \subseteq B$, $h_0|A = h$, such that h_0 *cannot* be extended to any submodule of B properly containing A_0. We finish by showing that $A_0 = B$. Indeed, suppose there exists an element $b \in B \backslash A_0$. Then

$$\mathfrak{A} := \{r \in R : \ br \in A_0\}$$

is a right ideal of R. We define $f : \mathfrak{A} \to I$ by

$$f(r) = h_0(br) \quad (\forall r \in \mathfrak{A}), \tag{3.8}$$

and check easily that $f \in \operatorname{Hom}_R(\mathfrak{A}, I)$. By assumption (see **Note** after (3.7)), there exists an element $i \in I$ such that $f(r) = ir$ for all $r \in \mathfrak{A}$. Now let $A_1 = A_0 + bR$, and define $h_1 : A_1 \to I$ by

$$h_1(a_0 + br) = h_0(a_0) + ir \quad (\forall a_0 \in A_0, \ r \in R).$$

To check that h_1 is well-defined, suppose $a_0 + br = a_0' + br'$. Then $b(r' - r) = a_0 - a_0' \in A_0$, so $r - r' \in \mathfrak{A}$, and hence $f(r' - r) = i(r' - r)$. On the other hand, by (3.8):

$$f(r' - r) = h_0(b(r' - r)) = h_0(a_0 - a_0') = h_0(a_0) - h_0(a_0') .$$

Therefore, we have $i(r'-r) = h_0(a_0) - h_0(a_0')$, and hence $h_0(a_0) + ir = h_0(a_0') + ir'$. Now that we know h_1 is well-defined, we check easily that $h_1 \in \mathrm{Hom}_R(A_1, I)$. Since h_1 clearly extends h_0, we get the desired contradiction. □

As a first application of Baer's Criterion, we offer the following result in the commutative setting.

(3.9) Proposition. *Let R be a commutative domain with quotient field K, and let I be a (right) K-vector space. Then I_R is an injective R-module.*

Proof. Let $f \in \mathrm{Hom}_R(\mathfrak{A}, I_R)$, where $\mathfrak{A}$ is an ideal of R. By (3.7), it suffices to check that f can be extended to R. For this purpose, we may, of course, assume that $\mathfrak{A} \neq 0$. Identify I with $\bigoplus_{j \in J} I_j$, where $I_j = K_K$, and let

$$f_j = \pi_j \circ f : \mathfrak{A} \longrightarrow I_j = K \quad (j \in J),$$

where π_j is the jth projection map. By (2.16), we know that f_j is multiplication by some element $b_j \in K$. Fix a nonzero element $a \in \mathfrak{A}$. Since $f(a) = (b_j a)_{j \in J}$ has only *finitely many* nonzero coordinates, almost all b_j 's are zero. Therefore, $b := (b_j)_{j \in J} \in I$, and we have $f(a) = ba$ for every $a \in \mathfrak{A}$, as desired. □

It is of interest to point out that this Proposition can equally well be proved by using (3.6A). Just note that, as a vector space over K, I_K is an injective module. With this observation, (3.6A) applies to give the injectivity of I_R since the quotient field K is flat as an R-module. (The functor $- \otimes_R K$ is naturally equivalent to localization at the multiplicative set $R \setminus \{0\}$, and the localization functor is always exact.) This argument applies to noncommutative localization as well; for more information on this, see Exercise (10.29).

§3B. *Self-Injective Rings*

In the study of projective modules, the right regular module R_R played a special role, since it is a free, and hence projective, module. In the context of injectivity, however, the situation is quite different. For most rings R, R_R is simply not injective. But there do exist rings R for which R_R is injective; we say that such rings are *right self-injective*. Some positive and some negative examples are given below.

(3.10A) Example. *The ring $\mathbb{Z}$ is clearly not self-injective.* In fact, $f : 2\mathbb{Z} \to \mathbb{Z}$ defined by $f(2n) = n$ (for every $n \in \mathbb{Z}$) clearly cannot be extended to a homomorphism $f' : \mathbb{Z} \to \mathbb{Z}$.

(3.10B) Example. Let R be the ring of $n \times n$ upper triangular matrices over a ring $k \neq 0$, where $n \geq 2$. *Then R is not right self-injective.* To simplify the notations, we work in the case $n = 2$. Consider the ideal $\mathfrak{A} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ and

define $f : \mathfrak{A} \to R$ by $f\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$. This is easily checked to be a right R-homomorphism. If f can be extended to R, there would exist a matrix $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$ such that

$$f\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xa \\ 0 & 0 \end{pmatrix} \qquad (\forall a \in k),$$

which is clearly impossible. This shows that R_R is not injective.

(3.11A) Example. Let $R = A \times B$, with $e = (1, 0) \in A$, so $A = eR = Re$. Let M_A be an A-module, viewed as an R-module via the projection $R \to A$. *Then M_A is injective iff M_R is injective.* The "if" part is a special case of Exercise 29, so we shall only prove the "only if" part here. Assume M_A is injective. To apply Baer's Test to M_R, consider $f \in \mathrm{Hom}_R(I, M)$, where $I \subseteq R$ is any right ideal. Let $\mathfrak{A} = I \cap A$ and $\mathfrak{B} = I \cap B$, so $I = \mathfrak{A} \oplus \mathfrak{B}$. Note that $f(\mathfrak{B}) = f(\mathfrak{B})e = f(\mathfrak{B}e) = f(0) = 0$. We can therefore extend f to $f' : R \to M$ by first extending f from $\mathfrak{A}$ to A (using the injectivity of M_A) and taking $f'|B = 0$. This checks that M_R is injective. In particular, the ideal A_R is injective iff A is a right self-injective ring.

(3.11B) Corollary. *Let $R = \prod_{j \in J} A_j$, where the A_j's are rings. Then R is right self-injective iff each A_j is. (Thus, for instance, any direct product of division rings is right and left self-injective.)*

Proof. By (3.4)(1), R_R is injective iff each $(A_j)_R$ is injective, iff each $(A_j)_{A_j}$ is injective. ☐

(3.11C) Remark. In the notation of (3.11B), assume each $A_j \neq 0$ is right self-injective, and let $\mathfrak{A} = \bigoplus_j A_j \subseteq R$. *If J is infinite, $\mathfrak{A}_R$ is not injective* (even though each $(A_j)_R$ is)! Indeed, if $\mathfrak{A}_R$ is injective, we would have $R = \mathfrak{A} \oplus \mathfrak{B}$ for a suitable right ideal $\mathfrak{B} \neq 0$. But for any $b = (b_j) \in \mathfrak{B}$:

$$b \cdot (0, \ldots, 1, 0, \ldots) = (0, \ldots, b_j, 0, \ldots) \in \mathfrak{B} \cap \mathfrak{A} = 0,$$

so all $b_j = 0$, a contradiction.

(3.11D) Corollary. *Let J be any set and let R be the Boolean ring of all subsets of J. (Sum is given by symmetric difference and product is given by intersection.) Then R is self-injective.*

Proof. Identify R with the direct product $\prod_{j \in J} A_j$ where each $A_j = \mathbb{F}_2$ (the field of two elements), and apply (3.11B). ☐

(3.12) Example. Let S be a PRID (principal right ideal domain), and $b \neq 0$ be an element of S such that $bS = Sb$. *Then the quotient ring $R := \bar{S} = S/bS$*

is always right self-injective. To check this, consider any right ideal $\mathfrak{A} = aS/bS$ of R. To apply Baer's Test, we must try to extend any given R-homomorphism $h : \mathfrak{A} \to R$ to R. Write $h(\bar{a}) = \bar{s}$, where $s \in S$. ("Bar" shall always mean taking image in $\bar{S}$.) Write $b = ac$, $c \in S\backslash\{0\}$. We have

$$0 = h(0) = h(\bar{b}) = h(\bar{a}\,\bar{c}) = \bar{s}\,\bar{c} = \overline{sc},$$

so, using $bS = Sb$, we can write $sc = tb = tac$ for some $t \in S$. Canceling c gives $s = ta$, so $h(\bar{a}) = \bar{s} = \bar{t}\,\bar{a}$. Now extend h to R by sending $\bar{1}$ to $\bar{t}$.

(3.13) Corollary. (1) $\mathbb{Z}/n\mathbb{Z}$ $(n > 0)$ *is a self-injective ring.* (2) *For any field k and any nonzero polynomial $f(t)$ in $k[t]$, $k[t]/(f(t))$ is a self-injective ring.*

The statements (1) and (2) in this Corollary are very useful conclusions. For instance, using (1), one can give a pretty quick proof of Prüfer's Theorem for abelian groups, which says that any abelian group killed by a fixed integer $n > 0$ is a direct sum of cyclic groups (each necessarily killed by the same n). The second conclusion (2) has, also, some nice applications to linear algebra. In fact, if T is a linear operator on a finite-dimensional vector space V over a field k, then the subalgebra $k[T] \subseteq \text{End}_k V$ generated by T is isomorphic to $k[t]/(f(t))$ where f is the minimal polynomial of T. The fact that $k[T]$ is self-injective can be used to give surprising alternative proofs to various facts about linear operators, for instance, the Jordan Canonical Form Theorem. These applications of (3.13) are given in more detail in the exercises of this section.

We shall now come to another class of examples of self-injective rings. *For the balance of this subsection, let R be a finite-dimensional algebra over a field k.* First we remark that the dual k-space[17] $\hat{R} = \text{Hom}_k(R, k)$ has the structure of an (R, R)-bimodule. The right and left R-actions on $\hat{R}$ are defined, respectively, by

$$(\varphi \cdot x)(r) = \varphi(xr), \quad \text{and} \quad (y \cdot \varphi)(r) = \varphi(ry),$$

where $\varphi \in \hat{R}$, and $x, y, r \in R$ (cf. the formation of $\tilde{M}$ in (3.5)), from which the bimodule law $(y\varphi)x = y(\varphi x)$ easily follows. It is of interest to compare, say, the right R-module $\hat{R}$ with the right regular module R_R. By definition, R is said to be a *Frobenius algebra* (over k) if we have an isomorphism $\hat{R} \cong R$ in $\mathfrak{M}_R$. Now recall from (3.6C) that $\hat{R}$ is always an *injective* right R-module. Therefore, we have:

(3.14) Proposition. *Any Frobenius algebra R over a field k is right self-injective.*

Now whether this can be used to generate good examples of right self-injective rings depends on what kinds of Frobenius algebras we can come up with. Before we give any examples, let us first provide some useful characterizations of Frobenius algebras. (The convention $\dim_k R < \infty$ remains in force.)

[17]For an R-module M, we write $\hat{M}$ for the k-dual of M, and reserve the notation M^* for the R-dual $\text{Hom}_R(M, R)$ of M.

(3.15) Theorem. *For any k-algebra R, the following are equivalent:*

(1) *R is a Frobenius algebra.*
(2) *There exists a nonsingular bilinear pairing $B : R \times R \to k$ with the following "associative" property: $B(xy, z) = B(x, yz)$ for all $x, y, z \in R$.*
(3) *There exists a hyperplane[18] H in the k-space R which contains no nonzero right ideal.*

Proof. (1)$\Longrightarrow$(2). Fix an isomorphism $f : R \to \hat{R}$ in $\mathfrak{M}_R$, so $f(xy) = f(x)y$. Define $B : R \times R \to k$ by $B(x, y) = f(x)(y) \in k$. Since f is an isomorphism, B is nonsingular. Also

$$B(xy, z) = f(xy)(z) = (f(x)y)(z) = f(x)(yz) = B(x, yz).$$

(2)$\Longrightarrow$(3). Let H be the hyperplane $\{z \in R : B(1, z) = 0\}$. Suppose H contains a right ideal $\mathfrak{A}$. Then, for any $x \in \mathfrak{A}$, $0 = B(1, xR) = B(x, R)$, so $x = 0$, and hence $\mathfrak{A} = 0$. (A similar proof also shows that H contains no nonzero left ideal.)

(3)$\Longrightarrow$(1). Fix a linear functional $\lambda : R \to k$ with $\ker \lambda = H$, and define $f : R \to \hat{R}$ by $f(x)(z) = \lambda(xz)$. If $f(x) = 0$, then $xR \subseteq \ker \lambda = H$ implies that $x = 0$. Therefore, f is one-one, and hence onto. Finally, for all $x, y, z \in R$:

$$(f(x)y)(z) = f(x)(yz) = \lambda(xyz) = f(xy)(z).$$

Therefore, $f(xy) = f(x)y$, so f is an isomorphism in $\mathfrak{M}_R$. □

Remark. We have defined the notion of a Frobenius algebra by working in $\mathfrak{M}_R$. The characterization (2) above (or the observation made in parentheses in the proof of (2)$\Longrightarrow$(3)) shows, however, that "Frobenius algebra" is a left-right symmetric concept. (Alternatively, given the pairing B in (2), one checks directly that $g : R \longrightarrow \hat{R}$ given by $(yg)(x) = B(x, y)$ is an isomorphism in ${}_R\mathfrak{M}$.) In particular, we see that a *Frobenius algebra must also be a left self-injective ring.*

(3.15)′ Corollary. *Suppose k is an infinite field. If R has only finitely many minimal right ideals, then R is a Frobenius algebra.*

Proof. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be *all* minimal right ideals, and fix a nonzero $a_i \in \mathfrak{A}_i$ for each i. Since k is infinite, an easy exercise in linear algebra shows the existence of a hyperplane $H \subset R$ avoiding $a_1, \ldots, a_n$. If $\mathfrak{A} \neq 0$ is any right ideal, then $\mathfrak{A} \supseteq \mathfrak{A}_i$ for some i. Then $a_i \in \mathfrak{A}$ so $\mathfrak{A} \not\subseteq H$. Now apply the theorem. □

(3.15A) Example. *Suppose k is infinite and the k-algebra R is a proper quotient of a Dedekind k-domain A. Then R is a Frobenius algebra.*[19] (Say, $R = A/I$;

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[18]By a *hyperplane*, we mean here a hyperplane containing the origin, that is, a linear subspace of codimension 1.

[19]This conclusion is in fact true without any assumption on k; see Exercise 15.

$I \neq 0$. Then there are only finitely many ideals of A containing I.) For instance, we can take $R = k[t]/(f(t))$ where $f \neq 0$.

(3.15B) Example. Let R be the commutative k-algebra $k[x, y]$ with relations $x^2 = y^2 = 0$, where k is *any* field. *Then R is a (local) Frobenius algebra.* To see this, let H be the hyperplane $k + kx + ky$. Suppose $H \supseteq \alpha R$, where $\alpha = a + bx + cy \in R$. Then H contains $\alpha xy = axy$, so $a = 0$. Similarly, H contains $\alpha x = cxy$ and $\alpha y = bxy$, so $b = c = 0$ and $\alpha = 0$. This argument can be generalized easily to the algebra $R = k[x_1, \ldots, x_r]$ with the relations $x_1^{n_1} = \cdots = x_r^{n_r} = 0$ (where all $n_i > 0$). Here, we take H to be $\ker(\lambda)$ where $\lambda : R \to k$ is the k-linear map sending $\bar{f} \in R$ to the coefficient of $x_1^{n_1 - 1} \cdots x_r^{n_r - 1}$ in f. In algebraic geometry, this algebra R arises naturally in the study of isolated singularities. For the polynomial

$$G(X_1, \ldots, X_r) = X_1^{n_1 + 1} + \cdots + X_r^{n_r + 1} \in \mathbb{C}[X_1, \ldots, X_r],$$

which defines a variety with an isolated singularity at the origin, the algebra R above arises as

$$\mathbb{C}[X_1, \ldots, X_r]/\left(\tfrac{\partial G}{\partial X_1}, \ \ldots, \ \tfrac{\partial G}{\partial X_r}\right).$$

This example becomes considerably more general if we take G to be any weighted homogeneous polynomial.

(3.15B′) Example. In contrast to (3.15B), we mention a couple of (commutative) non-Frobenius algebras. For instance, $R = k[x, y]/(x, y)^{n+1}$ (for $n \geq 1$) is not Frobenius. One easy way to check this is to apply Exercise 14 in this section: for our algebra R, the images of x^n and y^n both generate 1-dimensional (hence minimal) ideals. Another (non-Frobenius) example is $S = k[x, y]/(x^2, \ xy^{n+1}, \ y^{n+2})$. For this algebra, the images of xy^n and y^{n+1} both generate 1-dimensional ideals. (For more information about these two algebras, see (3.69) and (3.70), respectively.)

(3.15C) Example. Let $K \supseteq k$ be a field extension of degree $n < \infty$. Viewing K in the natural way as a (K, K)-bimodule, we can form the "trivial extension" $R = K \oplus K$ (defined in (2.22)(A)), with the multiplication

$$(a, b)(c, d) = (ac, ad + bc) \quad \text{for } a, b, c, d \in K.$$

This is a commutative k-algebra of dimension $2n$. The ideal $J := (0) \oplus K \subseteq R$ has square (0) with $R/J \cong K$, so R is a local algebra with unique maximal ideal J. Also, J is a minimal ideal of R, since any nonzero element $(0, b) \in J$ has a multiple $(ab^{-1}, 0)(0, b) = (0, a)$ for any $a \in K$. Therefore, the only ideals in R are (0), J, and R. Clearly, then, (3.15)(3) holds if we take H to be any k-hyperplane in R not containing J. It follows that R is a Frobenius k-algebra. This example can be extended to a *noncommutative* setting; see (16.60) and Exercise (16.22) below.

(3.15D) Example. *Any k-division algebra R is a Frobenius algebra.* (The only right ideals of R are (0) and R, so we can take H to be *any* hyperplane.) From this, one can show easily that any semisimple k-algebra is also a Frobenius algebra; see Exercise 12.

(3.15E) Example. *If R is the group algebra kG, where k is any field and G is any finite group, then R is a Frobenius algebra.* Of course, if char $k \nmid |G|$, then R is semisimple by *FC*–(6.1); in this case, we can use Exercise 12 mentioned above. For general characteristic, let $\lambda : kG \to k$ be defined by $\lambda(\sum \alpha_g g) = \alpha_1$. Then $\ker(\lambda)$ is the hyperplane

$$H = \left\{\alpha = \sum \alpha_g g : \alpha_1 = 0\right\}.$$

If $0 \neq \alpha \in H$, fix $g \neq 1$ such that $\alpha_g \neq 0$. Then

$$\alpha g^{-1} = \alpha_g \cdot 1 + \cdots \notin H,$$

so $\alpha R \not\subseteq H$. Thus, H has the property in (3.15)(3), so kG is always a Frobenius algebra. The proof of (3.15) also shows that a nonsingular bilinear function B on kG as in (3.15)(2) is given by $B(\alpha, \beta) = \lambda(\alpha\beta) \in k$.

(3.15F) Example. In commutative algebra, a finite-dimensional algebra R over a field k is said to be *étale* if the standard trace function "tr" on the algebra R has the property that the pairing $B : R \times R \to k$ defined by $B(x, y) = \mathrm{tr}(xy)$ is nonsingular. Such a pairing is always associative since

$$B(x, yz) = \mathrm{tr}(xyz) = B(xy, z).$$

Therefore, *an étale algebra is always a Frobenius algebra.* This is not surprising since an equivalent definition for R to be an étale algebra is that $R \cong K_1 \times \cdots \times K_n$, where the K_i 's are finite separable field extensions of k. Since each K_i is a Frobenius algebra by (3.15)(3), so is their direct product R by Exercise 12.

By (3.14), all of the above algebras are right (and left) self-injective. In general, however, *a ring may be left self-injective without being right self-injective.* An example of such a ring can be found in (3.74B).

The class of rings that are 1-sided or 2-sided self-injective has been under close scrutiny by ring theorists. These rings will emerge again in §13 (Chapter 5) in connection with the formation of the maximal rings of quotients. The study of two special classes of self-injective rings, called *Frobenius rings* and *quasi-Frobenius* (QF) *rings* (both generalizing finite-dimensional Frobenius algebras) will be taken up in §15 and §16 in Chapter 6. The above excursion on Frobenius algebras is intended to be just a quick preview of some of this deeper material to come.

§3C. Injectivity versus Divisibility

Returning now to the study of injective modules, we shall next investigate the close relationship between injectivity and the notion of "divisibility".

Let I be a right module over a ring R. If $u \in I$ and $a \in R$, we shall say that *u is divisible by a* if $u \in Ia$; that is, if there exists an element $v \in I$ such that $u = va$. For such an element v to exist, we have clearly the following necessary condition:

$$\text{For } x \in R, \quad ax = 0 \Longrightarrow ux = 0,$$

or, in the notation of annihilators, $\text{ann}_r(a) \subseteq \text{ann}(u)$. This observation leads to the following definition.

(3.16) Definition. We say that I_R is a *divisible module*[20] if, for any $u \in I$ and $a \in R$ such that $\text{ann}_r(a) \subseteq \text{ann}(u)$, u is divisible by a.

For any ring R, the following easy Proposition offers two slightly different statements to characterize the divisibility of a module.

(3.17) Proposition. *For any right R-module I, the following are equivalent:*

(1) *I is a divisible module.*
(2) *For any $a \in R$, $\text{ann}^l(\text{ann}_r(a)) = Ia$.*
(3) *For any $a \in R$, any R-homomorphism $f : aR \to I$ extends to an R-homomorphism from R_R to I.*

Proof. First note that (2) is just a symbolic way of expressing the definition of divisibility, so we really need not distinguish (2) from (1).

(1)$\Longrightarrow$(3). Let $f \in \text{Hom}_R(aR, I)$, and let $u = f(a) \in I$. Then

$$x \in \text{ann}_r(a) \Longrightarrow 0 = f(ax) = f(a)x = ux \Longrightarrow x \in \text{ann}(u).$$

By (1), $u = va$ for some $v \in I$, so f extends to $R_R \to I$ given by $1 \mapsto v$.

(3)$\Longrightarrow$(2). We need only prove "$\subseteq$" in (2). Let $u \in \text{ann}^l(\text{ann}_r(a))$. The map $f : aR \to I$ given by $f(as) = us$ ($\forall\ s \in R$) is then a well-defined R-homomorphism. By (3), f is left multiplication by some $v \in I$. Therefore, $u = f(a) = va$. $\square$

In Nicholson-Yousif [95], a module I_R satisfying the condition (3) above is said to be "principally injective". The preceding proposition says that this is just equivalent to I being divisible in the sense of (3.16). In view of Baer's Theorem (3.7), we see that these notions amount to a sort of "weakened" injectivity for the module in question. In particular, we have the following immediate consequence of (3.17), which puts the relationship between injectivity and divisibility in good perspective.

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[20]In many books, I_R is defined to be divisible if any $u \in I$ is divisible by any non-0-divisor of R. We are firmly convinced that this is not the "correct" definition. Our later results relating divisibility and injectivity will fully justify our use of Def. (3.16); see, for instance, (3.72), (3.73), Exer. (3.43), and so on.

(3.17)′ Corollary. *If I_R is injective, then it is divisible. The converse holds if R is a principal right ideal ring* (PRIR), *that is, a ring in which all right ideals are principal.*

The fact that in general a divisible module need not be injective is clear from the (first) "only if" part of the following observation.

(3.18) Proposition. *A ring R is von Neumann regular iff every right R-module is divisible, iff every cyclic right R-module is divisible.*

Proof. If R is von Neumann regular, every aR is a direct summand of R_R, so of course any $f \in \mathrm{Hom}_R(aR, I_R)$ can be extended to R. Thus, any I_R is divisible. Conversely, assume every *cyclic* right R-module is divisible. Then for any $a \in R$, aR is divisible so the identity map $aR \to aR$ can be extended to $R_R \to aR$. This simply means that the inclusion map $aR \hookrightarrow R_R$ splits, so every aR is a direct summand of R_R. This amounts to R being von Neumann regular. □

Note that, in the simple case when R is a *domain*, the condition for I_R to be divisible boils down to $I = Ia$ for every $0 \neq a \in R$. With this interpretation, it follows that, in this case, any epimorphic image of a divisible module is also divisible, and so are any direct sum or direct product of divisible modules.[21] This remark will be useful whenever we work with divisible modules over domains, for instance over the ring of integers.

(3.19) Proposition. *A $\mathbb{Z}$-module (i.e., an abelian group) is injective iff it is divisible. Any $\mathbb{Z}$-module A can be embedded in an injective $\mathbb{Z}$-module.*

Proof. The first statement is a special case of (3.17)′. To prove the second statement, we identify A with a quotient F/H, where F is a free abelian group and $H \subseteq F$ is a subgroup. Expressing F as a direct sum of $\mathbb{Z}$'s, we can embed F into a group K that is a direct sum of $\mathbb{Q}$'s. Since $\mathbb{Q}$ is divisible (as a $\mathbb{Z}$-module), K and hence also K/H are divisible. We are done by embedding $A = F/H$ into K/H, and noting that K/H is an injective $\mathbb{Z}$-module (by the first part). □

The Proposition we just proved leads us quickly to the following major result in the theory of injective modules.

(3.20) Theorem. *For any ring R, any $A \in \mathfrak{M}_R$ can be embedded in an injective right R-module.*

Proof. First view A as a $\mathbb{Z}$-module and embed it into an injective $\mathbb{Z}$-module M, by (3.19). Now recall that $\tilde{M} := \mathrm{Hom}_{\mathbb{Z}}(R, M)$ is a right R-module via the action:

$$(fr)(r') = f(rr') \quad (\text{for } f \in \tilde{M}, \quad r, r' \in R),$$

[21] This statement is certainly not true in general over non domains.

and that $\tilde{M}_R$ is, in fact, injective by (3.6B). We finish by showing that A can be embedded in $\tilde{M}$. Define $\varepsilon : A \to \tilde{M}$ by $\varepsilon(a)(r) = ar$ $(a \in A,\ r \in R)$. This is an R-homomorphism since

$$\varepsilon(as)(r) = a(sr) = \varepsilon(a)(sr) = (\varepsilon(a)s)(r)$$

for every $s \in R$. Finally $\varepsilon(a) = 0$ implies that $a = \varepsilon(a)(1) = 0$, so ε is the desired embedding. □

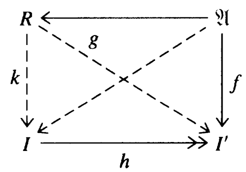
(3.21) Remark. A more symbolic way to present the preceding proof is simply to note:

$$A \cong \mathrm{Hom}_R(R, A) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, A) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M) = \tilde{M}.$$

The next question to ask in the theory of injectives is, naturally, whether there is any degree of uniqueness on the embedding of A in an injective module. Before we take up this important topic, let us first pause to give a couple of applications of (3.20). We have seen in §2 that a ring R is right hereditary iff submodules of right projective R-modules are projective. The following is the injective analogue of this characterization.

(3.22) Theorem. *A ring R is right hereditary iff quotients of right injective R-modules are injective.*

Proof. Assuming first that quotients of right injectives are injective, we must check that any right ideal $\mathfrak{A} \subseteq R$ is projective. Now that we have proved (3.20), we can apply the *Modified Projectivity Test* (Exercise 2.16) to achieve this goal; namely, we need only check that, for any epimorphism $h : I \to I'$, where I_R is *injective*, any $f \in \mathrm{Hom}_R(\mathfrak{A}, I')$ can be lifted to some $f' \in \mathrm{Hom}_R(\mathfrak{A}, I)$. But by assumption, I' is also injective, so f can be extended to $g : R \to I'$, and, since R_R is projective, g can be lifted to $k : R \to I$. We are done by choosing f' to be the restriction of k to $\mathfrak{A}$.



The converse is done simply by "dualizing" the above argument. Even the diagram is the same; we'll leave it as a fun exercise. (Baer's Criterion will play the role of the *Modified Injectivity Test.*) □

(3.23) Corollary. *Let R be a domain. If divisible right R-modules are all injective, then R is right hereditary.*

Proof. By (3.22), it suffices to check that a quotient I' of an injective module I_R is also injective. By (3.17)$'$, I is divisible, and since R is a domain, I' is also divisible. By assumption, I' must then be injective, as desired. □

(3.24) Corollary. *Let R be a commutative domain, with quotient field K. Then divisible R-modules are injective iff R is a Dedekind domain.*

Proof. The "only if" is the commutative case of (3.23). For the "if" part, assume R is Dedekind, and let M be a *divisible* R-module. We apply Baer's Criterion to test the injectivity of M. Thus, let $f : \mathfrak{A} \to M$, where $\mathfrak{A} \subseteq R$ is a right ideal. We may assume $\mathfrak{A} \neq 0$, so we have an equation $\sum a_i b_i = 1$, where $0 \neq a_i \in \mathfrak{A}$ and $b_i \in \mathfrak{A}^{-1} \subseteq K$. Since M is divisible, there exist $m_i \in M$ such that $f(a_i) = m_i a_i$. Now we have, for any $a \in \mathfrak{A}$:

$$\begin{aligned} f(a) &= f\Big(\sum a_i b_i a\Big) = \sum f(a_i)(b_i a) \qquad \text{(note } b_i a \in R) \\ &= \sum (m_i a_i)(b_i a) = \sum m_i (a_i b_i a) \\ &= \Big(\sum m_i (a_i b_i)\Big) a, \end{aligned}$$

so we can extend f to $f' : R \to M$ by taking $f'(1) = \sum m_i (a_i b_i)$. (*A small but subtle point:* It is best to keep the parentheses around $a_i b_i \in R$, since $(m_i a_i) b_i$ may not make sense!) □

While we are still on the subject of commutative domains, it is worth pointing out that there is another case in which we can prove that divisibility implies injectivity.

(3.25) Proposition. *Let R be a commutative domain, and M_R be a torsion-free module (i.e., $mr = 0 \Rightarrow m = 0$ or $r = 0$). Then M is injective iff it is divisible.*

Proof. ("If" part) Let M be divisible. We test again that any $f \in \text{Hom}_R(\mathfrak{A}, M)$ can be extended to R, where $\mathfrak{A}$ is a nonzero ideal in R. Fix an element $0 \neq a \in \mathfrak{A}$, and choose $m \in M$ such that $f(a) = ma$. For any $b \in \mathfrak{A}$, we have

$$f(b)a = f(ba) = f(ab) = f(a)b = mab = mba.$$

Since M is torsion-free, we get $f(b) = mb$ for all $b \in \mathfrak{A}$. □

Example. In view of (3.24) and (3.25), it is of interest to give an explicit example of a module M over a commutative domain R that is divisible but not injective. For this we can take the polynomial ring $R = \mathbb{Z}[x]$ and the R-module $M = K/R$, where $K = \mathbb{Q}(x)$ is the quotient field of R. Clearly M_R is divisible; we'll leave it to the reader to show that M_R is not injective.

§3D. Essential Extensions and Injective Hulls

Next we shall present the basic theory of injective hulls of arbitrary modules, due to Eckmann-Schöpf and Baer. Our exposition here follows that of Lambek [66] which seems difficult to improve upon. We begin by introducing the notion of an essential extension.

(3.26) Definition. A right R-module $E \supseteq M_R$ is said to be an *essential extension* of M if every nonzero submodule of E intersects M nontrivially. An essential extension $E \supseteq M$ is said to be *maximal* if no module properly containing E can be an essential extension of M.

If $E \supseteq M$ is an essential extension, we shall also say that M is an *essential* (or *large*) submodule of E, and write $M \subseteq_e E$ to denote this fact.[22] The notion of a large submodule is "opposite" to that of a *small* (or *superfluous*) submodule: recall from *FC*–(24.1) that a submodule $S \subseteq E$ is small (written $S \subseteq_s E$) if, for any submodule $N \subseteq E$, $S + N = E$ implies that $N = E$.

(3.27) Remarks.

(1) *$M \subseteq_e E$ iff, for any nonzero $a \in E$, there exists $r \in R$ such that $0 \neq ar \in M$.* This is a rather obvious fact that hardly requires a proof. Nevertheless, it provides a convenient way to verify $M \subseteq_e E$; in fact, most of the time, we check essentiality by applying this criterion.

(2) (Transitivity) *If $M \subseteq_e E$ and $E \subseteq_e E'$, then $M \subseteq_e E'$.* This follows quickly from (1), or directly from definition. The transitivity property for essentiality is regarded as basic; in the sequel, we shall sometimes apply it without explicit mention or reference.

The notion of an essential extension leads to a new interpretation of injectivity, as follows.

(3.28) Lemma. *A module M_R is injective iff it has no proper essential extensions.*

Proof. First assume M is injective, and consider any proper extension $E \supsetneq M$. By (3.4)(2), we have $E = M \oplus N$ for some submodule $N \neq 0$. Here $N \cap M = 0$, so $E \supseteq M$ is not an essential extension. Conversely, assume that M has no proper essential extensions, and embed M in an injective module I_R. By Zorn's Lemma, there exists a submodule $S \subseteq I$ maximal with respect to the property that $S \cap M = 0$. Then in the quotient I/S, any nonzero submodule S'/S intersects the image of M nontrivially, so $\mathrm{im}(M) \subseteq_e I/S$. By assumption, we must have

[22] There seems to be no universally accepted notation for essential extensions. Many different notations have been used in the literature. Our notation here follows that of McConnell-Robson [87] and Goodearl-Warfield [89].

$\text{im}(M) = I/S$. This means that $I = M \oplus S$, so M is an injective module by (3.4)(1). □

(3.29) Lemma. *Any module M_R has a maximal essential extension.*

Proof. Fix an injective module $I \supseteq M$, and consider any family of essential extensions of M in I that are linearly ordered by inclusion. By (3.27)(1), it is clear that the union of the family is also essential over M. By Zorn's Lemma, it follows that we can find a submodule E maximal with respect to the property that $M \subseteq_e E \subseteq I$. We claim that *$E$ is a maximal essential extension of M*. Indeed, if this is false, we would be able to find an embedding $E \subsetneq E'$ such that $M \subseteq_e E'$. (**Note.** E' is just *some* R-module; it may not be in I.) By the injectivity of I, the inclusion map $E \subseteq I$ can be extended to some $g : E' \to I$. Clearly $(\ker g) \cap M = 0$, so $M \subseteq_e E'$ implies that $\ker g = 0$. We can therefore identify E' with $g(E')$. But then $M \subseteq_e E'$ contradicts the maximal choice of E. □

Now we are ready for the main results of Eckmann-Schöpf and Baer.

(3.30) Theorem. *For modules $M \subseteq I$, the following are equivalent:*

(1) *I is maximal essential over M.*
(2) *I is injective, and is essential over M.*
(3) *I is minimal injective over M.*

Proof. (1)$\Longrightarrow$(2). By the Transitivity Property in (3.27)(2), (1) implies that I has no proper essential extension. Therefore, I is injective by (3.28).

(2)$\Longrightarrow$(3). Let I' be an injective module such that $M \subseteq I' \subseteq I$. By (3.4)(2), $I = I' \oplus N$ for some submodule $N \subseteq I$. Since $N \cap M = 0$, we must have $N = 0$ (since $M \subseteq_e I$), so $I' = I$.

(3)$\Longrightarrow$(1). Assume I is minimal injective over M. The proof of (3.29) yields a submodule $E \subseteq I$ that is maximal essential over M. Using (1)$\Longrightarrow$(2), we know that E is injective, and therefore $E = I$, which proves (1). □

(3.31) Definition. If the modules $M \subseteq I$ satisfy the (equivalent) properties (1), (2), (3) above, we say that I is an *injective hull* (or *injective envelope*) of M. (By (3.29), any module M has an injective hull.)

(3.32) Corollary. *Any two injective hulls, I, I' of M are isomorphic over M; that is, there exists an isomorphism $g : I' \to I$ which is the identity on M.*[23]

(From now on, we shall write $E(M)$ for "the" injective hull of M.)

[23] In general, however, this isomorphism is not unique.

Proof. By the injectivity of I, we can find $g : I' \to I$ extending the inclusion map $M \to I$. As in the proof of (3.29), we have $\ker g = 0$, since $M \subseteq_e I'$. Therefore, $g(I')$ is an injective submodule of I containing M. Now the property (3) in (3.30) implies that $g(I') = I$, so $g : I' \to I$ is the desired isomorphism. □

Similar arguments, together with (3.27)(2), can be used to prove the following related results.

(3.33) Corollary. (1) *If I is an injective module containing M, then I contains a copy of $E(M)$.* (2) *If $M \subseteq_e N$, then N can be enlarged into a copy of $E(M)$. In fact, $E(N) = E(M)$.*

The injective hull of M is an injective module I for which there is a monomorphism $M \to I$ whose image is "large". The projective cover of M is a projective module P for which there is an epimorphism $P \to M$ whose kernel is "small" (cf. *FC*–p. 361). These are, therefore, dual notions. We have just seen that the injective hull of a module always exists. However, we have seen earlier (in *FC*–§24) that the projective cover of a module exists only over a very specific class of rings!

We now give some examples of injective hulls of modules.

(3.34) Example. Consider $M \subseteq I = E(M)$, and let N be any module such that either $N \subseteq_e M$ or $M \subseteq N \subseteq I$. Then $E(N) = I$ as well. (This follows from (3.33)(2).)

(3.35) Example. Let R be a commutative domain with quotient field K. From (3.9), we know that K_R is injective, and, by checking (3.27)(1), we know that $R \subseteq_e K$ (as R-modules). Therefore, $E(R) = K$. More generally, consider any torsion-free module M_R. Localizing at the multiplicative set $S = R\backslash\{0\}$, we get

$$M \otimes_R K = S^{-1}M \supseteq M,$$

which is again easily seen to be an essential extension. Now, $M \otimes_R K$ is a K-vector space, so by (3.9) it is injective as an R-module. From this and what we said above, it follows that $E(M) = M \otimes_R K = S^{-1}M$.

(3.36) Example. In the case $R = \mathbb{Z}$, $E(M)$ is what is usually known as the "divisible hull" of the abelian group M. Let C_n denote the cyclic group of order n. For any prime p, let C_{p^∞} (the "Prüfer p-group") be the ascending union of the groups

$$C_p \subset C_{p^2} \subset C_{p^3} \subset \dots \,. \tag{3.37}$$

Then C_{p^∞} is p-divisible, and hence divisible. (It is isomorphic to the p-primary part of $\mathbb{Q}/\mathbb{Z}$.) By (3.19), C_{p^∞} is $\mathbb{Z}$-injective, and by (3.27)(1), C_{p^∞} is essential over any C_{p^i} $(i \geq 1)$. Therefore, $E(C_{p^i}) = C_{p^\infty}$ for all $i \geq 1$.

(3.38) Example. In general, over any ring R, *if $M_j \subseteq E_j$ for all $j \in J$, then $\bigoplus M_j \subseteq_e \bigoplus E_j$ iff $M_j \subseteq_e E_j$ for all j.* The "only if" part is obvious. For the

"if" part, it suffices to check the case of a *finite* direct sum (by (3.27)(1)). Writing $J = \{1, 2, \ldots, n\}$, and using the Transitivity Property, we need only check that

$$M_1 \oplus E_2 \oplus \cdots \oplus E_n \subseteq_e E_1 \oplus E_2 \oplus \cdots \oplus E_n$$

whenever $M_1 \subseteq_e E_1$. This case is quickly checked again by using (3.27)(1). Now assume that all the E_j's are injective. If $|J| < \infty$, then by (3.4)(1) $\bigoplus_{j \in J} E_j$ is also injective, so we get

(3.39) $$E\Big(\bigoplus_{j \in J} M_j\Big) = \bigoplus_{j \in J} E(M_j) \qquad (|J| < \infty).$$

Specializing to $R = \mathbb{Z}$, all the E_j's are divisible abelian groups. The direct sum $\bigoplus_{j \in J} E_j$ is also divisible, and hence $\mathbb{Z}$-injective, for an *arbitrary* indexing set J. Therefore, (3.39) holds for $\mathbb{Z}$-modules without any assumptions on J. In particular, if we take $J = \{\text{all primes}\}$ and $M_p = C_p$ for $p \in J$, then this gives

(3.40) $$E(C_2 \oplus C_3 \oplus C_5 \oplus \cdots) = C_{2^\infty} \oplus C_{3^\infty} \oplus C_{5^\infty} \oplus \cdots .$$

The latter group is isomorphic to $\mathbb{Q}/\mathbb{Z}$, and also to the torsion subgroup of the circle group S^1.

(3.41) Example. Let R be a finite-dimensional algebra over a field k. We have shown that $\hat{R} = \mathrm{Hom}_k(R, k)$, viewed as a right R-module as in (3.6C), is injective. *We shall show that $\hat{R}$ is in fact the injective hull of the right R-module $R/\mathrm{rad}\ R$*, where $\mathrm{rad}\ R$ is the Jacobson radical of R. To see this, let S be the socle of $(\hat{R})_R$, i.e. the sum of all simple R-submodules of $\hat{R}$. Since any nonzero submodule contains a simple submodule, we see that $S \subseteq_e \hat{R}$. Therefore, $E(S) = \hat{R}$. It remains to identify the isomorphism type of S as a right R-module. Using *FC*–Exercise (4.18), we have

$$\begin{aligned} S &= \{f \in \hat{R} :\ f \cdot \mathrm{rad}\ R = 0\} \\ &= \{f \in \hat{R} :\ f(\mathrm{rad}\ R) = 0\} \\ &\cong (R/\mathrm{rad}\ R)\hat{}. \end{aligned}$$

Now, the semisimple algebra $R/\mathrm{rad}\ R$ is a Frobenius k-algebra by Exercise 12; that is, $(R/\mathrm{rad}\ R)\hat{} \cong R/\mathrm{rad}\ R$. This is an isomorphism of right $R/\mathrm{rad}\ R$-modules, and hence also an isomorphism of right R-modules. We have thus shown that $S \cong (R/\mathrm{rad}\ R)_R$ and so $\hat{R} \cong E((R/\mathrm{rad}\ R)_R)$. In particular, we see that *$(\hat{R})_R$ is independent of the choice of k*. This fact has an interesting consequence: it shows that the property of R being a Frobenius k-algebra is actually independent of k (as long as R is a finite-dimensional k-algebra). (For a much more precise version of this, see (16.21).)

Before we give more examples of injective hulls, we first describe another useful method for checking the injectivity of a module.

(3.42) Lemma. *Let R be a subring of a ring S, and $\mathfrak{B}$ be a nonempty subset of R such that $S \cdot \mathfrak{B} \subseteq R$. Let I_S be a right S-module on which $\mathfrak{B}$ has zero annihilator*

(i.e., for $i \in I$, $i \cdot \mathfrak{B} = 0 \Longrightarrow i = 0$). If I is injective as an S-module, then I is also injective as an R-module.

Proof. It suffices to show that for any right ideal $\mathfrak{A} \subseteq R$, any $f \in \mathrm{Hom}_R(\mathfrak{A}, I)$ can be extended to some $g \in \mathrm{Hom}_S(\mathfrak{A} \cdot S, I)$ for g can then be extended to S by the injectivity of I_S. We construct g as follows:

$$g\left(\sum a_i s_i\right) = \sum f(a_i)s_i \in I \qquad (a_i \in \mathfrak{A},\ \ s_i \in S).$$

To show that this is well-defined, suppose $\sum a_i s_i = 0$. For any $b \in \mathfrak{B}$, we have $s_i b \in S \cdot \mathfrak{B} \subseteq R$, so from $\sum a_i s_i b = 0$, we get $\sum f(a_i)(s_i b) = 0$ by applying f. This means that $\sum f(a_i)s_i \in I$ is killed by every $b \in \mathfrak{B}$, so by assumption, $\sum f(a_i)s_i = 0$. This shows that the g above is well-defined, from which it is clear that g is an S-homomorphism. This completes Baer's Test for the injectivity of I_R. □

(3.43) Example. Let $S = \mathbb{M}_n(k)$, where k is a semisimple ring, and let $\mathfrak{B}$ be the left ideal of S consisting of all matrices with nonzero entries only on the nth column. Then the left annihilator of $\mathfrak{B}$ in S is clearly zero, since a nonzero matrix cannot left-annihilate every column vector. And, of course, we have $S \cdot \mathfrak{B} = \mathfrak{B}$. Thus, we can apply (3.42) with $I = S_S$ and R *any subring of S containing* $\mathfrak{B}$. Note that S is a semisimple ring so all right S-modules, in particular S_S, are injective. Moreover, the fact that, for $s \in S$, $s \cdot \mathfrak{B} = 0 \Longrightarrow s = 0$ implies that $R_R \subseteq_e S_R$, by (3.27)(1). Therefore, applying (3.42), *we arrive at the conclusion that $E(R_R) = S_R$*. This conclusion is possibly surprising, since all we need to assume about the subring R is that it contains $\mathfrak{B}$. For instance, R may not even contain k, the subring of all scalar matrices. Some explicit examples of R are given below.

(3.43A) $R =$ subring of all upper triangular $n \times n$ matrices over k.

(3.43B) $R =$ subring of matrices with nonzero entries only on the diagonal and on the nth column.

(3.43C) $R =$ subring of the ring in (3.43B) consisting of matrices with a constant diagonal.

(3.43D) $(n = 3)$ R consisting of all matrices (a_{ij}) with $a_{31} = a_{32} = 0$.

(3.43E) $(n = 3)$ R consisting of all matrices (a_{ij}) with $a_{12} = a_{31} = a_{32} = 0$.

(3.43F) $(n = 3)$ $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} & \mathbb{Q} \\ 0 & 0 & \mathbb{Q} \end{pmatrix}$ with $k = \mathbb{Q}$, or $R = \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{R} \\ \mathbb{R} & \mathbb{Q} & \mathbb{R} \\ 0 & 0 & \mathbb{R} \end{pmatrix}$ with $k = \mathbb{R}$.

Let us analyze a couple of these examples in more detail. The most basic example is perhaps (3.43A). Thus, let R be the ring of all $n \times n$ upper triangular matrices. For simplicity, let us assume that k is a division ring. In this case, we have the following decompositions:

$$(3.44) \qquad S_S = E_1 \oplus \cdots \oplus E_n, \qquad R_R = P_1 \oplus \cdots \oplus P_n,$$

where E_i is the (minimal) right S-ideal consisting of matrices with nonzero entries only on the ith row, and $P_i = E_i \cap R$. (The P_i 's are the principal indecomposable right R-modules; see *FC*–p.377.) From $R \subseteq_e S$, we have $P_i \subseteq_e E_i$ for all i, by (3.38). Also, since S_R is injective, each $(E_i)_R$ is injective, so we have $E(P_i) = E_i$ for all i. Here, all E_i's are isomorphic as S-modules, and therefore also as R-modules. The fact that the P_i 's have the same injective hull is not surprising, since each P_i is isomorphic to an essential R-submodule of P_1. (Note that $P_1 = E_1$ is both projective and injective in $\mathfrak{M}_R$, as well as in $\mathfrak{M}_S$.)

In the example above, we also have $E({}_R R) = S$, by a similar argument with left modules. For this, we have to apply the *left-module* version of (3.42), choosing $\mathfrak{B}$ to be the *right* ideal of S consisting of matrices with nonzero entries only on the first row.

The situation is different, however, with the subring in (3.43B). To avoid confusion, let us rename this subring T. While we have $E(T_T) = S$ on the general ground of (3.43), we *do not* have $E({}_T T) = S$, for $n \geq 3$. In fact, for the matrix unit $E_{12} \in S$, we see easily that $T \cdot E_{12} \cap T = (0)$, so $S \supseteq {}_T T$ is not even an essential extension. Thus, (3.43) does not yield any information on the injective envelope $E({}_T T)$.

In (3.35), we have seen that, for a commutative domain R, $E(R_R)$ is the quotient field of R. In (3.43), we have constructed many examples of rings R for which $E(R_R)$ has the form $\mathbb{M}_n(k)$; in particular, $E(R_R)$ turns out to have a ring structure that is compatible with the right R-module structure on $E(R_R)$. However, the following remarkable example from Osofsky [64] shows that, in general, $E(R_R)$ may *not* have such a ring structure.

(3.45) Osofsky's Example. Let A be the ring $\mathbb{Z}/4\mathbb{Z}$, $\mathfrak{A} = 2A$, and let R be the "triangular ring" $\begin{pmatrix} A & \mathfrak{A} \\ 0 & A \end{pmatrix}$ of 32 elements (see *FC*–(1.14)). We claim that *any* injective module $E_R \supseteq R$ cannot be given a ring structure compatible with the right R-module structure on E. (In particular, $R \neq E(R_R)$.) Indeed, consider the ideal $I = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{A} \end{pmatrix}$. Viewing I as a right ideal of R, we can find a copy of $E(I) \supseteq I$ inside E. There exists an element $x \in E(I)$ solving the equation

$$x \cdot \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

since $E(I)$ is divisible, and it is easy to check that the right R-annihilator of $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ is contained in that of $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Similarly, we can check that there

exists $y \in E$ such that

$$y \cdot 2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

We claim that $x \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 0$. Indeed, assume $x \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$. Then, since $I \subseteq_e E(I)$, $x \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} R \cap I \neq 0$. But

$$\begin{aligned} x \cdot \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} R &= \left\{ x \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} : \ a, b, c \in A \right\} \\ &= \left\{ x \begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix} : \ a \in A \right\} \\ &= \left\{ 0, x \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \end{aligned}$$

Therefore, we must have $x \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ (the only nonzero element in I). But right multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ leads to $x \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 0$, a contradiction. If E has a ring structure compatible with its right R-module structure, it would follow that

$$\begin{aligned} 0 = x \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} y &= x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} 2y \\ &= x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ &= x \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \end{aligned}$$

a final contradiction. (For more information on the ring R, see (7.6)(6) and Exercise (8.16).)

In the above example, the ring R is not Jacobson semisimple; in fact, $\text{rad}(R)$ is given by $\begin{pmatrix} \mathfrak{A} & \mathfrak{A} \\ 0 & \mathfrak{A} \end{pmatrix}$. In a later paper, Osofsky [67] has also constructed examples with similar properties that are Jacobson semisimple.

§3E. *Injectives over Right Noetherian Rings*

As a first application of the formation of the injective hull, we prove the following result which gives a somewhat surprising characterization of right noetherian rings. More results of a similar spirit will be given in (3.48).

(3.46) Theorem (Bass, Papp). *For any ring R, the following statements are equivalent:*

(1) *Any direct limit of injective right R-modules is injective.*
(2) *Any direct sum of injective right R-modules is injective.*
(3) *Any countable direct sum of injective right R-modules is injective.*
(4) *R is a right noetherian ring.*

Proof. We shall prove that (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4)$\Longrightarrow$(1).

(1)$\Longrightarrow$(2) follows from the fact that any direct sum of right R-modules can be re-interpreted as a direct limit of its finite partial sums.

(2)$\Longrightarrow$(3) is a tautology.

(3)$\Longrightarrow$(4). Consider any ascending chain of right ideals $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$. Let $\mathfrak{A}$ be their union, and let

$$(*) \qquad E = \bigoplus_{i \geq 1} E(R/\mathfrak{A}_i).$$

Define the map $f : \mathfrak{A} \to E$ by sending $a \in \mathfrak{A}$ to $(a + \mathfrak{A}_i)_{i \geq 1}$ where $a + \mathfrak{A}_i$ is viewed as an element in $R/\mathfrak{A}_i \subseteq E(R/\mathfrak{A}_i)$. Notice that, since $a \in \mathfrak{A}_i$ for sufficiently large i, $f(a)$ is indeed in the direct sum E in $(*)$. By our hypothesis (3), E_R is injective, so the R-homomorphism f can be expressed in the form:

$$f(a) = ea \qquad (\forall a \in \mathfrak{A}),$$

where $e = (e_i)_{i \geq 1}$ is a suitable element in E. Now for sufficiently large i, we have $e_i = 0$, so we also have, for any $a \in \mathfrak{A}$, $0 = f(a)_i = a + \mathfrak{A}_i$. This means that $\mathfrak{A} = \mathfrak{A}_i$ for sufficiently large i, so we have proved that right ideals of R satisfy ACC.

(4)$\Longrightarrow$(1). Let I be a direct limit $\varinjlim I_\alpha$, where each $I_\alpha \in \mathfrak{M}_R$ is injective, and α ranges over a directed set. To apply Baer's Test to I_R, consider any $f \in \text{Hom}_R(\mathfrak{A}, I)$, where $\mathfrak{A}$ is any right ideal of R. Since R is right noetherian, $\mathfrak{A}_R$ is f.g., so $f(\mathfrak{A})$ is contained in $\text{im}(I_\alpha)$ for some α. Pick a f.g. submodule $A \subseteq I_\alpha$ that maps *onto* $f(\mathfrak{A})$ in the direct limit, and let B be defined by the short exact sequence

$$0 \to B \to A \to f(\mathfrak{A}) \to 0.$$

Since A_R is f.g., so is B_R (*FC*–(1.21)). This together with the fact that B "becomes" zero in the direct limit imply that, for some $\beta \geq \alpha$, B maps to zero in I_β. Letting A' be the image of A in I_β, we have then $A' \cong f(\mathfrak{A})$ under the map $i_\beta : I_\beta \to I$. Therefore, we can "factor" f through a homomorphism $g : \mathfrak{A} \to I_\beta$ (so that $f = i_\beta \circ g$). Since I_β is injective, g can be extended to R; clearly this implies that f can be extended to R as well. This completes Baer's Test, so we have proved the injectivity of I. □

(3.47) Remarks.

(A) If we are willing to forgo Condition (1) in the Theorem, the clinching implication (4)$\Longrightarrow$(2) is considerably easier. Indeed, if $\{I_\alpha\}$ are injective modules and R is right noetherian, then for any right ideal $\mathfrak{A} \subseteq R$ and any $f \in \text{Hom}_R(\mathfrak{A}, \bigoplus I_\alpha)$,

$\text{im}(f)$ is contained in some $I_{\alpha_1} \oplus \cdots \oplus I_{\alpha_n}$ since $\mathfrak{A}_R$ is f.g. Using the injectivity of $I_{\alpha_1} \oplus \cdots \oplus I_{\alpha_n}$, we can then extend f to R.

(B) Let $\{M_\alpha\}$ be right modules over a right noetherian ring R. Then $\bigoplus_\alpha E(M_\alpha)$ is injective by (3.46). But we also have $\bigoplus_\alpha M_\alpha \subseteq_e \bigoplus_\alpha E(M_\alpha)$ by (3.38). Therefore, we get $E(\bigoplus_\alpha M_\alpha) \cong \bigoplus_\alpha E(M_\alpha)$ in this case.

The theorem of Bass and Papp (ca. 1959) was among the first major results on direct sums of injective modules, though the implication (4)$\Longrightarrow$(1) in the theorem was known earlier to Cartan and Eilenberg (see [56: p. 17, Exercise 8]). Developing this theme further, we next obtain another result characterizing right noetherian rings R in terms of the decomposition of injective right R-modules. In the following theorem, (1)$\Longleftrightarrow$(2) is due to Matlis [58] and Papp [59], while (1)$\Longleftrightarrow$(3) is due to Faith-Walker [67].

(3.48) Theorem. *For any ring R, the following are equivalent:*

(1) *R is right noetherian.*
(2) *Any injective module M_R is a direct sum of indecomposable (injective) submodules.*
(3) *There exists a cardinal number α such that any injective module M_R is a direct sum of (injective) submodules of cardinality $\leq \alpha$.*

Proof. (1)$\Longrightarrow$(2). We first show that, given (1), any injective module $E_R \neq 0$ contains an indecomposable injective submodule. Indeed, taking a nonzero $x \in E$, it suffices to consider the case when $E = E(xR)$ (see (3.33)(1)). In view of (1), xR cannot contain an infinite direct sum. Since $xR \subseteq_e E$, E also cannot contain an infinite direct sum. From this, we see easily that E contains an indecomposable injective submodule. Now, for any injective module M_R, consider all families of indecomposable injective submodules of M whose sum is direct. By Zorn's Lemma, there exists such a family $\{M_i : i \in I\}$ that is maximal. Then

$$M = E \oplus \left(\bigoplus_i M_i\right) \qquad \text{(for some } E \subseteq M)$$

since $\bigoplus_i M_i$ is injective by (4)$\Longrightarrow$(2) in (3.46). The first part of the proof implies that the injective module E must be zero, so $M = \bigoplus_i M_i$ as desired.

(2)$\Longrightarrow$(3). Consider any indecomposable injective module E_R, and let $0 \neq x \in E$. In view of (3.33)(1), $E = E(xR)$, so $E \cong E(R/\mathfrak{A})$ for some right ideal $\mathfrak{A} \subseteq R$. Thus, (2) implies that any injective M_R is a direct sum of submodules isomorphic to the ones in the following list:

$$\{E(R/\mathfrak{A}) : \ \mathfrak{A} \subseteq R \text{ is a right ideal}\}.$$

Since this is a set, (3) follows by taking α to be, say, $\sum |E(R/\mathfrak{A})|$.

(3)$\Longrightarrow$(1). Let α be a cardinal as in (3). By (3.46), it suffices to show that, for any nonzero injective right modules $M_1, M_2, \ldots,$ $M := M_1 \oplus M_2 \oplus \cdots$ is injective.

Let $\beta = \alpha + |M|$ (an infinite cardinal), and

$$(3.49) \qquad M' = \prod_{j=1}^{\infty} M_j, \qquad M'' = \prod_C M',$$

where $|C| > \beta$. By (3.4), M' and M'' are *injective*. Therefore, by (3), $M'' = \bigoplus_{b \in B} I_b$ for suitable submodules I_b with cardinality $\leq \alpha$. We shall construct disjoint subsets $B_1, B_2, \cdots \subseteq B$ with $|B_j| \leq \beta$ such that each M_j can be embedded (necessarily as a direct summand) in $\bigoplus_{b \in B_j} I_b$. This will show that M embeds as a direct summand in $\bigoplus_{b \in B} I_b = M''$, and hence M is injective! To construct the B_j's, we use induction. Suppose $B_1, \ldots, B_n$ have been constructed, and let

$$N = \bigoplus \{I_b : \ b \in B_1 \cup \cdots \cup B_n\} \subseteq M''.$$

Since $|B_1 \cup \cdots \cup B_n| \leq \beta$ and each $|I_b| \leq \alpha \leq \beta$, we have $|N| \leq \beta$. From (3.49), $M'' \supseteq \bigoplus_{c \in C} X_c$, where each $X_c \cong M_{n+1}$. If $N \cap X_c \neq 0$ for each $c \in C$, we would have $|N| \geq |C| > \beta$, which is not the case. Therefore, there exists $c \in C$ such that $N \cap X_c = 0$. Hence X_c can be embedded in

$$M''/N = \bigoplus \{I_b : \ b \in B \backslash (B_1 \cup \cdots \cup B_n)\}.$$

Since $|X_c| = |M_{n+1}| \leq \beta$, $X_c \cong M_{n+1}$ can already be embedded in $\bigoplus_{b \in B_{n+1}} I_b$ for a suitable $B_{n+1} \subseteq B \backslash (B_1 \cup \cdots \cup B_n)$ with $|B_{n+1}| \leq \beta$. This completes the inductive construction of the B_j's. □

It is nice to have proved the above characterization theorem for right noetherian rings in terms of the decomposition properties of their injective right modules. For most practical purposes, however, the following special case of the theorem will be sufficient. This result depends only on (1)$\Longrightarrow$(2) of (3.48).

(3.50) Corollary. *Let N be a f.g. right module over a right noetherian ring. Then $E(N)$ is a finite direct sum of indecomposable injectives.*

Proof. By (1)$\Longrightarrow$(2) of (3.48), $E(N) = \bigoplus_i M_i$ where the M_i's are indecomposable. Since N is f.g., we have $N \subseteq M_{i_1} \oplus \cdots \oplus M_{i_n}$ for suitable $i_1, \ldots, i_n$. But this is an essential extension, since $N \subseteq_e E(N)$. Therefore, we must have $E(N) = M_{i_1} \oplus \cdots \oplus M_{i_n}$. □

It can be shown that this Corollary is already true for any *noetherian* module N over any ring R. The proof of this uses some of the material in the next subsection; see Exercise 21, or more generally, (6.12).

§3F. Indecomposable Injectives and Uniform Modules

In view of (3.48) and (3.50), the important role of the indecomposable injective modules is now apparent (at least over right noetherian rings). We need a good

working list of alternative characterizations for such modules. This is preceded by a couple of necessary definitions.

Definition. A nonzero module M_R is called *uniform* if any two nonzero submodules of M intersect nontrivially (equivalently: any nonzero submodule of M is indecomposable, or else: any nonzero submodule of M is essential in M). A right ideal $\mathfrak{A} \subsetneq R$ is called (right) *meet-irreducible* if the cyclic module $(R/\mathfrak{A})_R$ is uniform (or equivalently, for right ideals $\mathfrak{B}$, $\mathfrak{B}' \supseteq \mathfrak{A}$, $\mathfrak{B} \cap \mathfrak{B}' = \mathfrak{A}$ implies that $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{B}' = \mathfrak{A}$).

(3.51) Examples.

(1) For any M_R, we have the obvious implications:

$$\text{simple} \Longrightarrow \text{uniform} \Longrightarrow \text{indecomposable}.$$

Over a semisimple ring, all three notions coincide. Over $R = \mathbb{Z}$, however, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Z}/p^n\mathbb{Z}$ ($n \geq 2$, $p =$ prime) are all uniform, but not simple. Over the commutative $\mathbb{Q}$-algebra $R = \mathbb{Q}[u, v]$ defined by the relations $u^2 = v^2 = uv = 0$, the right regular module R_R is indecomposable, but is not uniform as it contains the direct sum of the two nonzero ideals $\mathbb{Q}u$ and $\mathbb{Q}v$. (R here is a special case of the first algebra in Example (3.15B$'$).)

(2) For a *commutative* ring R, a prime ideal $\mathfrak{p} \subset R$ is always meet-irreducible. For, if $\mathfrak{B}$, $\mathfrak{B}' \supseteq \mathfrak{p}$ and $\mathfrak{B} \cap \mathfrak{B}' = \mathfrak{p}$, then $\mathfrak{B}\mathfrak{B}' \subseteq \mathfrak{p}$, so we must have $\mathfrak{B} = \mathfrak{p}$ or $\mathfrak{B}' = \mathfrak{p}$. However, a primary ideal need not be meet-irreducible, even in a noetherian ring R. For instance, in the polynomial ring $\mathbb{Q}[x, y]$,

$$\mathfrak{A} = (x^2, xy, y^2) = (x, y)^2$$

is a primary ideal, but as we saw in (1), $\mathbb{Q}[x, y]/\mathfrak{A}$ is not uniform, so $\mathfrak{A}$ is *not* meet-irreducible. On the other hand, over a commutative noetherian ring, any meet-irreducible ideal is primary; see Zariski-Samuel [58: Vol. 1, p. 209], or (3.80) below.

(3) If R is *not* commutative, a prime ideal $\mathfrak{p}$ need not be (right) meet-irreducible. For instance, in $R = \mathbb{M}_2(\mathbb{Q})$, the zero prime ideal can be written as $\mathfrak{B} \cap \mathfrak{B}'$ where $\mathfrak{B} = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$, $\mathfrak{B}' = \begin{pmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$. *(Where does the proof in* (2) *break down??)* For more information about this, see Exercise 55.

(3.52) Theorem. *For any injective right module M over a ring R, the following conditions are equivalent:*

(1) *M is indecomposable.*
(2) *$M \neq 0$, and $M = E(M')$ for any nonzero submodule $M' \subseteq M$.*
(3) *M is uniform.*
(4) *$M = E(U)$ for some uniform module U.*
(5) *$M = E(R/\mathfrak{A})$ for some meet-irreducible right ideal $\mathfrak{A} \subset R$.*
(6) *M is strongly indecomposable; that is, $E = \mathrm{End}(M_R)$ is a local ring.*

Proof. (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4) are obvious (or very easy).

(4)$\Longrightarrow$(5). Let $V \neq 0$ be a cyclic submodule of U. Since U is uniform, $V \subseteq_e U$. But then $V \subseteq_e E(U)$ by (3.27)(2), and this gives $M = E(V)$. We are done by identifying V with $R/\mathfrak{A}$ for a (necessarily meet-irreducible) right ideal $\mathfrak{A} \subset R$.

(5)$\Longrightarrow$(6). Here $M = E(U)$ where $U \cong R/\mathfrak{A}$ is uniform. If α is a nonunit in $E = \text{End}(M_R)$, then $\ker \alpha \neq 0$. (If $\ker \alpha = 0$, then $\text{im}(\alpha) \subsetneq M$. But $\text{im}(\alpha) \cong M$ is injective, so $M = \text{im}(\alpha) \oplus A$ for some $A \neq 0$. Both $\text{im}(\alpha)$ and A must intersect U nontrivially, contradicting the fact that U is uniform.) Therefore, $U \cap \ker \alpha \neq 0$. If β is another nonunit in E, then likewise $U \cap \ker \beta \neq 0$ and we have

$$\ker(\alpha + \beta) \supseteq (U \cap \ker \alpha) \cap (U \cap \ker \beta) \neq 0.$$

This implies that $\alpha + \beta$ is a nonunit in E; hence E is a local ring, by FC–(19.1).

(6)$\Longrightarrow$(1). If E is a local ring, it does not have nontrivial idempotents. □

(3.53) Corollary. *If an injective module I_R can be written as $M_1 \oplus \cdots \oplus M_n$ where the M_i's are indecomposable, then n is uniquely determined, and (up to a permutation) so are the isomorphism types of the indecomposable summands $M_1, \ldots, M_n$. (This conclusion applies, in particular, to the direct decomposition of $I = E(N)$, where N is a f.g. right module over a right noetherian ring; see* (3.50).*)*

Proof. Since each M_i is strongly indecomposable, the Krull-Schmidt-Azumaya Theorem (FC–(19.21)) applies. □

Remark. Using an infinite version of the KS-Azumaya Theorem, it can be seen that the uniqueness statement in (3.53) also holds for infinite direct sums of indecomposable injectives. However, in FC, we have only proved the finite version of the KS-Azumaya Theorem, so we have to settle with (3.53) here.

With the help of (3.52), we can make a good start toward the classification of the right indecomposable injectives over a ring R. To this end, we introduce the notion of the associated prime ideals of a right module, which is of interest in its own right. (For the definition and characterization of prime ideals in an arbitrary ring, see FC–(10.2).) First we need a lemma.

(3.54) Lemma. *We say that a module N_R is prime if $N \neq 0$, and* $\text{ann}(N) = \text{ann}(N')$ *for any nonzero submodule $N' \subseteq N$. For any such prime module N,* $\mathfrak{p} := \text{ann}(N)$ *is always a prime ideal in R.*

Proof. Clearly $\mathfrak{p} \neq R$, since $N \neq 0$. Now assume there exist ideals $\mathfrak{A}, \mathfrak{B} \supsetneq \mathfrak{p}$ such that $\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{p}$. Then, $N' := N\mathfrak{A}$ is a nonzero submodule of N with $\text{ann}(N') \supseteq \mathfrak{B} \supsetneq \mathfrak{p}$, a contradiction. □

Definition. Let M be a right R-module. An ideal $\mathfrak{p}$ of R is called an *associated prime*[24] of M if there exists a prime submodule $N \subseteq M$ such that $\mathfrak{p} = \text{ann}(N)$. (Note that $\mathfrak{p}$ is indeed a prime ideal by (3.54).) The set of associated primes of M is denoted by $\text{Ass}(M)$. For instance, $\text{Ass}(0) = \emptyset$; and, if N is a prime module, then $\text{Ass}(N) = \{\text{ann}(N)\}$.

(3.55) Example. *Let* $\mathfrak{p}$ *be an ideal in* R. *Then* $N = (R/\mathfrak{p})_R$ *is a prime module iff* $\mathfrak{p}$ *is a prime ideal,* in which case we'll have $\text{Ass}(N) = \{\mathfrak{p}\}$. Ineed, if N is a prime module, then, as we have observed above, $\text{ann}(N) = \mathfrak{p}$ is a prime ideal. Conversely, assume $\mathfrak{p}$ is a prime ideal and consider any nonzero submodule $N' = \mathfrak{A}/\mathfrak{p} \subseteq R/\mathfrak{p}$, where $\mathfrak{A} \supsetneq \mathfrak{p}$ is a right ideal. We have for any $r \in R$:

$$N' \cdot r = 0 \iff \mathfrak{A} \cdot r \subseteq \mathfrak{p} \iff \mathfrak{A} \cdot (rR) \subseteq \mathfrak{p} \iff r \in \mathfrak{p}.$$

Hence $\text{ann}(N') = \mathfrak{p} = \text{ann}(N)$, so N is a prime module, with $\text{Ass}(N) = \{\mathfrak{p}\}$.

In commutative algebra, $\text{Ass}(M)$ is defined a little differently. The following lemma reconciles this difference.

(3.56) Lemma. *Assume that R is commutative, and M is a right R-module. Then, a prime ideal* $\mathfrak{p}$ *belongs to* $\text{Ass}(M)$ *iff* $\mathfrak{p} = \text{ann}(m)$ *for some* $m \in M$.

Proof. First, let $\mathfrak{p} \in \text{Ass}(M)$, so $\mathfrak{p} = \text{ann}(N)$ for a suitable prime submodule $N \subseteq M$. Fixing any element $m \neq 0$ in N, we have $\mathfrak{p} = \text{ann}(mR) = \text{ann}(m)$ (by commutativity). Conversely, suppose $\mathfrak{p}$ is a prime of the form $\text{ann}(m)$, where $m \in M$. Then $mR \cong R/\mathfrak{p}$ is a prime module by (3.55), so its annihilator $\mathfrak{p} \in \text{Ass}(M)$. □

You can perhaps try to remember (3.56) as follows: basically, it says that, in the commutative case, $\mathfrak{p}$ is an associated prime of M iff you can find "a copy of" $R/\mathfrak{p}$ in M. A few more remarks (in the general case) should help.

(3.57) Remarks and Examples (over an *arbitrary* ring R).

(1) *If* M' *is a submodule of* M_R, *then* $\text{Ass}(M') \subseteq \text{Ass}(M)$.

(2) *If* $M' \subseteq_e M$, *then, using* (1) *and the definition of associated primes, we see easily that* $\text{Ass}(M') = \text{Ass}(M)$. In particular, we have always $\text{Ass}(M) = \text{Ass}(E(M))$.

(3) *If* $0 \to M' \to M \to M'' \to 0$ *is exact, then*

$$(*) \qquad \text{Ass}(M) \subseteq \text{Ass}(M') \cup \text{Ass}(M'').$$

For, let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} = \text{ann}(N)$ where N is a prime submodule of M. If $N' := N \cap M' \neq 0$, then N' is also a prime submodule of M', and

[24] The reader should perhaps be warned that there do exist other somewhat different definitions of associated primes of a module in the literature.

$\mathfrak{p} = \text{ann}(N') \in \text{Ass}(M')$. If $N \cap M' = 0$, then N is isomorphic to a (prime) submodule of M'', and it follows that $\mathfrak{p} = \text{ann}(N) \in \text{Ass}(M'')$. In general, however, we may not have equality in (*). For instance, over $R = \mathbb{Z}$, if we take $M = \mathbb{Z}$, $M' = 2\mathbb{Z}$ and $M'' = \mathbb{Z}/2\mathbb{Z}$, then the prime $(2) \subset \mathbb{Z}$ associated with M'' is not associated with M, since $\text{Ass}(M) = \{(0)\}$.

(4) From (1), (3) and induction, it follows readily that

$$\text{Ass}(M_1 \oplus \cdots \oplus M_n) = \text{Ass}(M_1) \cup \cdots \cup \text{Ass}(M_n).$$

(5) *In general,* $\text{Ass}(M)$ *may very well be empty for a f.g. module* M. For instance, let R be the commutative local ring $\mathbb{Q}[x_1, x_2, \ldots]$ with the relations $x_1^2 = x_2^2 = \cdots = 0$. Since the only prime ideal of R is $\mathfrak{p} = (x_1, x_2, \ldots)$ and $\text{ann}(\mathfrak{p}) = 0$, we see from (3.56) that any ideal $M \subseteq R$ (e.g. $M = R$ or $M = x_i R$) has the property that $\text{Ass}(M) = \emptyset$.

The last example shows that, to guarantee that $\text{Ass}(M)$ be nonempty, some kind of "finiteness condition" is needed. The following lemma gives such a finiteness condition.

(3.58) Lemma. *Let* $M_R \neq 0$. *If* $\text{ann}(N_0)$ *is a maximal member in the family* $\{\text{ann}(N)\}$ *where* N *ranges over all nonzero submodules of* M, *then* N_0 *is a prime submodule and* $\text{ann}(N_0)$ *is an associated prime of* M. *In particular, if* R *is a ring whose ideals satisfy* ACC *(e.g.,* R *is a left or a right noetherian ring), then, for any nonzero module* M_R, $\text{Ass}(M) \neq \emptyset$.

Proof. Clear. □

Essential to the applications we have in mind for indecomposable injectives is the following general observation on the associated primes for uniform modules.

(3.59) Lemma. *If* M *is a uniform right module over any ring* R, *then* $|\text{Ass}(M)| \leq 1$.

Proof. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Ass}(M)$, say $\mathfrak{p}_i = \text{ann}(N_i)$ for suitable prime submodules N_i $(i = 1, 2)$. Then $N := N_1 \cap N_2 \neq (0)$ since M is uniform, and we have $\mathfrak{p}_1 = \text{ann}(N) = \mathfrak{p}_2$. □

Note that the converse of (3.59) is not true in general. For instance, over the (commutative noetherian) ring $R = \mathbb{Q}[u, v]$ $(u^2 = v^2 = uv = 0)$ in (3.51)(1), the module $M = R_R$ is not uniform, but by (3.58), $\text{Ass}(M) = \{\mathfrak{p}\}$ for $\mathfrak{p} := (u, v)$, since $\mathfrak{p}$ is the unique prime of R. (Of course, $\mathfrak{p} = \text{ann}(u) = \text{ann}(v)$ also shows that $\mathfrak{p} \in \text{Ass}(M)$ on account of (3.56).)

We now come back to the study of indecomposable injective modules. In the following, let us write $\mathcal{I}(R)$ for the set of isomorphism classes of the right indecomposable injectives over any ring R, and write $\text{Spec}\ R$ for the "prime spectrum" of R (the set of all prime ideals of R).

(3.60) Theorem. *Let R be a right noetherian ring. Then there is a natural surjection $\alpha : \mathcal{I}(R) \to \text{Spec}(R)$. In general, however, α is not a bijection.*

Proof. For any class $[M] \in \mathcal{I}(R)$, M is uniform by (3.52), so M has exactly one associated prime according to (3.58) and (3.59). We can therefore define $\alpha[M]$ to be this associated prime. To show that α is *onto*, consider any $\mathfrak{p} \in \text{Spec } R$, and let $M = (R/\mathfrak{p})_R$. By (3.50), $E(M) = M_1 \oplus \cdots \oplus M_n$ where the M_i 's are indecomposable injective modules.[25] Using (1), (2) in (3.57) and then (3.55), we have:

$$\text{Ass}(M_i) \subseteq \text{Ass}(E(M)) = \text{Ass}(M) = \{\mathfrak{p}\}.$$

Therefore, $\alpha[M_i] = \mathfrak{p}$ for any i.

To show that α is not a bijection in general, consider the case of a simple noetherian domain R that is not a division ring. *Here,* Spec R *is a singleton, consisting of the zero prime ideal, but we claim that* $|\mathcal{I}(R)| \geq 2$. To see this, first note that R_R is uniform. (For, if otherwise, there would exist x, $y \neq 0$ in R such that $xR \cap yR = 0$. From this, we see easily that R_R would contain an infinite direct sum

$$xR \oplus yxR \oplus y^2xR \oplus \cdots,$$

contradicting the fact that R is right noetherian.) Therefore, $[E(R_R)] \in \mathcal{I}(R)$ by (3.52). On the other hand, fix any simple right R-module U. We finish by showing that $[E(R_R)] \neq [E(U)]$ in $\mathcal{I}(R)$. Indeed, if $E(R_R) \cong E(U)$, this uniform (injective) module would contain a copy of R_R and a copy of U; since U is simple, R_R must contain a copy of U. But then, by *FC*–(3.10), R is a right artinian domain, and hence a division ring, a contradiction. □

It is possible to formulate the precise conditions on a right noetherian ring R under which the map α above will be a bijection. However, we do not pursue this matter in detail here. In the following, we shall only cover the two most important cases in which it is known that α *is* a bijection. The first case is when we impose the stronger assumption that R be right artinian. In this case, the result is as follows.

(3.61) Theorem. *Let R be a right artinian ring. Then $\alpha : \mathcal{I}(R) \to$ Spec R is a bijection. If $\{V_1, \ldots, V_n\}$ is a complete set of simple right R-modules, then $\{E(V_1), \ldots, E(V_n)\}$ is a complete set of indecomposable injective right R-modules, up to isomorphism.*

Proof. By the Hopkins-Levitzki Theorem (*FC*–(4.15)), R is right noetherian, so we can apply (3.60). Let n be the number of simple components of $R/\text{rad } R$. Then R has exactly n simple right modules, say, $V_1, \ldots, V_n$. Consider any $[M] \in \mathcal{I}(R)$. For any $x \neq 0$ in M, $x \cdot R$ has a composition series, so it contains a simple

[25]The M_i 's here are actually isomorphic to one another. The proof of this fact, however, depends on Goldie's Theorem; see (11.25) below.

module, say, V_i. Then, by (3.52), $M = E(V_i)$. Also, $i \neq j \Longrightarrow E(V_i) \not\cong E(V_j)$, since the $E(V_i)$'s are uniform. Therefore, $\mathcal{I}(R)$ has exactly n elements, namely, $[E(V_1)], \ldots, [E(V_n)]$. On the other hand, since rad R is nilpotent, prime ideals of R correspond to prime ideals of the semisimple ring $R/\text{rad}\ R$. From this, we see that $|\text{Spec}(R)| = n$, which clearly implies that α is a bijection. (Of course, we can explicitly describe the inverse map $\beta : \text{Spec}\ R \to \mathcal{I}(R)$ too. For $\mathfrak{p} \in \text{Spec}\ R$, $R/\mathfrak{p}$ is a simple artinian ring. If V is "the" simple right $R/\mathfrak{p}$-module, viewed as a simple right R-module, we can define $\beta(\mathfrak{p}) = [E(V)] \in \mathcal{I}(R)$.) □

The next case is when we assume the noetherian ring R to be commutative. Here, we have the following well known result in commutative algebra.

(3.62) Matlis' Theorem. *If R is a commutative noetherian ring, then*

$$\alpha : \mathcal{I}(R) \to \text{Spec}\ R$$

is a bijection. Moreover,

$$\{E(R/\mathfrak{p}) : \ \mathfrak{p} \in \text{Spec}\ R\}$$

gives a complete list of indecomposable injective R-modules, up to isomorphism.

Proof. For $\mathfrak{p} \in \text{Spec}\ R$, $R/\mathfrak{p}$ is a uniform R-module by (3.51)(2). Therefore, we can define $\beta : \text{Spec}\ R \to \mathcal{I}(R)$ by

$$\beta(\mathfrak{p}) = [E(R/\mathfrak{p})] \in \mathcal{I}(R).$$

Clearly, $\alpha\beta(\mathfrak{p}) = \mathfrak{p}$, by (3.57). We finish by showing that $\beta\alpha[M] = [M]$ for any $[M] \in \mathcal{I}(R)$. Say $\text{Ass}(M) = \{\mathfrak{p}\}$. By (3.56), $\mathfrak{p} = \text{ann}(m)$ for some $m \in M$. Then $mR \cong R/\mathfrak{p}$ as R-modules. But by (3.52)(2), $M = E(mR)$. Therefore,

$$[M] = [E(R/\mathfrak{p})] = \beta(\mathfrak{p}) = \beta\alpha[M].$$

□

(3.63) Example. Let R be a commutative PID with quotient field K. Let P be a complete set of nonzero prime elements of R (up to associates). According to (3.62), $\mathcal{I}(R)$ consists of the classes of $E(R) = K$ and $E(R/pR)$ for $p \in P$. The latter modules can be easily constructed as follows. Let $(K/R)_p$ $(p \in P)$ denote the p-primary torsion submodule of K/R. Then $K/R = \bigoplus_{p \in P}(K/R)_p$. Since K is a divisible R-module, so are K/R and $(K/R)_p$ for each $p \in P$. By (3.17)′, each $(K/R)_p$ is an injective R-module. We see easily that $(K/R)_p$ is an essential extension of $p^{-1}R/R \cong R/pR$, so we have $E(R/pR) \cong (K/R)_p$ for each $p \in P$. A complete set of indecomposable injectives over R is therefore $\{K,\ (K/R)_p\ (p \in P)\}$, generalizing the classification of indecomposable divisible objects in the category of abelian groups.

In the above example, the indecomposable injective module $(K/R)_p$ has the filtration

$$(0) \subsetneq p^{-1}R/R \subsetneq p^{-2}R/R \subsetneq \cdots,$$

with each filtration factor $\cong R/pR$. It turns out that, in the general commutative noetherian case, an indecomposable injective (other than $E(R) \cong K$) has a rather "similar" feature. This discovery is also due to Matlis, who developed the detailed structure theory of indecomposable injectives in the commutative noetherian case in his Chicago thesis in the late 50s. In the interest of first completing our discussion of the general theory, we shall postpone the presentation of Matlis' results to a later subsection. Readers who are eager to see this material right away may of course proceed directly to §3I at this point.

§3G. Injectives over Some Artinian Rings

If R in (3.61) is a general right artinian ring (or even a right and left artinian ring), the injective hulls $E(V_i)$ there need not be f.g. R-modules, or equivalently, they may not be of finite (composition) length (see Exercise 34). Rosenberg and Zelinsky [59] have studied the precise conditions which would ensure the finite generation of the $E(V_i)$'s. We do not go into the details of their results here; instead, we shall present a couple of classical results in the positive case where we can indeed draw a "f.g." conclusion. These positive cases include (1) R is commutative, (2) R is a finite-dimensional algebra over a field, and (3) R is 1-sided self-injective. We shall treat the first two cases in this subsection; the last case will become clear when we study quasi-Frobenius rings in §15 (see Exer. (15.13)).

We start now with the case when R is *commutative.*

(3.64) Theorem. *Let R be a commutative artinian ring. Keeping the notations in* (3.61), *let $E_i = E(V_i)$ and let $M = E_1 \oplus \cdots \oplus E_n$. Then:*

(1) *M is a faithful R-module;*
(2) *M is f.g. with* $\text{length}_R(M) = \text{length}_R(R)$; *and*
(3) *for any f.g. R-module N, $E(N)$ is also f.g.*

Since artinian rings are noetherian, some of these results can be deduced from Matlis' general analysis of the injective indecomposables over a commutative noetherian ring, which we shall present later in §3I. But the artinian case is really simpler, and the conclusions are also much sharper. Thus, we may as well give a quick ad hoc exposition of it here, without worrying about the efficiency issue. Besides, the theorem above goes well with the remaining results and examples in this subsection on modules over finite-dimensional algebras.

Proof of (3.64). (1) Let $0 \neq a \in R$ be such that $Ma = 0$. Since aR has a simple quotient, there exists a nonzero R-homomorphism

$$f : aR \longrightarrow V_1 \oplus \cdots \oplus V_n \subseteq M.$$

By the injectivity of M, there exists $m \in M$ such that $f(a) = ma$. But then $Ma = 0$ implies that $f(a) = 0$, a contradiction. This proves the faithfulness of M.

(2) Let $(0) = I_0 \subsetneq \cdots \subsetneq I_t = R$ be a composition series for R_R, and let $M_i = \{m \in M : mI_i = 0\}$. To prove (2), it suffices to show that

$$(0) = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_0 = M$$

is a composition series for M. Taking "Hom" from the exact sequence of R-modules

$$0 \longrightarrow I_{i+1}/I_i \longrightarrow R/I_i \longrightarrow R/I_{i+1} \longrightarrow 0,$$

into the injective module M, we get an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(R/I_{i+1}, M) \longrightarrow \mathrm{Hom}_R(R/I_i, M) \longrightarrow \mathrm{Hom}_R(I_{i+1}/I_i, M) \longrightarrow 0,$$

where the "Hom" groups are still R-modules since R is commutative. Upon identifying the first two modules with M_{i+1} and M_i, we obtain an R-isomorphism

$$M_i/M_{i+1} \cong \mathrm{Hom}_R(I_{i+1}/I_i,\ M).$$

Say $I_{i+1}/I_i \cong V_j$ (for some j). Then the above R-module is isomorphic to

$$\mathrm{Hom}_R(V_j,\ M) = \prod_k \mathrm{Hom}_R(V_j,\ E_k) = \prod_k \mathrm{Hom}_R(V_j,\ V_k) = \mathrm{End}_R(V_j).$$

Let $V_j \cong R/\mathfrak{m}$ where $\mathfrak{m} \subset R$ is a maximal ideal. Then

$$\mathrm{End}_R(V_j) \cong \mathrm{End}_{R/\mathfrak{m}}(R/\mathfrak{m}) \cong R/\mathfrak{m}\,,$$

and this is a simple R-module, as desired,

(3) Let N_R be f.g. Then, by (3.50) and (3.61), $E(N)$ is a direct sum of copies of the E_i 's. This direct sum must be finite. Since each E_i is f.g. by the above, it follows that $E(N)$ is also f.g. □

The module M in (3.64) has a special significance. In the terminology of §19, it is a "minimal injective cogenerator" for R (and such a module is always faithful, by (19.7)). The interesting role such a module plays in duality theory will be examined in detail in §19.

A second case where we have positive results on the finite generation of injective hulls is the case of finite-dimensional (not necessarily commutative) algebras.

(3.65) Theorem. *Let N be a f.g. right module over a finite-dimensional k-algebra R (where k is a field). Then* $\dim_k E(N) < \infty$.

Proof. We need only modify slightly the proof of (3.20). Using the left R-module structure on R, we can make $\mathrm{Hom}_k(R, N)$ into a right R-module, which is injective according to (the paragraph following) (3.5). Since we can embed $N \cong$

$\text{Hom}_R(R, N)$ into $\text{Hom}_k(R, N)$ as R-modules, it follows that

$$\dim_k E(N) \le \dim_k \text{Hom}_k(R, N) = (\dim_k R)(\dim_k N) < \infty.$$

□

There is also another (somewhat different) way to prove (3.65), by using the formation of k-duals. For any left R-module M, let $\hat{M}$ denote its (first) dual $\text{Hom}_k(M, k)$. Recall that $\hat{M}$ has a natural *right* R-module structure defined by

$$(fr)(m) = f(rm) \qquad (f \in \hat{M}, \ \ r \in R, \ \ m \in M).$$

This gives an exact (contravariant) functor $G : {}_R\mathfrak{M} \to \mathfrak{M}_R$, and we have a similar functor $F : \mathfrak{M}_R \to {}_R\mathfrak{M}$. Now, let $N \in \mathfrak{M}_R$ be f.g., and fix an epimorphism $R^n \to \hat{N}$ in ${}_R\mathfrak{M}$. Applying G, we get a monomorphism

$$N = N^{\wedge\wedge} \longrightarrow (R^n)^\wedge \cong (\hat{R}_R)^n$$

in $\mathfrak{M}_R$. Since $(\hat{R})_R$ is injective (by (3.6C)), $\dim_k E(N) \le n \cdot \dim_k R < \infty$.

Using the functors F, G, we also see the following:

$$M = \text{simple left } R\text{-module} \Longrightarrow \hat{M} = \text{simple right } R\text{-module},$$

$$M = \text{indecomposable left } R\text{-module} \Longrightarrow \hat{M} = \text{indecomposable right } R\text{-module}.$$

These observations will enable us to describe explicitly the injective hulls $E(V_i)$ for the simple right R-modules $\{V_i\}$. We proceed as follows. Let $e_1, \ldots, e_n$ be a set of primitive idempotents in R such that $Re_1, \ldots, Re_n$ give a complete set of principal indecomposable left R-modules (see *FC*–§25), and let $J = \text{rad } R$. Then Re_i/Je_i $(1 \le i \le n)$ give a complete set of simple left R-modules (*FC*–(25.3)), and consequently $V_i = (Re_i/Je_i)^\wedge$ $(1 \le i \le n)$ give a complete set of simple right R-modules. The surjections $Re_i \to Re_i/Je_i$ induce injections $(Re_i/Je_i)^\wedge \to (Re_i)^\wedge$, so each V_i is a submodule of $(Re_i)^\wedge$. Also, Re_i being indecomposable implies that $(Re_i)^\wedge$ is indecomposable, and by (3.6C), Re_i being projective in ${}_R\mathfrak{M}$ implies that $(Re_i)^\wedge$ is injective in $\mathfrak{M}_R$. Therefore, by (3.52), we have $(Re_i)^\wedge = E(V_i)$ for each i, and (3.61) yields the following.

(3.66) Corollary (Nagao-Nakayama). *Let R be a finite-dimensional algebra over a field k, and let $e_1, \ldots, e_n$ be primitive idempotents of R such that $Re_1, \ldots, Re_n$ give a complete set of principal indecomposable left R-modules. Then*

$$(Re_1)^\wedge, \ldots, (Re_n)^\wedge$$

give a complete set of indecomposable injective right R-modules, and any injective right R-module is ("uniquely") a direct sum of these.

The "duality" between left and right R-modules provided by the functor "$\wedge$" also puts certain other facts in a better perspective. For instance, we know from *FC*–(21.18) that Je_i is the *unique* maximal submodule of Re_i. Dualizing this, we obtain the fact that V_i is the *unique* simple submodule of $(Re_i)^\wedge$. Of course, from the

perspective of injective modules, this follows from the fact that the indecomposable injective module $(Re_i)^\wedge$ is *uniform.*

(3.67) Example. Let us compute the indecomposable injectives over the ring R of upper triangular $n \times n$ matrices over a field k. We shall first carry out this computation without using duality. Let e_i be the matrix unit E_{ii}. Then $P_i = e_i R$ consists of matrices in R with only nonzero entries on the ith row, and $\{P_i : 1 \leq i \leq n\}$ gives a complete set of principal indecomposable right R-modules. For convenience, let us write $P_{n+1} = 0$, and let $J = \text{rad } R$. As in FC–(25.11),

$$M_i = P_1 J^{i-1} \cong P_i \qquad (1 \leq i \leq n+1),$$

and $U_i := M_i/M_{i+1}$ $(1 \leq i \leq n)$ are the simple right R-modules. We know (from the paragraph following (3.44)) that M_1 is injective. Since R is right hereditary, M_1/M_{i+1} is also injective (cf. (2.36), (3.22)). It is easy to see that

$$U_i = M_i/M_{i+1} \subseteq_e M_1/M_{i+1}, \quad \text{so} \quad E(U_i) = M_1/M_{i+1}.$$

Therefore, *the indecomposable injectives are given by* M_1/M_{i+1} *for* $1 \leq i \leq n$. Now let us bring in the duality theory. The principal indecomposable *left* modules are Re_i (consisting of matrices of R with nonzero entries only on the ith column), for $1 \leq i \leq n$. By (3.66), we know that the indecomposable injective right R-modules are given by $(Re_i)^\wedge$ $(1 \leq i \leq n)$. By dimension comparison, we see that

$$(3.68) \qquad (Re_i)^\wedge \cong E(U_i) = M_1/M_{i+1} \quad (1 \leq i \leq n),$$

and consequently

$$(Re_i/Je_i)^\wedge \cong U_i = M_i/M_{i+1} \cong e_i R/e_i J \quad (1 \leq i \leq n).$$

Of course, it is also easy to establish directly the duality between the ith simple left module Re_i/Je_i and the ith simple right module $e_i R/e_i J$, from which we can then deduce (3.68) from (3.66). The duality between Re_i and M_1/M_{i+1} asserted in (3.68) can likewise be seen directly as follows. We identify $M_1 = P_1$ with the space of row vectors and Re_n with the space of column vectors, and define the pairing

$$f : M_1 \times Re_n \longrightarrow k$$

by taking usual inner products. This pairing is middle R-linear, so it induces a right R-module isomorphism $(Re_n)^\wedge \cong M_1$. If we further identify Re_i with the submodule of Re_n consisting of column vectors of the form $(b_1, \ldots, b_i, 0, \ldots, 0)^t$, then the pairing f above induces a new pairing

$$f_i : M_1/M_{i+1} \times Re_i \longrightarrow k,$$

which yields directly $(Re_i)^\wedge \cong M_1/M_{i+1}$ for $1 \leq i \leq n$.

Next, we return to two commutative non-Frobenius algebras R introduced in Example (3.15B′); we shall compute the injective hulls of R_R for both of these algebras.

(3.69) Example. Let A be the polynomial algebra $k[x, y]$ over a field k, and let $R = A/\mathfrak{A}$, where

$$\mathfrak{A} = (x, y)^{n+1} = \textstyle\sum_{i+j=n+1} x^i y^j A \quad (n \geq 0).$$

Then R is a (commutative) local k-algebra (with unique maximal ideal $\bar{x}R + \bar{y}R$), with k-basis $\mathcal{B} = \{\bar{x}^i \bar{y}^j : i + j \leq n\}$. In particular,

$$\dim_k R = 1 + 2 + \cdots + (n+1) = (n+1)(n+2)/2.$$

Certainly, R is not self-injective. (For instance, the R-homomorphism $\bar{x}^n R \to R$ mapping $\bar{x}^n$ to $\bar{y}^n$ does not extend to R.) Let us apply (3.66) to determine the injective R-modules. Here, $e = 1$ is the only primitive idempotent, and there is only one simple R-module $V = k$ (with $V\bar{x} = V\bar{y} = 0$). Therefore, by (3.66), every injective R-module is a direct sum of the *unique* indecomposable injective, given by $E(V) = (Re)^{\wedge} = \hat{R}$. We claim that the R-module $\hat{R}$ is given by

$$I := (x, y)^n / (x^{n+1}A + y^{n+1}A).$$

Initially, I is only an A-module. Since obviously $I\,\mathfrak{A} = 0$, we may indeed view I as an R-module. To see that $I \cong \hat{R}$, first note that a k-basis for I is given by

$$\mathcal{B}' = \{\bar{x}^p \bar{y}^q : \; p + q \geq n; \;\; p \leq n, \, q \leq n\}.$$

(The "bars" here refer to modulo $x^{n+1}A + y^{n+1}A$.) An easy counting shows that

$$\dim_k I = |\mathcal{B}'| = (n+1)(n+2)/2 = \dim_k R = \dim_k \hat{R}.$$

Now, there is a k-bilinear pairing $\beta : I \times R \longrightarrow k$ defined by

$$\beta(f, g) = \text{coefficient of } x^n y^n \text{ in } fg.$$

(Note that $\beta(f, g) = 0$ whenever $f \in x^{n+1}A + y^{n+1}A$ or $g \in \mathfrak{A}$, so β is well-defined.) Under the pairing β, $\mathcal{B}'$ and $\mathcal{B}$ are visibly a pair of dual bases, with $\bar{x}^{n-i}\bar{y}^{n-i} \in \mathcal{B}'$ dual to $\bar{x}^i \bar{y}^j \in \mathcal{B}$ (for $i + j \leq n$). Also, clearly, $\beta(fh, g) = \beta(f, hg)$ for any $h \in A$. Putting all this information together, we see that $I \cong \hat{R}$ as R-modules. The indecomposable injective R-module I is the injective hull of its unique simple R-submodule generated by $\bar{x}^n \bar{y}^n$. We can now also compute the injective hull of R itself. The socle of R (sum of its simple submodules) is given

$$S = \bigoplus_{i+j=n} \bar{x}^i \bar{y}^j k \cong V \oplus \cdots \oplus V \quad (n+1 \text{ copies}).$$

Since R is artinian, this socle is essential in R, so we have

$$E(R) = E(S) \cong E(V \oplus \cdots \oplus V) \cong E(V) \oplus \cdots \oplus E(V) \cong (n+1) \cdot \hat{R}.$$

The following picture of $\mathcal{B}'$ and $\mathcal{B}$ in the case $n = 3$ illustrates very well the duality between these two k-bases on I and on R, respectively. (For convenience,

we have omitted the "bars" in this picture.)

$$\begin{array}{ccccccc} & & & x^3y^3 & & & \\ & & x^2y^3 & & x^3y^2 & & \\ & xy^3 & & x^2y^2 & & x^3y & \\ y^3 & & xy^2 & & x^2y & & x^3 \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \end{array}$$

(3.70) Example. Let A and n be as above, and let $R = A/(x^2,\ xy^{n+1},\ y^{n+2})$. (Here, the denominator denotes the ideal generated by the three indicated elements in A.) Now, $\dim_k R = 2n + 3$, with k-basis

$$\mathcal{B} = \{\bar{1};\ \bar{x}, \bar{y};\ \bar{x}\bar{y}, \bar{y}^2;\ \ldots\ ;\ \bar{x}\bar{y}^n, \bar{y}^{n+1}\}.$$

Using the same notation V for the unique simple R-module as in (3.69), we can compute the unique injective indecomposable $E(V)$. The calculations are similar to those in (3.69), so we shall just describe the answer here and omit the details. The claim is that $\hat{R}$ is isomorphic, as an R-module, to $I := (x^2, xy, y^2)/(x^2, y^{n+3})$. The k-pairing $\beta : I \times R \longrightarrow k$ is now given by

$$\beta(f,\ g) = \text{coefficient of } xy^{n+2} \text{ in } fg.$$

The (ordered) dual k-basis on I is given by

$$\mathcal{B}' = \{\bar{x}\bar{y}^{n+2};\ \bar{y}^{n+2}, \bar{x}\bar{y}^{n+1};\ \ldots\ ;\ \bar{y}^2, \bar{x}\bar{y}\},$$

leading to the following configuration (again with the "bars" suppressed);

$$\begin{array}{ccc} & xy^{n+2} & \\ y^{n+2} & & xy^{n+1} \\ y^{n+1} & & xy^n \\ \vdots & & \vdots \\ y^2 & & xy \end{array} \qquad \begin{array}{ccc} & 1 & \\ x & & y \\ xy & & y^2 \\ \vdots & & \vdots \\ xy^n & & y^{n+1} \end{array}$$

Finally, note that the socle S of R is $\bar{x}\bar{y}^n k \oplus \bar{y}^{n+1}k \cong V \oplus V$, so the injective hull of R is given by

$$E(R) \cong E(V) \oplus E(V) \cong \hat{R} \oplus \hat{R}.$$

In this example, we could have taken $n = 0$, for which $R = k[x, y]/(x^2, xy, y^2)$. Here, we get $\hat{R} \cong (x^2, xy, y^2)/(x^2, y^3)$. On the other hand, if we let $n = 1$ in (3.69), we get $\hat{R} \cong (x, y)/(x^2, y^2)$. It is an easy exercise to see directly that these two descriptions of $\hat{R}$ are consistent. We note also that, although R fails to be a Frobenius algebra (being not self-injective), its quotient

$$R/\bar{y}^{n+1}R \cong k[x, y]/(x^2,\ y^{n+1})$$

is a Frobenius algebra by (3.15B).

The methods used for computing $\hat{R}$ in the examples above may look a bit ad hoc. Later, a more canonical method dealing with the same type of examples will be presented in §3J; see (3.92) and (3.93).

§3H. Simple Injectives

In this subsection, we shall present a couple of standard results on simple injective modules. The case of commutative rings is easier, so we'll start with that. Later in the subsection, we'll return to noncommutative rings, and briefly discuss the notion of right V-rings (rings all of whose simple right modules are injective).

We begin by recalling some well-known characterizations of von Neumann regular commutative rings. The following result was Exercise (4.15) in *FC*. For the reader's convenience, a full proof is included here.

(3.71) Theorem. *For any commutative ring R, the following conditions are equivalent:*

(1) *R is von Neumann regular.*
(2) *R is a reduced ring of Krull dimension* 0.
(3) *At every maximal ideal $\mathfrak{m} \subset R$, the localization $R_{\mathfrak{m}}$ is a field.*

Proof. (1)$\Longrightarrow$(2). For $a \in R$, (1) implies that $a = axa = a^2x$ for some $x \in R$. Therefore, $a^2 = 0 \Longrightarrow a = 0$, so R is reduced. To prove the rest of (2), we have to show that any prime ideal $\mathfrak{p}$ is maximal. We do this by showing that $\bar{R} = R/\mathfrak{p}$ is a field. For $a \notin \mathfrak{p}$, we have $a = a^2x$ for some x. Since $\bar{a} \in \bar{R}$ is not a 0-divisor, cancellation of $\bar{a}$ yields $\bar{a}\bar{x} = \bar{1} \in \bar{R}$.

(2)$\Longrightarrow$(3). Let $\mathfrak{m}$ be a given maximal ideal. If R is reduced, so is $R_{\mathfrak{m}}$. If R has Krull dimension 0, so does $R_{\mathfrak{m}}$. But then $\mathfrak{m}_{\mathfrak{m}}$ is the only prime ideal in $R_{\mathfrak{m}}$, so $\mathfrak{m}_{\mathfrak{m}} = \text{Nil}(R_{\mathfrak{m}}) = 0$. Since $(R_{\mathfrak{m}}, \mathfrak{m}_{\mathfrak{m}})$ is a local ring, it follows that $R_{\mathfrak{m}}$ is a field.

(3)$\Longrightarrow$(1). Assume (3) and consider $a \in R$. At any maximal ideal $\mathfrak{m} \subset R$, we have

$$(aR/a^2R)_{\mathfrak{m}} \cong (aR)_{\mathfrak{m}}/(a^2R)_{\mathfrak{m}} \cong aR_{\mathfrak{m}}/a^2R_{\mathfrak{m}}.$$

Since $R_{\mathfrak{m}}$ is a field, $aR_{\mathfrak{m}} = a^2R_{\mathfrak{m}}$. This implies that $(aR/a^2R)_{\mathfrak{m}} = 0$, for all $\mathfrak{m}$. Therefore, $aR/a^2R = 0$, which means that $a = a^2x$ for some $x \in R$. □

The next result gives criteria for a *simple* module over any commutative ring to be injective. The equivalence of (1) and (3) appeared in Rosenberg-Zelinsky [59]. Our choice of (3.16) as the definition of divisible modules enables us to add (2) to the list of equivalences.

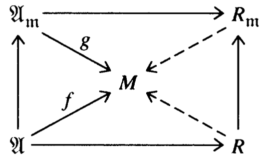
(3.72) Theorem. *Let R be a commutative ring, and $M = R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal in R. Then the following conditions are equivalent:*

(1) *M_R is injective.*
(2) *M_R is divisible.*
(3) *The localization $R_{\mathfrak{m}}$ is a field.*

Proof. (1)$\Longrightarrow$(2) holds for all rings, by (3.17)′.

(2)$\Longrightarrow$(3). Let $r \in \mathfrak{m}$. The equation $\bar{1} = \bar{v}r$ clearly has no solution for $\bar{v} \in M = R/\mathfrak{m}$. Since M is divisible, there must exist $x \in R$ such that $rx = 0$ but $0 \neq \bar{1} \cdot x = \bar{x}$. This shows that every $r \in \mathfrak{m}$ is killed by some element outside of $\mathfrak{m}$. From this, it follows that $\mathfrak{m}_\mathfrak{m} = 0$, so $R_\mathfrak{m}$ is a field.

(3)$\Longrightarrow$(1). Let $f \in \mathrm{Hom}_R(\mathfrak{A}, M)$, where $\mathfrak{A}$ is any ideal in R. Note that $M = R/\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$ may be viewed as an $R_\mathfrak{m}$-module, so f induces the $R_\mathfrak{m}$-homomorphism g in the following commutative diagram



Since $R_\mathfrak{m}$ is a field, $\mathfrak{A}_\mathfrak{m}$ is either (0) or $R_\mathfrak{m}$. Clearly, then, g can be extended to $R_\mathfrak{m}$, from which it follows (see diagram) that f can be extended to R. □

The first part of the Corollary below is due to I. Kaplansky. The second part is a bonus we derived from our choice of (3.16) as the "correct" definition of divisible modules.

(3.73) Corollary. *A commutative ring R is von Neumann regular iff all simple R-modules are injective, iff all simple R-modules are divisible.*

Proof. This follows immediately from (3.71) and (3.72). □

Of course, (3.72) and (3.73) are results special for commutative rings. In the literature, a ring R is called a *right V-ring* (after O. Villamajor) if every simple right R-module is injective. Thus, (3.73) says that, in the category of commutative rings, the V-rings are exactly the (commutative) von Neumann regular rings. For *noncommutative* rings, the situation is quite a bit more subtle. The following example shows that, over a general von Neumann regular ring E, there may exist a simple right module which is divisible but not injective; in particular, E may not be a right V-ring.

(3.74A) Example. Let k be a division ring and ${}_kV$ be a left k-vector space of *infinite* dimension. Let $E = \mathrm{End}({}_kV)$, defined as a ring of *right* operators on V. Then E is a von Neumann regular ring (*FC*–(4.27)), and V is a simple right E-module. We claim that V_E *is divisible but not injective.* (In particular, E is not a right V-ring.)

To see that V_E is *divisible*, let $u \in V$ and $a \in E$ be such that $u \notin Va$. By working with a suitable basis, we can define an endomorphism $x \in E$ that vanishes on Va but not on u. Then $(Va)x = 0$ implies $ax = 0 \in E$, but we have $ux \neq 0$. This shows that, as long as $ax = 0 \Longrightarrow ux = 0$, we can solve the equation $u = va$ for $v \in V$.

To show that V_E is *not injective*, fix a k-basis $\{v_i : i \in I\}$ on V. For each i, let $\pi_i \in E$ be defined by $v_j \pi_i = \delta_{ij} v_i$, where δ_{ij} are the Kronecker deltas. Let $\mathfrak{A} = \sum_i \pi_i E$ (a right ideal in E), and let $f : \mathfrak{A} \to V_E$ be defined by:

$$f\Big(\sum_i \pi_i e_i\Big) = \sum_i v_i e_i \in V,$$

where $e_i \in E$ are almost all zero. To show that f is well-defined, we have to prove that

$$\pi_{i_1} e_{i_1} + \cdots + \pi_{i_n} e_{i_n} = 0 \in E \implies v_{i_1} e_{i_1} + \cdots + v_{i_n} e_{i_n} = 0 \in V.$$

This follows by applying $\pi_{i_1} e_{i_1} + \cdots + \pi_{i_n} e_{i_n}$ to $v_{i_1} + \cdots + v_{i_n}$. If V_E is injective, there would exist a vector $v \in V$ such that $v_i = f(\pi_i) = v\pi_i$ for every $i \in I$. But if $v = \alpha_{i_1} v_{i_1} + \cdots + \alpha_{i_n} v_{i_n}$, then for any $i \notin \{i_1, \ldots, i_n\}$, we have $v\pi_i = 0$, a contradiction. This shows that V_E is not injective, as desired.

Pursuing these ideas a little further, we also get the following interesting information.

(3.74B) Example. *The von Neumann regular ring E constructed above is left self-injective, but not right self-injective*! To see this, we keep the notations introduced before, and *fix an index $i \in I$*. We have a natural E-epimorphism $g : E \to V_E$ defined by $g(e) = v_i e$ ($\forall e \in E$). It is easy to see that $g' : V_E \to E$ sending $v_i e$ to $\pi_i e$ ($\forall e \in E$) is a well-defined E-homomorphism splitting g. Therefore, V_E is isomorphic to the direct summand $\pi_i E$ of E_E; in particular, V_E is projective. However, we knew that V_E is not injective. It follows from (3.4)(1) that E_E is also *not* injective.

It remains to show that ${}_E E$ is injective. This can be deduced from a judicious use of the Injective Producing Lemma (3.5). In fact, let us think of ${}_E E$ as $\mathrm{Hom}_k({}_k V_E, {}_k V)$, where k-homomorphisms are written on the right. Here, ${}_k V$ is surely an injective k-module since k is a division ring, and, as we have shown above, V_E is a projective E-module. Therefore, by (3.5) (or more precisely the dual version of (3.5) with "left" and "right" interchanged), we can conclude that $\mathrm{Hom}_k({}_k V_E, {}_k V) = {}_E E$ is an injective left E-module. Therefore, E is a left self-injective ring, as claimed.[26]

For those readers who prefer to work with rings in more concrete terms, we observe that, using the basis $\{e_i : i \in I\}$ on V and following the usual procedures of linear algebra, we can express E as the ring of "row-finite" matrices over k whose rows and columns are indexed by the set I indexing the given basis of V.

To conclude this subsection, let us present a nice characterization theorem for right V-rings, due to Villamajor. The condition (3) below involves the notion of the

[26]After showing that E is left self-injective, one can actually infer that E is *not* right self-injective, since it will be shown (in (6.49)) that a left and right self-injective ring must be Dedekind-finite, and the ring E in question here is not.

radical of a module M_R: by definition, $\text{rad}(M)$ is the intersection of all maximal submodules of M; if there are no maximal submodules, $\text{rad}(M)$ is defined to be M itself. This is a straightforward generalization of the notion of the Jacobson radical of a ring since, upon viewing R as a right module over itself, $\text{rad}\, R_R$ is just the usual Jacobson radical of the ring R.

(3.75) Theorem. *For any ring R, the following are equivalent:*

(1) *R is a right V-ring;*
(2) *any right ideal $A \subsetneq R$ is an intersection of maximal right ideals;*
(3) *for any right R-module M,* $\text{rad}(M) = 0$.

Proof. (3) $\Longrightarrow$ (2) follows by applying (3) to the module $M = R/A$.

(2) $\Longrightarrow$ (1). We shall show that any simple right R-module S is injective by applying Baer's Test to S. Thus, consider any homomorphism $f : B \to S$, where $B \subseteq R$ is any right ideal. In order to extend f to R, we may assume that $f \neq 0$. Fix an element $x \notin A := \ker(f)$. By (2), there exists a maximal right ideal $\mathfrak{m} \supseteq A$ not containing x. Since $B/A \cong S$ is simple, we have $\mathfrak{m} \cap B = A$, and clearly, $B + \mathfrak{m} = R$. We can then extend f to $g : R \to S$ by defining $g(b + m) = f(b)$ for any $b \in B$ and any $m \in \mathfrak{m}$.

(1) $\Longrightarrow$ (3). We are supposed to show here that any $x \in M \setminus \{0\}$ is excluded by some maximal submodule. The cyclic module xR certainly has a maximal submodule, so there exists a surjection $h : xR \to S$ for some simple module S. Since S is assumed to be injective, h extends to some homomorphism $h' : M \to S$. Now $\ker(h')$ is a maximal submodule of M excluding x. □

(3.75)′ Corollary. *If R is a right V-ring, then any quotient of R is a Jacobson semisimple ring, and any nonzero right R-module has a maximal submodule.*

In the case of *commutative* rings R, we can combine the above information with (3.73) to make the following sharper statement: *R is a V-ring iff it is a von Neumann regular ring, iff all quotients of R are Jacobson semisimple rings.*

In the general case, however, right V-rings need not be von Neumann regular; also, they need not be left V-rings. Since we only have a passing interest in right V-rings, we shall not digress to give the necessary examples here.

§3I. Matlis' Theory

In this subsection, we return to the theme of (3.62) and present Matlis' theory of indecomposable injectives over a commutative noetherian ring R. Recall from (3.62) that the set of indecomposable injectives over R is given (up to isomorphisms) by

$$\{E(R/\mathfrak{p}) : \mathfrak{p} \in \text{Spec}\, R\}.$$

For each $\mathfrak{p} \in \operatorname{Spec} R$, we would like to describe the injective hull $E(R/\mathfrak{p})$, and compute its endomorphism ring. The first step in this program is to pass to the localization $R_\mathfrak{p}$, which is a commutative noetherian local ring with unique maximal ideal $\mathfrak{p}R_\mathfrak{p}$. The advantage of passing to the localization is that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a field, in fact the quotient field of the integral domain $R/\mathfrak{p}$.

Let us first work in a local setting. If R is (any) local ring with unique maximal ideal $\mathfrak{m}$, we shall often refer to R by writing $(R, \mathfrak{m})$. The injective hull $E({}_R(R/\mathfrak{m}))$ of the unique simple left R-module ${}_R(R/\mathfrak{m})$ is an important object in local algebra. It is convenient to refer to it by some name, so let us call it the "standard module" of the local ring for now. (In more formal terminology, to be introduced later in §19, this module is called the (left) *minimal injective cogenerator* of R. We hesitate to use this term here since we have not yet defined the notion of a cogenerator.) Some concrete examples of standard modules over finite-dimensional local algebras have already appeared in (3.69) and (3.70).

The following Proposition establishes a nice "double annihilator" result for the left ideals in a local ring R with respect to the standard module. For the record, we note that neither commutativity nor noetherianness is needed for this result.

(3.76) Proposition. *Let $(R, \mathfrak{m})$ be a local ring, and $E = E(R/\mathfrak{m})$ be the (left) standard module of R. Then for any left ideal $\mathfrak{A} \subseteq R$, we have $\operatorname{ann}^R(\operatorname{ann}^E(\mathfrak{A})) = \mathfrak{A}$.*

Proof. Let $A = \operatorname{ann}^E(\mathfrak{A}) = \{e \in E : \mathfrak{A}e = 0\}$. Of course we have $\mathfrak{A} \subseteq \operatorname{ann}^R(A)$. To prove the equality, assume for the moment that there exists $r \in R \backslash \mathfrak{A}$ with $rA = 0$. The cyclic nonzero submodule $R \cdot \bar{r}$ in $R/\mathfrak{A}$ certainly has a maximal submodule, so there is a surjection, say, f, from it to the copy of ${}_R(R/\mathfrak{m})$ in E. By the injectivity of E, f extends to an R-homomorphism $g : R/\mathfrak{A} \to E$. Let $a := g(\bar{1}) \in A$. Then $f(\bar{r}) = rg(\bar{1}) = ra = 0$, a contradiction. □

(3.76)′ Corollary. *In the notation of the above Proposition, the standard module $E = E(R/\mathfrak{m})$ is a faithful R-module.*

Proof. Let $\mathfrak{A} = 0$ in (3.76). Then $\operatorname{ann}^E(\mathfrak{A}) = E$, so it follows that $\operatorname{ann}^R(E) = 0$. □

In §19, it will be shown that any "cogenerator" module is always faithful; (3.76)′ is a special case of this.

Let us now return to the *commutative* setting, and try to explain how we may "localize" the study of the injective indecomposables. This step also does not require R to be noetherian.

(3.77) Proposition. *Let $\mathfrak{p} \in \operatorname{Spec} R$, and let $R_\mathfrak{p}$ denote the localization of R at $\mathfrak{p}$. Then $E(R/\mathfrak{p})$ has a natural structure as an $R_\mathfrak{p}$-module, and as such, it is isomorphic to the standard module of the local ring $R_\mathfrak{p}$.*

Proof. We first show that any $r \in R \backslash \mathfrak{p}$ acts as an automorphism on $E := E(R/\mathfrak{p})$. If this is true, then E becomes an $R_\mathfrak{p}$-module by letting $r^{-1}s$ act as the composite of the action of s followed by the inverse of the action of r. Say $rx = 0$, where $x \in E$. If $x \neq 0$, then in view of $R/\mathfrak{p} \subseteq_e E$, there exists $s \in R$ such that $0 \neq sx \in R/\mathfrak{p} \subseteq E$. Then $r(sx) = s(rx) = 0$ implies $sx = 0$ (since r certainly acts as an R-monomorphism on $R/\mathfrak{p}$), a contradiction. Next consider $r \cdot E$. By the above, $r \cdot E \cong E$ is injective as an R-module, so by the indecomposability of E, we must have $r \cdot E = E$, as desired.

Viewing E now as an $R_\mathfrak{p}$-module, *we claim that it is also injective.* In fact, consider any $R_\mathfrak{p}$-module $X \supseteq E$. Then $X = E \oplus Y$ for some R-module Y. This Y is automatically an $R_\mathfrak{p}$-module. For, if $y \in Y$ and $r \in R \backslash \mathfrak{p}$, we have $r^{-1}y = e + y'$ for some $e \in E$ and $y' \in Y$. Then $y = re + ry'$ implies that $y = ry'$, so $r^{-1}y = y' \in Y$. This proves the injectivity of E as an $R_\mathfrak{p}$-module. Now consider the element $\overline{1} \in R/\mathfrak{p} \subseteq E$. Since $\mathfrak{p}R_\mathfrak{p}$ kills $\overline{1}$, $R_\mathfrak{p} \cdot \overline{1}$ in E is clearly isomorphic to $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. From $R/\mathfrak{p} \subseteq_e E$ (as R-modules), we have $R_\mathfrak{p} \cdot \overline{1} \subseteq_e E$ as $R_\mathfrak{p}$-modules. Therefore, we have $E \cong E(R_\mathfrak{p} \cdot \overline{1})$, which is the standard module of the local ring $R_\mathfrak{p}$. □

Before we go over to a local setting, we shall prove another useful result, using an argument due essentially to Emmy Noether.

(3.78) Theorem. *Let R be a commutative noetherian ring, and M be a uniform left R-module. Let $\mathfrak{p} \in \operatorname{Spec} R$ be the unique associated prime of M (see* (3.58), (3.59)*). Then, for any $c \in M$, we have $\mathfrak{p}^n c = 0$ for some positive integer n.*

Proof. We may assume that $c \neq 0$, so that $\mathfrak{q} := \operatorname{ann}^R(c) \neq R$. Then $R/\mathfrak{q} \cong R \cdot c$ is also uniform, so the ideal $\mathfrak{q}$ is meet-irreducible. *We claim that every $b \in \mathfrak{p}$ has a power in* $\mathfrak{q}$. If so, then, since $\mathfrak{p}$ is a finitely generated ideal, we have $\mathfrak{p}^n \subseteq \mathfrak{q}$ for some n, and so $\mathfrak{p}^n c = 0$ as desired. To prove our claim, note that

$$\operatorname{Ass}(R/\mathfrak{q}) = \operatorname{Ass}(R \cdot c) \subseteq \operatorname{Ass}(M) = \{\mathfrak{p}\}$$

implies that $\operatorname{Ass}(R/\mathfrak{q}) = \{\mathfrak{p}\}$ (again in view of (3.58)). Therefore, by (3.56), $\mathfrak{p} = \operatorname{ann}^R(\overline{d})$ for some nonzero element $\overline{d}$ in the R-module $R/\mathfrak{q}$. In particular, for any $b \in \mathfrak{p}$, we have $bd \in \mathfrak{q}$. *Assume that $b^n \notin \mathfrak{q}$ for all n.* The ideals $\mathfrak{q} : b^n = \{r \in R : rb^n \in \mathfrak{q}\}$ form an ascending chain, so $\mathfrak{q} : b^n = \mathfrak{q} : b^{n+1}$ for some n. We shall now establish the following equation:

$$\mathfrak{q} = (\mathfrak{q} + Rb^n) \cap (\mathfrak{q} + Rd). \tag{3.79}$$

It suffices to prove the inclusion "$\supseteq$". Let x be any element in the RHS, say

$$x = q_1 + y_1 b^n = q_2 + y_2 d,$$

where $q_i \in \mathfrak{q}$, and $y_i \in R$. Then

$$xb = q_1 b + y_1 b^{n+1} = q_2 b + y_2 db \in \mathfrak{q}\,,$$

since $db \in \mathfrak{q}$. Therefore, $y_1 b^{n+1} \in \mathfrak{q}$. But then $y_1 b^n$ is already in $\mathfrak{q}$, and we get $x \in \mathfrak{q}$. This establishes (3.79), which expresses $\mathfrak{q}$ as the intersection of two bigger ideals (since b^n, $d \notin \mathfrak{q}$). This contradicts the meet-irreducibility of $\mathfrak{q}$. □

(3.80) Historical Note. The above argument was essentially the one used by Emmy Noether to show that, *in a commutative noetherian ring R, any meet-irreducible ideal $\mathfrak{q}$ in R is $\mathfrak{p}$-primary with respect to the radical $\mathfrak{p}$ of $\mathfrak{q}$*. Noether used this result to derive the Lasker-Noether Decomposition Theorem for any commutative ring satisfying what she called the "finiteness condition" ("Endlichkeitsbedingung") for ideals: the time was 1921.[27]

A typical indecomposable injective $E = E(R/\mathfrak{p})$ ($\mathfrak{p}$ prime) is always uniform (by (3.52)) with $\text{Ass}(E) = \{\mathfrak{p}\}$, so we can apply (3.78) to E. Letting $E_n = \text{ann}^E(\mathfrak{p}^n)$ $(n \geq 0)$, we have a filtration

(3.81) $$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n \subseteq \cdots,$$

with $\bigcup_{n \geq 0} E_n = E$ according to (3.78). Note also that, upon viewing E as an $R_\mathfrak{p}$-module, we have $E_n = \text{ann}^E((\mathfrak{p}R_\mathfrak{p})^n)$. In particular, the E_n 's are $R_\mathfrak{p}$-submodules of E. We shall now pass to the localization $R_\mathfrak{p}$, so that we can work with E as the standard module of $R_\mathfrak{p}$. It is easy to see that, for any $R_\mathfrak{p}$-modules A, B, $\text{Hom}_{R_\mathfrak{p}}(A, B)$ is the same as $\text{Hom}_R(A, B)$. Therefore, for the purposes of computing the endomorphism ring of E, the localization from R to $R_\mathfrak{p}$ is also harmless. Having said the above, we shall now assume R is a (commutative, noetherian) *local* ring, with maximal ideal $\mathfrak{m}$ (which is our former $\mathfrak{p}R_\mathfrak{p}$). We have $E_n = \text{ann}^E(\mathfrak{m}^n)$, and, thanks to (3.76), we have now also $\mathfrak{m}^n = \text{ann}^R(E_n)$. (If we insist on working in the original ring, the annihilator of the E_n 's would be the "contractions" of the $\mathfrak{m}^n$'s, which are called the "symbolic powers" of $\mathfrak{p}$.)

Working in the local setting, we write K for the residue class field $R/\mathfrak{m}$. Objects like E_n/E_{n-1} and $\mathfrak{m}^{n-1}/\mathfrak{m}^n$ are now K-vector spaces, and the latter is finite-dimensional over K since $\mathfrak{m}$ is a finitely generated ideal. Note that $E_1 = E_1/E_0$ is 1-dimensional over K (since E is a uniform R-module). This means that $E_1 = R/\mathfrak{m}$ in E. Using the standard module E, we can form, for any R-module M, the E-dual $M^* := \text{Hom}_R(M, E)$, which is also an R-module. (Usually, the $*$ denotes the R-dual; here we have to use it to denote the E-dual.) Since E is

[27] Should any reader have any trouble ever in recalling in what year Noether did this famous work, just count the number of letters in the not-so-long-in-German word "Endlichkeitsbedingung"! The Lasker-Noether Theorem was originally named after the chess master and mathematician Emanuel Lasker and Emmy's father, Max Noether. The version of this theorem for a commutative noetherian ring should perhaps be more appropriately called the "Lasker-Noether-Noether Theorem". Incidentally, Noether herself never knew that the rings satisfying her *Endlichkeitsbedingung* were to be christened *Noetherian* rings. This term was coined by Claude Chevalley only in 1943 (in his paper on local rings); Emmy Noether died in 1935, at the age of 53, shortly after escaping from the Nazis and emigrating to the United States.

injective, dualizing R-modules is an exact (contravariant) functor from R-modules to R-modules. With these observations, we are now in a position to compute the filtration factors in (3.81) and the duals of the E_n 's (in the local case).

(3.82) Theorem. (1) $E_n/E_{n-1} \cong (\mathfrak{m}^{n-1}/\mathfrak{m}^n)^* \cong \mathfrak{m}^{n-1}/\mathfrak{m}^n$ *as* R*-modules (or equivalently, as* K*-vector spaces).* (2) $E_n^* \cong R/\mathfrak{m}^n$ *as* R*-modules.*

Proof. (1) Taking an idea from the proof of (3.64)(2), consider the exact sequence

$$0 \to \mathfrak{m}^{n-1}/\mathfrak{m}^n \to R/\mathfrak{m}^n \to R/\mathfrak{m}^{n-1} \to 0,$$

which induces an exact dual sequence:

$$0 \to (R/\mathfrak{m}^{n-1})^* \to (R/\mathfrak{m}^n)^* \to (\mathfrak{m}^{n-1}/\mathfrak{m}^n)^* \to 0. \tag{3.83}$$

Now $(R/\mathfrak{m}^i)^* = \operatorname{Hom}_R(R/\mathfrak{m}^i,\, E)$ can be identified with E_i via the map $\varphi \mapsto \varphi(\overline{1})$. Making this identification for $i = n$ and $i = n - 1$ in (3.83), we see that $(\mathfrak{m}^{n-1}/\mathfrak{m}^n)^* \cong E_n/E_{n-1}$. This proves the first isomorphism in (1). Next, note that $(\mathfrak{m}^{n-1}/\mathfrak{m}^n)^*$ is just $\operatorname{Hom}_R(\mathfrak{m}^{n-1}/\mathfrak{m}^n,\, E_1)$. Since $E_1 \cong K$, this is the K-dual of the finite-dimensional K-vector space $\mathfrak{m}^{n-1}/\mathfrak{m}^n$, which is K-isomorphic to $\mathfrak{m}^{n-1}/\mathfrak{m}^n$ (albeit not naturally).

For (2), consider the natural R-homomorphism

$$\alpha_n : R/\mathfrak{m}^n \to E_n^* = \operatorname{Hom}_R(E_n,\, E)$$

given by sending $\overline{r}$ to the multiplication by r on E_n. If $rE_n = 0$, then $r \in \operatorname{ann}^R(E_n) = \mathfrak{m}^n$ by (3.76). Therefore, α_n is a monomorphism; we need to prove that it is an *isomorphism*. We proceed by induction on n, the case $n = 0$ being clear. Assuming that α_{n-1} is an isomorphism, we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}^{n-1}/\mathfrak{m}^n & \longrightarrow & R/\mathfrak{m}^n & \longrightarrow & R/\mathfrak{m}^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \\
0 & \longrightarrow & (E_n/E_{n-1})^* & \longrightarrow & E_n^* & \longrightarrow & E_{n-1}^* & \longrightarrow & 0
\end{array}$$

where β_n is taken to be the restriction of α_n. Since α_n is a monomorphism, so is β_n. But by (1), the domain and range of β_n have the same (finite) K-dimension. Therefore, β_n must be an isomorphism. Since α_{n-1} is also an isomorphism, it follows from a diagram chase (simple case of the 5-Lemma) that α_n is an isomorphism, as desired. ☐

We now come to the main result of this subsection, which is a computation of the endomorphism ring of the standard module E. The key example to keep in mind here is the Prüfer group C_{p^∞} (the direct limit of the $\mathbb{Z}/p^n\mathbb{Z}$'s): its endomorphism ring is well-known to be the ring of p-adic integers, which is the inverse limit of the $\mathbb{Z}/p^n\mathbb{Z}$'s. Matlis' result below is a direct generalization of this: note that R continues to denote a *local* commutative noetherian ring (while the nonlocal case can be treated by first applying a localization).

(3.84) Theorem. *The endomorphism ring* $\text{End}_R(E)$ *is isomorphic to* $\tilde{R}$*, the* $\mathfrak{m}$*-adic completion*[28] *of* R*. (By definition,* $\tilde{R}$ *is the inverse limit* $\varprojlim R/\mathfrak{m}^n$*.)*

Proof. We have observed before that $E = \bigcup_{n\geq 0} E_n$. Taking full advantage of the notation of direct and inverse limits, we have the following sequence of ring isomorphisms:

$$\begin{aligned} \text{Hom}_R(E, E) &= \text{Hom}_R(\varinjlim E_n, E) \\ &\cong \varprojlim \text{Hom}_R(E_n, E) \\ &\cong \varprojlim R/\mathfrak{m}^n \\ &\cong \tilde{R}, \end{aligned}$$

as desired. □

Incidentally, this theorem implies that the endomorphism ring for the standard module E is also commutative. This fact did not seem to be obvious at the outset.

Much more can be said about the module E and the "duality" relationship of its submodules with the ideals of R, especially in the *complete* local case. We shall come back to this in §19 when we have the proper terminology with which to discuss duality theory. Here, we'll just content ourselves with a few corollaries of the results so far obtained.

(3.85) Corollary. *Let* R *be a commutative noetherian (but not necessarily local) ring,* $\mathfrak{m}$ *be a maximal ideal of* R*, and* $E = E(R/\mathfrak{m})$*. Then:*

(1) *Each* R*-module* E_n *(annihilator of* $\mathfrak{m}^n$*) has finite length.*
(2) *Each f.g.* R*-submodule* $M \subseteq E$ *has finite length.*
(3) E *is countably generated.*

Proof. To prove (1), note that if we go to the localization $R_\mathfrak{m}$, the residue class ring does not change: $R/\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. From what we have proved in (3.82), each E_i/E_{i-1} is finite-dimensional over this field, and therefore it has finite length over R. Since $E_0 = 0$, it follows by induction on n that E_n has finite length. In particular, E_n is f.g., so E itself is countably generated. Finally, (2) also follows from (1) since each f.g. submodule M is contained in some E_n. □

Using this, we can retrieve our earlier result (3.64) for commutative artinian rings as the "Krull dimension zero" case of the Matlis theory. In fact we have the following 2-way statement, also due to Matlis.

[28] In other words, the completion of R with respect to the filtration given by the powers of $\mathfrak{m}$.

(3.86) Corollary. *Let R be a commutative noetherian ring. Then R is an artinian ring iff every injective indecomposable module over R is f.g.*

Proof. First assume that R is artinian. Represent a typical injective indecomposable module in the form $E = E(R/\mathfrak{p})$, where $\mathfrak{p} \in \text{Spec } R$. Since R is artinian, the prime ideal $\mathfrak{p}$ must be maximal, so we can try to apply (3.85). There exists some n such that $\mathfrak{p}^n = \mathfrak{p}^{n+1} = \cdots$. Therefore, we have $E_n = E_{n+1} = \cdots$ in E. Since $E = \bigcup_{i \geq 0} E_i$, it follows that $E = E_n$, and by (3.85), this module is f.g.

Conversely, assume that every injective indecomposable (i.e., every $E = E(R/\mathfrak{p})$ where $\mathfrak{p} \in \text{Spec } R$) is f.g. Since R is noetherian, it follows that $E_1 \subseteq E$ is f.g. But E_1 is just $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ which is the quotient field of $R/\mathfrak{p}$. It is easy to see that this quotient field is a f.g. module over $R/\mathfrak{p}$ only if $R/\mathfrak{p}$ is already a field. Therefore, it follows that every prime ideal in R is maximal. For a (commutative) noetherian ring R, this means that R is actually artinian. $\square$

Notice that the argument for the "if" part above also goes through if we only assume every $E(R/\mathfrak{p})$ to be *noetherian.* Therefore, the Corollary remains good if we replace the words "finitely generated" there by "noetherian".

Another remark: as we saw in the proof of (3.64)(3), once we know the "only if" statement in (3.86), the finite generation of the injective hull of any f.g. module over a commutative artinian ring follows as a result.

§3J. Some Computations of Injective Hulls

All rings will be assumed to be commutative in this subsection, and modules will be taken to be left modules.

If R is not an artinian ring, the injective hull of a f.g. R-module is often *not* f.g., so it is generally not easy to construct (or "get our hands on") such an injective hull. Of course, we have the example that $E(R)$ is the quotient field of R when R is a domain, and we have the examples of injective hulls of cyclic modules over a PID. In this subsection, we would like to offer some explicit computations of $E(V)$ where ${}_RV$ is cyclic but not free, and R is noetherian but not a PID. In fact, R will be of the form $k[x_1, \ldots, x_r]/J$ where J is an ideal contained in $(x_1, \ldots, x_r)$, and V will be the R-module k on which all x_i 's act trivially. (This action of R/J on k will be fixed throughout.) My thanks are due to D. Eisenbud and C. Huneke who encouraged me to include these interesting examples here, and gave me generous help toward the write-up of the following exposition. In presenting this material, I have also benefited from consulting a paper of D. G. Northcott [74].

We begin by recalling the following well known result in commutative algebra.

(3.87) Artin-Rees Lemma. *For any two ideals I, J in a noetherian ring A, there exists an integer s such that for any $d \geq s$, $I \cap J^d = J^{d-s}(I \cap J^s) \subseteq J^{d-s}I$.*

A proof of this lemma can be found on p. 107 of Atiyah-Macdonald [69]. Using this lemma, we obtain the following basic result on injective modules over a noetherian ring.

(3.88) Proposition. *Let A be a noetherian ring, and M be an injective A-module. For any ideal $J \subseteq A$,*

$$N := \{x \in M : J^n x = 0 \ \ for\ some\ \ n \geq 0\} = \bigcup_{n \geq 0} \operatorname{ann}^M(J^n)$$

is an injective A-submodule of M.

Proof. (The proof relies partly on Exercise 28, so the reading of this proof should be preceded by a quick consultation of that exercise.) Let $M_n = \operatorname{ann}^M(J^n)$, which is an A-submodule of M. To check that $N = \bigcup_{n \geq 0} M_n$ is injective, we apply Baer's Test (3.7). Let $\lambda : I \to N$ be an A-homomorphism, where $I \subseteq A$ is any ideal. Since A is noetherian, I is f.g. Therefore, $\lambda(I) \subseteq M_n$ for a suitable integer n. By (3.87) above, for a sufficiently large integer $d \geq n$, we have $I \cap J^d \subseteq J^n I$. This gives

$$\lambda(I \cap J^d) \subseteq \lambda(J^n I) \subseteq J^n \lambda(I) \subseteq J^n M_n = 0,$$

so λ induces an A-homomorphism

$$\lambda_1 : I/(I \cap J^d) \longrightarrow M_n \subseteq M_d\,,$$

which in turn induces $\lambda_2 : (I + J^d)/J^d \to M_d$. Now, by Exercise 28(1), M_d is an injective A/J^d-module. Therefore, λ_2 extends to an A/J^d-homomorphism $A/J^d \to M_d$. Composing this with $A \to A/J^d$, we get an A-homomorphism $A \to M_d \subseteq N$ which extends λ. This completes Baer's Test, thus proving the injectivity of N. □

(3.89) Remark. There is one special case of the Proposition that is particularly worth mentioning. Apply it to $M = E_A(A/J)$, where $J \subset A$ is a prime ideal. Let N be defined as above. Since $N \supseteq M_1 \neq 0$ and M is indecomposable, the injectivity of N implies that $M = N = \bigcup_{n \geq 0} M_n$. This conclusion has been obtained before by a different method in §3I.

For the rest of this subsection, let $A = k[x_1, \ldots, x_r]$, where k is a field, and let $\mathfrak{m}$ be the maximal ideal $(x_1, \ldots, x_r)$ in A. We shall first compute the injective hull $E_A(k)$. To carry out this computation, we start with $M := \hat{A}$ (the k-dual of A). By (3.6C), this is an *injective* A-module. Using the k-decomposition

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots,$$

where A_n denotes the space of homogeneous polynomials of degree n, we have $M = \prod_{n \geq 0} \hat{A}_n$. (Note that, here, the k-dual $\hat{A}_n$ is identified with the space of functionals on A that vanish on A_i for all $i \neq n$.) Now by (3.88), $N := \bigcup_{n \geq 0} \operatorname{ann}^M(\mathfrak{m}^n)$ is an injective A-module. Since $\mathfrak{m}^n = A_n \oplus A_{n+1} \oplus \cdots$, we

have

$$\begin{aligned}\mathrm{ann}^M(\mathfrak{m}^n) &= \{f \in \hat{A} : f(\mathfrak{m}^n) = 0\} \\ &= \hat{A}_0 \oplus \hat{A}_1 \oplus \cdots \oplus \hat{A}_{n-1},\end{aligned}$$

so $N = \bigoplus_{n \geq 0} \hat{A}_n$. It is easy to see that $\hat{A}_0$ is an A-submodule of M isomorphic to ${}_A k$, and that it is essential in N. Therefore, we have arrived at the injective hull $E_A(k)$: it is given by the A-module N. This basic observation leads to several nice computations of injective hulls, which we put together in the following result.

(3.90) Theorem. *Let J be an ideal of A contained in the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_r)$, and let $R = A/J$. Then:*

(1) $E_R(k) = \mathrm{ann}^N J = \{f \in \hat{A} : f(J + \mathfrak{m}^n) = 0$ *for some* $n\}$;
(2) *as an A-module, $E_R(k)$ has injective hull N;*
(3) *if R is a local artinian ring, then $E_R(k) = \hat{R}$, and $E_A(\hat{R}) = N$. If R is a Frobenius k-algebra, then in fact $E_A(R) = N$.*

Proof. (1) Since N is injective over A, Exercise 28(1) implies that $P := \mathrm{ann}^N J$ is injective over R. Now $k \cong \hat{A}_0 \subseteq_e P$ (as A-modules, and hence as R-modules), so $P = E_R(k)$. (Of course, we can also think of P as the R-module $\mathrm{Hom}_A(R, N)$. Note that, in the special case when $J = 0$, we get back $E_A(k) = N$.)

(2) Since $\mathrm{ann}^N J$ lies between $\hat{A}_0$ and N, its A-injective hull is of course N.

(3) Assuming that R is a local artinian ring, we have $\mathfrak{m}^n \subseteq J$ for some n, so (1) simplifies to

$$E_R(k) = \{f \in \hat{A} : f(J) = 0\} = \hat{R}.$$

(This can also be deduced as a special case of (3.41).) Thus, the first conclusion in (3) follows from (2). This also yields the second conclusion since, in the case when R is a Frobenius algebra, $R \cong \hat{R}$ as R-modules, and hence as A-modules. □

Next we offer some other useful descriptions for the two A-modules M and N, which can actually be traced back to the early work of F. S. Macaulay. Let us consider the following polynomial ring and formal power series ring

$$T := k[x_1^{-1}, \ldots, x_r^{-1}] \quad \text{and} \quad S := k[[x_1^{-1}, \ldots, x_r^{-1}]]$$

in the variables $x_1^{-1}, \ldots, x_r^{-1}$. To properly understand these two rings (and their relationship to A), let

$$D := \{x_1^{d_1} \cdots x_r^{d_r} : d_i \geq 0\};$$

this is the set of monomials in the x_i's, which forms a k-basis for A. Now form D^{-1}, whose elements are known as "inverse monomials"; this set forms a k-basis for T. In the literature, the elements of T are called "inverse polynomials" in the x_i's. Similarly, the elements of S are called "inverse formal power series" in the x_i's.

We shall now define an A-module structure on T (and subsequently on S). For $\alpha, \beta \in D$, let $\alpha * \beta^{-1}$ be $\alpha\beta^{-1}$ if $\alpha\beta^{-1} \in D^{-1}$, and zero otherwise. Strictly speaking, it is necessary to check that this indeed gives an A-module structure on T; we leave this to the reader. (In any case, Exercise 51 greatly clarifies this point.[29]) Notice that essentially the same construction defines an A-module structure on $S = k[[x_1^{-1}, \dots, x_r^{-1}]]$: although the elements in S are formally infinite linear combinations of the inverse monomials, the A-action on T can be *formally* extended to S so that S becomes an A-module containing the A-submodule T.

(3.91) Proposition.

(1) *In the category of A-modules, we have $E_A(k) = N \cong T$, and $M \cong S$.*
(2) *In case k has characteristic 0, N (resp. M) is also isomorphic to the A-module $T' := k[y_1, \dots, y_r]$ (resp. $S' := k[[y_1, \dots, y_r]]$), where we let each x_i act on T' (resp. S') as the partial differential operator $\partial/\partial y_i$ (noting that these operators are k-linear, and that they commute on both T' and S').*

Proof. (1) First let us verify that $N \cong T$. We construct a map $\varepsilon : T \to N \subseteq M$ by taking

$$\varepsilon(\beta^{-1})(\gamma) = \delta_{\beta,\gamma} \quad (\text{for } \beta, \gamma \in D),$$

where "δ" means the Kronecker deltas. Clearly, ε is a k-linear isomorphism from T to N, so it only remains to show that ε is an A-homomorphism, that is, to show that

$$\varepsilon(\alpha * \beta^{-1})(\gamma) = \varepsilon(\beta^{-1})(\alpha\gamma) \quad \text{for any } \alpha, \beta, \gamma \in D.$$

We check this in the following two cases:

Case 1. $\alpha\beta^{-1} \notin D^{-1}$. Here, the LHS above is zero, and so is the RHS since $\beta \neq \alpha\gamma$.

Case 2. $\alpha\beta^{-1} \in D^{-1}$. Here, the LHS is $\delta_{\alpha^{-1}\beta,\gamma}$, and the RHS is $\delta_{\beta,\alpha\gamma}$, and these are clearly equal.

Clearly, the isomorphism ε can also be extended formally to give an A-isomorphism from S to M. (The inverse of this isomorphism $\varepsilon^{-1} : M \to S$ can be described as follows: $\varepsilon^{-1}(f) = \sum_{\beta \in D} f(\beta)\beta^{-1}$, for every k-linear functional $f \in M = \hat{A}$.)

(2) We have a k-isomorphism $\varphi : T \to T'$ defined by

$$\varphi(x_1^{-d_1} \cdots x_r^{-d_r}) = \frac{y_1^{d_1} \cdots y_r^{d_r}}{d_1! \cdots d_r!} \quad (\text{for all } d_1, \dots, d_r \geq 0).$$

[29] In that exercise, T is portrayed as a certain subquotient of the injective A-module $k(x_1, \dots, x_r)$.

A routine calculation shows that this is an isomorphism of A-modules. The same construction gives an A-module isomorphism from S to S'. □

In practice, it is convenient to think of ε in the proof of (3.91) as an identification map, so that $S = k[[x_1^{-1}, \ldots, x_r^{-1}]]$ can be identified with M (the space of all functionals on A), and $T = k[x_1^{-1}, \ldots, x_r^{-1}]$ can be identified with N (the space of functionals on A that vanish on sufficiently high powers of $\mathfrak{m}$). In particular, we can then think of $D^{-1} \subseteq T$ as the "dual basis" of $D \subseteq A$, although this dual basis generates only N, and not M.

Given this convenient viewpoint, let us use (3.90) to compute a couple of examples of $E_R(k)$ for quotient rings $R = A/J$, where $J \subseteq (x_1, \ldots, x_r)$. For comparison purposes, let us take two earlier examples of $R = A/J$ ((3.69) and (3.70)) for which we have already computed $E_R(k)$ by ad hoc methods. We'll show how we can "recompute" these two injective hulls by using the more efficient machinery of this subsection. In the following, we take $r = 2$, and write $A = k[x, y]$. In both examples, R will be a local artinian ring, so by (3.90)(3) (or (3.41)), $E_R(k)$ is given by $\hat{R}$. The crux of the matter is to determine $\hat{R}$ as an A-module.

(3.92) Example. Let $J = (x, y)^{n+1} \subseteq A$, where $n \geq 0$. Here $R = A/J$ has a k-basis

$$C = \{x^i y^j : i, j \geq 0, \ \ i + j \leq n\}.$$

Therefore, $\hat{R}$ is the A-submodule in T spanned by the dual basis C^{-1}. For instance, for $n = 3$, these dual bases can be displayed as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & x^{-1} & & y^{-1} & & \\ & x^{-2} & & x^{-1}y^{-1} & & y^{-2} & \\ x^{-3} & & x^{-2}y^{-1} & & x^{-1}y^{-2} & & y^{-3} \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \end{array}$$

Here, the A-module structure on $\hat{R}$ is clear: on its k-basis C^{-1} displayed above, the action of x pushes the inverted monomials in the northeast direction, and the action of y pushes them in the northwest direction. (Of course, if x or y pushes some β^{-1} into "blank space", the corresponding action on β^{-1} is interpreted as zero.) Now in (3.69), we have obtained (for the case of a general n) another model for $\hat{R}$, namely, $\hat{R} = (x, y)^n/(x^{n+1}, y^{n+1})$. An explicit A-isomorphism from one model to the other is provided by multiplication by $x^n y^n$. A bonus conclusion (from (3.90)(1)) is that $E_A((x, y)^n/(x^{n+1}, y^{n+1})) \cong N$.

(3.93) Example. Let $J = (x^2, xy^{n+1}, y^{n+2}) \subseteq A$, where $n \geq 0$. Here $R = A/J$ has a k-basis

$$C := \{1, x, y, xy, y^2, \ldots, xy^n, y^{n+1}\},$$

so $\hat{R}$ is the A-submodule in T spanned by the dual basis C^{-1}. These two bases can be displayed as follows:

$$\begin{array}{cc} x^{-1} & 1 \\ x^{-1}y^{-1} & y^{-1} \\ \vdots & \vdots \\ x^{-1}y^{-n} & y^{-n} \\ & y^{-(n+1)} \end{array} \qquad\qquad \begin{array}{cc} x & 1 \\ xy & y \\ \vdots & \vdots \\ xy^n & y^n \\ & y^{n+1} \end{array}$$

Here, the A-module structure on $\hat{R}$ is simply that x pushes the inverted monomials to the east, and y pushes them to the north. (Again, pushing something into blank space means a zero action.)

Returning to the general notations introduced earlier in this subsection, we shall now conclude §3J with another supplementary remark concerning the A-module of inverted polynomials $T = k[x_1^{-1}, \ldots, x_r^{-1}]$. Since $T \cong N \cong E_A(k)$, T is the standard module of the localization of A at $\mathfrak{m} = (x_1, \ldots, x_r)$, by (3.77). Now by Matlis' Theory, $\text{End}_{A_\mathfrak{m}}(T) = \text{End}_A(T)$ is the completion of $A_\mathfrak{m}$, which is the ring of formal power series

$$P := k[[x_1, \ldots, x_r]].$$

The action of P on T is easy to describe: there is already an A-action on T, under which any inverted polynomial in T is killed by all monomials of a sufficiently high degree in $x_1, \ldots, x_r$. Such an A-action obviously extends to a P-action on T; this is the desired action. According to Exercise 49, *T is also the standard module of the complete local ring $P = k[[x_1, \ldots, x_r]]$.* Thus, for the power series ring P, we have also managed to compute the injective hull $E_P(k)$, where, as usual, the x_i 's act trivially on k.

§3K. Applications to Chain Conditions

Having led the reader through a rather long section on injective modules, we shall now regale him/her with a couple of nice applications of such modules! These applications are made to the study of the ascending/descending chain conditions on a ring and on its subrings.

First we recall a well-known situation. Let $S \subseteq R$ be rings (with the same identity) such that R is a f.g. as a right module over S. Suppose the ring S is right noetherian (resp. artinian). Then R_S is a noetherian (resp. artinian) module, so it follows that the ring R is also right noetherian (resp. artinian). In the case when R is a *commutative* ring, P. Eakin [68] proved the converse of this in the noetherian case: *if R is noetherian, then S must also be noetherian.* The same result was obtained independently by M. Nagata [68]. This has become a standard result in commutative algebra, and is known as the Eakin-Nagata Theorem. In the following, we shall present a noncommutative version of this theorem, due to D. Eisenbud. Our exposition here follows closely Eisenbud [70]; the gist of his

proof is an application of the Bass-Papp Criterion for right noetherian rings in (3.46).

We start with two preparatory lemmata.

(3.94) Lemma. *Let P be an (S, S)-bimodule (over any ring S) such that $P = u_1S + \cdots + u_nS$, where the $u_i \in P$ are such that $su_i = u_is$ for all $s \in S$. For any right S-module M, let $\tilde{M} =$ $\mathrm{Hom}_S(P_S, M_S)$, which is viewed as a right S-module via the left S-action on P. Then, for any right S-module N, $M \subseteq_e N \Rightarrow \tilde{M} \subseteq_e \tilde{N}$.*

Proof. Take any nonzero $f \in \tilde{N}$. Then $(f(u_1), \ldots, f(u_n)) \in N^n \backslash \{0\}$. Assuming $M \subseteq_e N$, we also have $M^n \subseteq_e N^n$ by (3.38), so there exists $s \in S$ such that

$$(f(u_1)s, \ldots, f(u_n)s) \in M^n \backslash \{0\}.$$

Therefore,

$$(fs)(u_i) = f(su_i) = f(u_is) = f(u_i)s \in M$$

are not all zero. Since $P = u_1S + \cdots + u_nS$, we see that $fs \in \tilde{M} \backslash \{0\}$, so it follows that $\tilde{M} \subseteq_e \tilde{N}$. □

(3.95) Lemma. *Let $S \subseteq R$ be two rings such that $R = u_1S + \cdots + u_nS$, where each $u_i \in R$ commutes elementwise with S. For any right S-module M, let $\tilde{M} =$ $\mathrm{Hom}_S(R_S, M_S)$, which is a right R-module via the left R-action on R (in the first variable). Then M_S is injective iff $\tilde{M}_R$ is injective.*

Proof. The "only if" part is a just special case of the "Injective Producing Lemma" (see (3.6B)). To prove the converse, assume now $\tilde{M}_R$ is injective. Taking the injective hull $N := E(M_S)$, our goal is to show that $M = N$. Since $M_S \subseteq_e N_S$, the lemma above (applied with $P = {}_SR_S$) implies that $\tilde{M}_S \subseteq_e \tilde{N}_S$. In particular, we have $\tilde{M}_R \subseteq_e \tilde{N}_R$. Since $\tilde{M}_R$ is injective, this implies that $\tilde{M} = \tilde{N}$. Consider now the following commutative diagram:

$$\text{(3.96)} \qquad \begin{array}{ccc} \tilde{M} = \mathrm{Hom}_S(R, M) & \xrightarrow{\ h\ } & \mathrm{Hom}_S(R, N) = \tilde{N} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M = \mathrm{Hom}_S(S, M) & \xrightarrow{\ k\ } & \mathrm{Hom}_S(S, N) = N \end{array}$$

where f, g are defined by restrictions, and h is an isomorphism as observed above. Since N_S is injective, g is onto. It follows that k is also onto, so $M = N$ is injective over S, as desired. □

To obtain a consequence of (3.95), we need the notion of a semiprimary ring: we say that a ring A is *semiprimary* if its Jacobson radical $\mathrm{rad}\ A$ is nilpotent, and $A/\mathrm{rad}\ A$ is a semisimple ring.

(3.97) Corollary. *Let $S \subseteq R$ be two rings as in* (3.95). *If R is a semisimple (resp. semiprimary) ring, then so is S.*

Proof. First assume R is semisimple. Take any right S-module M. The right R-module $\tilde{M}_R$ is certainly injective, by the semisimplicity of R. In view of the Lemma, M_S must also be injective. Since this holds for all M_S, S is a semisimple ring (by FC–(2.9)). Next, assume that R is semiprimary instead. Let $J = \text{rad } R$, and $J_0 = J \cap S$. Clearly, the pair of rings $S/J_0 \subseteq R/J$ satisfies the same hypothesis as that imposed on $S \subseteq R$. Since R/J is semisimple, the foregoing implies that S/J_0 is also semisimple. On the other hand, the nilpotency of J implies that of J_0. It follows that rad $S = J_0$, and that S is semiprimary. □

Note that the converse of the Corollary is not true, at least in the semisimple case. For instance, if we take S to be a field, and R to be the commutative extension $S[u]$ with the relation $u^2 = 0$, then S is semisimple, but R is obviously not.

We now come to the main result of this subsection.

(3.98) Theorem (Eakin-Nagata-Eisenbud). *Let* $S \subseteq R$ *be two rings as in* (3.95). *Then* R *is right noetherian iff* S *is.*

Proof. We have already discussed the motivating "if" part (which is true without the commuting condition on the u_i 's). For the converse, assume R is right noetherian. To show the same for S, it suffices to show, according to the Bass-Papp Theorem (3.46), that if M_i $(i \in I)$ are injective right S-modules, then so is $M := \bigoplus_{i \in I} M_i$. By Lemma (3.95), we need only show that $\tilde{M}$ is injective as a right R-module. Now

$$\tilde{M} = \text{Hom}_S(R_S, \textstyle\bigoplus_i M_i) \cong \bigoplus_i \text{Hom}_S(R_S, M_i) = \bigoplus_i \tilde{M}_i. \tag{3.99}$$

Here, the isomorphism in the middle is valid since R_S is a f.g. module. (This implies that any S-homomorphism $f : R \to \bigoplus_i M_i$ has image in a finite direct sum of the M_i 's and therefore f corresponds to essentially a finite number of $f_i : R \to M_i$.) Since M_i is injective over S, $\tilde{M}_i$ is injective over R by the Injective Producing Lemma. Over the right noetherian ring R, this implies that $\bigoplus_i \tilde{M}_i$ is injective, so by (3.99), $\tilde{M}$ is injective over R, as desired. □

There is also an artinian analogue of the above result, which we shall next prove. Here we need two basic facts about right artinian rings established in FC–§4. The first of these is that *right artinian rings are semiprimary* (FC–(4.12), (4.14)); the second is the all-important Hopkins-Levitzki Theorem (FC–(4.15)), which says that *a semiprimary ring is right noetherian iff it is right artinian.* With these good tools at our disposal, the proof of the following result is a breeze.

(3.100) Theorem (Eisenbud-Robson). *Let* $S \subseteq R$ *be two rings as in* (3.95). *Then* R *is right artinian iff* S *is.*

Proof. As before, it suffices to prove the "only if" part of the Theorem, so assume R is right artinian. Then R is semiprimary as noted above, and therefore so is S

by (3.97). In view of the Hopkins-Levitzki Theorem, (3.100) simply reduces back to (3.98). □

Note that the results (3.95)–(3.100) apply well, for instance, when R is a ring that is f.g. as a module over its own center. In this case, we get good descent theorems from R down to its center. Throughout the above analysis, we have assumed that the elements u_i in the equation $R = \sum_i u_i S$ commute with all elements of S. The following example shows that this assumption is actually essential for the truth of all of the results.

(3.101) Example (J.-E. Björk). Let $k \subset K$ be a field extension of infinite degree, and let S be the subring $\begin{pmatrix} K & K \\ 0 & k \end{pmatrix}$ of $R = \mathbb{M}_2(K)$. Let $u_1, u_2 \in R$ be the matrix units E_{11} and E_{21}, respectively. Then $u_1 S = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ and $u_2 S = \begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}$, so we have $u_1 S + u_2 S = R$. Here R is semisimple, but S is neither right noetherian nor right artinian (by *FC*–(1.22)), let alone semisimple. The problem here is that the elements u_1, u_2 fail to centralize the subring S in R.

One final thought concerns the hypothesis that the number of generators u_i be finite. This hypothesis turns out to be essential as well. The following rather trivial example shows that there are no results possible if we allow even a *countable* number of (central) generators.

(3.102) Example. Let S be a countable commutative valuation domain that is not a discrete valuation ring, and let R be its quotient field. Then R is also countable and so $R = \sum_{i=1}^{\infty} u_i S$ for $u_1, u_2, \ldots$ ranging over the elements of R. Here R is a field, but S is again neither noetherian nor artinian, let alone semisimple.

In the literature, there are various other versions of theorems relating the noetherian/artinian conditions of a ring to those of its subrings, due to E. Formanek, J.-E. Björk, and others. For some of these versions, we refer the reader to pp. 18-19 in Matsumura [86]. Later in §6 (especially §6F), we shall return to study in more detail the fascinating relationship between various finiteness conditions of a ring and those of its subrings.

Exercises for §3

1. Let R be a commutative domain that is not a field. If a module M_R is both projective and injective, show that $M = 0$.

2. Let R be a right self-injective ring.

   (1) Show that an element of R has a left inverse iff it is not a left zero-divisor in R.

   (2) If R has no nontrivial idempotents, show that R is a local ring, and that

the unique maximal (left, right) ideal $\mathfrak{m}$ of R consists of all left 0-divisors of R.

(3) If R is a domain, show that it must be a division ring.

3. In a ring theory text, the following exercise appeared: "Every simple projective module is injective." Find a counterexample.

4. *True or False*: If I_S is injective and $f : S \to R$ is a ring homomorphism, then $I \otimes_S R$ is injective as a right R-module.

5. Let a, b be elements in a ring R such that $ab = 1$ and $bR \subseteq_e R_R$. Show that $ba = 1$.

The next three exercises collect a few important properties of essential submodules of a module. These exercises will be used freely in the sequel, so the reader will be well advised to do them at this point.

6. (A) If $M_i \subseteq_e E$ for $1 \leq i \leq n$, show that $\bigcap_{i=1}^{n} M_i \subseteq_e E$. Does the same statement hold for an infinite family of essential submodules?

   (B) (Prompted by (3.38).) If $M_i \subseteq_e E_i$ for $i \in \mathbb{N}$, does it follow that $\prod_i M_i \subseteq_e \prod_i E_i$?

7. Let $f : E' \to E$ be a homomorphism of right R-modules. If $M \subseteq_e E$, show that $f^{-1}(M) \subseteq_e E'$. (In particular, if $E' \subseteq E$, then $M \subseteq_e E$ implies that $M \cap E' \subseteq_e E'$.) Use this to give a proof for the first part of Exercise 6.

8. Let U be an R-module that contains a direct sum $\bigoplus_\alpha V_\alpha$, and let $V_\alpha \subseteq_e E_\alpha \subseteq U$ for every α. Show that the sum $\sum_\alpha E_\alpha$ must also be a *direct* sum.

9. Show that a module M_R is semisimple iff no submodule $N \neq M$ is essential in M.

10. (Matlis) Show that a ring R is right hereditary iff the sum of two injective submodules of any right R-module is injective. (**Hint.** Use (3.22).)

11. (Osofsky) Show that a ring R is semisimple iff the intersection of two injective submodules of any right R-module is injective.

12. If R, S are Frobenius k-algebras, show that $R \times S$ and $R \otimes_k S$ are also Frobenius k-algebras. Using this together with (3.15D) and the Wedderburn-Artin Theorem, show that any finite-dimensional semisimple k-algebra is a Frobenius algebra.

13. In the commutative case, generalize (3.12) as follows. If S is any Dedekind domain and $\mathfrak{B} \subseteq S$ is any nonzero ideal, show that $R = S/\mathfrak{B}$ is a self-injective ring. (**Hint.** Note that R is unchanged if we localize S to a suitable semilocal Dedekind domain.)

14. For any finite-dimensional commutative local algebra R over a field k, show that the following are equivalent:

(1) R is a Frobenius k-algebra;
(2) R is self-injective;
(3) R has a unique minimal ideal.

(For much stronger versions of this exercise, see (15.27), and Exer. (16.1) below.)

15. Show that the k-algebra R in (3.15A) is Frobenius for *any* field k.

16. Let K/k be a field extension, and let R be a finite-dimensional k-algebra. Show that R is a Frobenius algebra over k iff $R^K = R \otimes_k K$ is a Frobenius algebra over K. (**Hint.** For the "if" part, use the Noether-Deuring Theorem (*FC*–(19.25)).)

17. Let R be a finite-dimensional algebra over a field k. The interpretation of $(\hat{R})_R$ as $E((R/\text{rad } R)_R)$ in (3.41) shows that the right R-module $\hat{R} = \text{Hom}_k(R, k)$ does not depend on the choice of the ground field k (as long as R is a finite-dimensional k-algebra). Prove the following partial generalization of this fact. Let K be a field extension of k within the center of R (so R is also a finite-dimensional K-algebra). Then for any f.g. left R-module M, $\text{Hom}_k(M, k)$ and $\text{Hom}_K(M, K)$ are isomorphic as right R-modules.

18. Use (3.13)(1) to prove *Prüfer's Theorem*: Any abelian group G of finite exponent n is isomorphic to a direct sum of cyclic groups, necessarily of exponents dividing n. (Your proof should show, in particular, that any element of order n generates a direct summand of G.)

19. Explain how (3.13)(2) would impact upon the proof of the Jordan Canonical Form Theorem.

20. Let S be a submodule of a right module M over a ring R. Show that there exists a submodule $C \subseteq M$ such that $E(M) \cong E(S) \oplus E(C)$.

21. For any noetherian right module M over any ring R, show that $E(M)$ is a finite direct sum of indecomposable injective R-modules. (**Hint.** By "noetherian induction", show that any submodule M' can be expressed as $M_1 \cap \cdots \cap M_n$ in such a way that each M/M_i is uniform. Then apply this to $M' = (0)$.)

22. Show that any injective module is the injective hull of a direct sum of cyclic modules.

23. Let $H = \text{End}(I_R)$ where I is an injective right R-module. For $f, h \in H$, show that $f \in H \cdot h$ iff $\ker(h) \subseteq \ker(f)$.

24. Let I_R be any injective module. If every surjective endomorphism of I is an automorphism, show that every injective endomorphism of I is an automorphism ("hopfian $\Longrightarrow$ cohopfian"). How about the converse?

25. Let M_R be any module, and let $f \in \text{End}_R(E(M))$. If $f|_M$ is an automorphism of M, show that f is an automorphism of $E(M)$.

26. Show that Baer's Criterion for Injectivity (3.7) can be further modified as follows: To check that a module I_R is injective, it is sufficient to show that, for any right ideal $\mathfrak{A} \subseteq_e R_R$, any $f \in \mathrm{Hom}_R(\mathfrak{A}, I)$ can be extended to R.

27. (P. Freyd) Give a direct proof for the validity of the modified Baer's Criterion in the last exercise by using the fact that a module I_R is injective iff it has no proper essential extensions.

28. (1) For an R-module M_R and an ideal $J \subseteq R$, let $P = \{m \in M : mJ = 0\}$. If M is an injective R-module, show that P is an injective R/J-module. (2) Use the above to give a new proof for the fact that any proper quotient of a commutative PID is a self-injective ring.

29. Let M_R be an R-module, and $J \subseteq R$ be an ideal such that $MJ = 0$. By Exercise 28, if M_R is injective, then $M_{R/J}$ is injective. Is the converse also true (cf. (3.11A))? (There is a "quasi-injective" analog for this and the last exercise: see Exer. (6.27A).)

30. Let $S = R[X]$ where X is any commuting set of indeterminates over R. For any essential right ideal $\mathfrak{A} \subseteq_e R_R$, show that $\mathfrak{A}[X] \subseteq_e S_S$. What if $S = R\langle Y\rangle$ where Y is a *noncommuting* set of indeterminates?

31. ("Schröder-Bernstein for Injectives": Bumby [65].) Let A and B be injective R-modules that can be embedded in each other. Show that $A \cong B$.

32. Suppose A, B are R-modules which can be embedded in each other. Show that $E(A) \cong E(B)$, but that we may not have $A \cong B$.

33. Let $J = \mathrm{rad}\, R$, where R is a semilocal ring (i.e., R/J is semisimple). Let V_R be a semisimple module, and $E = E(V)$. Show that there is an R-isomorphism $E/V \cong \mathrm{Hom}_R(J, E)$. (Here, the right R-action on $\mathrm{Hom}_R(J, E)$ comes from the left R-action on J.) If, moreover, $J^2 = 0$, show that $E/V \cong \mathrm{Hom}_R(J, V)$.

34. (Big injective hulls over artinian rings: Rosenberg-Zelinsky [59].) Let $A \subseteq B$ be division rings such that $\dim({}_A B) < \infty$ but $\dim(B_A) = \infty$. (Such pairs of division rings were first constructed by P. M. Cohn, in answer to a question of E. Artin.) Let $R = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix}$. By *FC*–(1.22), R is an artinian ring. Since $J = \mathrm{rad}\, R = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ with $R/J \cong A \times B$, the two simple right R-modules V', V may be thought of as A and B, respectively, with R acting via the projection $R \to A \times B$.

(1) Show that V' is injective.

(2) Determine the quotient module $E(V)/V$ and conclude that $E(V)_R$ is *not* f.g.

(**Comment.** Right artinian rings R over which $E(V)$ is f.g. for every simple right R-module V have a special significance in Morita Duality Theory: see (19.74) below.)

35. Over a right noetherian right self-injective ring R, show that any projective module P_R is injective. (For more general results, see §15B below.)

36. *True or False*: Let $R \subseteq S$ be rings such that $R \subseteq_e S_R$ and S_S is injective. Then $S = E(R_R)$.

37. (Douglas-Farahat) Let M be an additive abelian group, and let $R = \text{End}_{\mathbb{Z}}(M)$ (operating on the left of M). Show that ${}_RM$ is a projective module in case (1) M is a f.g. abelian group, or (2) $n\,M = 0$ for some positive integer n. (**Hint.** Use (2.12B) and Exercise 18 above.)

38. Use the fact that injective modules are divisible to prove the following: Let $E = E(R/\mathfrak{A})$ where $\mathfrak{A} \subsetneq R$ is a left ideal, and let $s \in R$. If $sE = 0$, then $ts = 0$ for some $t \in R\backslash\mathfrak{A}$. Deduce that, if $R\backslash\mathfrak{A}$ consists of non 0-divisors (e.g., R is a domain, or R is a local ring with maximal ideal $\mathfrak{A}$), then E is a faithful R-module. (In the local ring case, this gives another proof for the faithfulness of the "standard module".)

39. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring. Use the faithfulness of the standard module $E(R/\mathfrak{m})$ to show that $\bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0$. (This is a special case of the well-known "Krull Intersection Theorem" in commutative algebra.)

The following exercises, (40A) *through* (40G), *collect a few basic facts about associated primes and primary decompositions, mostly in a commutative setting. Some of these facts will prove to be useful in future chapters, when we consider the case of commutative rings.*

40A. Let R be a commutative noetherian ring, and let $\mathfrak{q}$ be a meet-irreducible ideal in R. By Noether's Theorem (see (3.80)), $\mathfrak{q}$ must be a primary ideal (i.e., $\mathfrak{q} \neq R$, and $xy \in \mathfrak{q} \Longrightarrow x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some n), say with radical $\mathfrak{p}$. Show that $E(R/\mathfrak{q}) \cong E(R/\mathfrak{p})$. Does this isomorphism still hold if $\mathfrak{q}$ is only assumed to be a primary ideal?

40B. Let I be an ideal in a commutative noetherian ring R. Show that

(1) I is primary iff $|\text{Ass}(R/I)| = 1$.

(2) By the Lasker-Noether Theorem (which we assume), there exists an (irredundant) primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ where the $\mathfrak{q}_i$'s are primary ideals with distinct radicals $\mathfrak{p}_i$'s. ("Irredundant" here means I is not the intersection of a smaller set of the $\mathfrak{q}_i$'s.) Show that $\text{Ass}(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

(3) Show that any prime minimal over I lies in $\text{Ass}(R/I)$. Conclude from this that the minimal members in $\text{Ass}(R/I)$ (with respect to inclusion) are exactly the primes of R that are minimal over I.

(4) Show that I is a radical ideal (i.e., $I = \sqrt{I}$) iff $\mathfrak{q} = \mathfrak{p}_i$ for all i. In

this case, show that *each* $\mathfrak{p}_i$ is a minimal prime over I.
(5) Part (2) above shows that the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ arising as radicals of the $\mathfrak{q}_i$'s in a primary decomposition depend only on I, and not on the decomposition chosen. Give an example to show, however, that the $\mathfrak{q}_i$ themselves may not be uniquely determined by I.

(**Hint.** Prove the "only if" part in (1) first; then deduce the "if" part there from (2). For (4), use Emmy Noether's example

$$I = (y^2, xy) = (y) \cap (y^2, x + ay) \qquad (\forall\, a \in \mathbb{C})$$

in the polynomial ring $\mathbb{C}[x, y]$, ca. 1921. For more information about the ring R/I, see (12.23)(a).)

40C. Use (2) of the above exercise to show that the conclusion of (3.78) holds already for *any* R-module M (over a commutative noetherian ring R) with $\mathrm{Ass}(M) = \{\mathfrak{p}\}$.

40D. Let M be a f.g. module over a commutative noetherian ring R, and let $I = \mathrm{ann}(M)$. In a commutative algebra monograph, it was claimed that $\mathrm{Ass}(M)$ is the same as $\mathrm{Ass}(R/I)$. Find a counterexample to this statement; then state (and prove) a corrected version thereof.

40E. Let M be a module over a commutative ring R.

(1) Show that any maximal member of the family $\{\mathrm{ann}(m) : 0 \neq m \in M\}$ is in $\mathrm{Ass}(M)$. (This is a commutative version of (3.58).)
(2) If R is noetherian, show that $\bigcup\{\mathfrak{p} : \mathfrak{p} \in \mathrm{Ass}(M)\}$ is precisely the set of elements of R which act as 0-divisors on M.
(3) Does (2) still hold if R is not noetherian?

40F. Let M_R be a noetherian module over an *arbitrary* ring R.

(1) Show that $|\mathrm{Ass}(M)| < \infty$.
(2) If R has ACC on ideals and $M \neq (0)$, show that there exists a filtration

$$(0) = M_0 \subsetneq \cdots \subsetneq M_n = M$$

such that each filtration factor M_i/M_{i-1} is a prime module.
(3) If R is commutative and noetherian, show that the filtration for M above may be chosen such that each $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for a suitable prime ideal $\mathfrak{p}_i \subset R$.

40G. Let R be the Boolean ring $\prod_{i \in I} \mathbb{Z}_2$.

(1) Show that $\mathrm{Ass}(R)$ consists of all prime ideals of the form $\mathfrak{p}_J = \prod_{i \in J} \mathbb{Z}_2$, where J is any subset of I such that $|I \setminus J| = 1$.
(2) If I is infinite, show that there exist primes of R that are *not* in $\mathrm{Ass}(R)$.

(**Comment** (not needed for the solution of the exercise). In general, the primes of R that are not associated primes correspond to the "nonprincipal ultrafilters" on the indexing set I.)

41. Let R be a commutative noetherian ring, and $E = E(R/\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of R. For any ideal $J \subseteq \mathfrak{p}$, let $\overline{R} = R/J$, $\overline{\mathfrak{p}} = \mathfrak{p}/J$, and let $E' := \text{ann}^E(J) \subseteq E$. Show that E' is isomorphic to the injective hull of the $\overline{R}$-module $\overline{R}/\overline{\mathfrak{p}}$.

42. (Vámos, Faith-Walker) Show that a ring R is right artinian iff every injective right R-module is a direct sum of injective hulls of simple R-modules.

43. Define a module I_R to be *fully divisible* if the following condition is satisfied: For any families $\{u_\alpha\} \subseteq I$ and $\{a_\alpha\} \subseteq R$ such that

$$\sum a_\alpha x_\alpha = 0 \ \text{(finite sum, } x_\alpha \in R) \implies \sum u_\alpha x_\alpha = 0,$$

there exists $v \in I$ such that $u_\alpha = v a_\alpha$ for all α. Show that I_R is fully divisible iff it is injective.

44. A ring R is defined to be *right principally injective* if R_R is a principally injective (or equivalently, divisible) module.[30] For instance, a right self-injective ring is right principally injective.

(1) Show that any von Neumann regular ring R is right principally injective.
(2) Give an example of a right principally injective ring that is neither von Neumann regular nor right self-injective.

The next three exercises below are taken from Nicholson-Yousif [95].

45. Prove the following for any right principally injective ring R:

(1) (generalizing Exercise 2) $a \in R$ has a left inverse iff a is not a left 0-divisor;
(2) R is Dedekind-finite iff any non left 0-divisor in R is a unit.

46. Let R be a right principally injective ring, and $f = f^2 \in R$. If $I \subseteq R$ is a right ideal isomorphic to the right ideal fR, show that $I = eR$ for some $e = e^2 \in R$. (For more information about this conclusion, see Exercises 35-38 in §6.)

47. For any right principally injective ring R, prove the following:

(1) If we have a direct sum of principal left ideals $\bigoplus_{i=1}^n Ra_i$ in R, then any R-homomorphism $g : \sum_i a_i R \to R$ extends to an endomorphism of R_R.
(2) If $\bigoplus_{i=1}^n A_i$ is a direct sum of ideals in R, then for any left ideal B in R, $B \cap \left(\bigoplus_i A_i\right) = \bigoplus_i (B \cap A_i)$.

48. Let R be a commutative noetherian complete local ring, and E be its standard module. For any R-submodules $A, B \subseteq E$, show that any R-homomorphism from A to B is given by a multiplication by an element of R.

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[30] A good alternative name for such R would be a *right divisible ring*.

49. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring with $\mathfrak{m}$-adic completion $\tilde{R}$ and standard module $E = E(R/\mathfrak{m})$. (You may assume that $\tilde{R}$ is also a noetherian local ring.) Upon identifying $\mathrm{End}_R(E)$ with $\tilde{R}$ by Matlis' Theorem (3.84), show that

    (1) the $\tilde{R}$-module E can be identified with the standard module $\tilde{E}$ of $\tilde{R}$, and

    (2) the R-submodules of E are the same as its $\tilde{R}$-submodules.

    (**Comment.** It will be clear from Morita's duality theory to be developed later (see (19.56)) that $\tilde{E}$ is an *artinian* $\tilde{R}$-module. From this, it follows from (2) that E is also an artinian R-module, without a completeness assumption on R. A further extension of this fact to a *nonlocal* setting can be found in Exercise (19.8).)

50. In the notation of the last exercise, let T be an R-submodule of E (so that it is also an $\tilde{R}$-submodule). Show that $T = E$ iff T is a faithful $\tilde{R}$-module. (You may use the result mentioned in the Comment on Exercise 49.)

51. Let $A = k[x_1, \ldots, x_r]$ where k is a field. In §3J, it is shown that the A-module k (with trivial x_i-action for all i) has injective hull $T = k[x_1^{-1}, \ldots, x_r^{-1}]$. Show that T is isomorphic to a quotient of the A-module of Laurent polynomials $k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$. In the case $r = 1$ (where we write $R = k[x]$), show that T is isomorphic to the x-primary component of the torsion A-module $k(x)/k[x]$.

52. Let $A = k[x_1, \ldots, x_r]$, $P = k[[x_1, \ldots, x_r]]$, and $T = k[x_1^{-1}, \ldots, x_r^{-1}]$, where k is a field. In §3J, we observed that there is a P-module structure on T extending its A-module structure. Show directly that T is a faithful P-module, and use Exercises (49) and (50) above to give an alternative proof for the fact that $T = E_A(k) = E_P(k)$, where k denotes the A-module (resp. P-module) with trivial x_i-action for all i.

53. For any r-tuple $a = (a_1, \ldots, a_r)$ over a field k, let $\mathfrak{m}_a$ be the maximal ideal $(x_1 - a_1, \ldots, x_r - a_r)$ in the polynomial ring $A = k[x_1, \ldots, x_r]$. Let $k_a = A/\mathfrak{m}_a$, so that k_a is the A-module k on which each x_i acts as multiplication by a_i. Construct the injective hull $E_A(k_a)$.

54. Lat $(A, \mathfrak{m}_A)$ be a commutative noetherian local ring, with (right) standard module ω_A. Let $(B, \mathfrak{m}_B)$ be a right artinian local ring that is a module-finite algebra over A such that $1_B \cdot \mathfrak{m}_A \subseteq \mathfrak{m}_B$. Show that the right standard module $\omega_B := E((B/\mathfrak{m}_B)_B)$ of B is given by $\mathrm{Hom}_A(B, \omega_A)$. [Here, B is viewed as an (B, A)-bimodule, and the right B-module structure on $\mathrm{Hom}_A(B, \omega_A)$ comes from the left B-structure on B.]

55. Let R be a right noetherian ring, and $\mathfrak{p}$ be a prime ideal of R. If $R/\mathfrak{p}$ is a domain, show that $\mathfrak{p}$ is right meet-irreducible (i.e., if A, B are right ideals such that $A \cap B = \mathfrak{p}$, then $A = \mathfrak{p}$ or $B = \mathfrak{p}$). (**Comment.** The converse is true too, but is much deeper. Its proof requires Goldie's Theorem to be proved in §11; see (11.25) below.)

Chapter 2

Flat Modules and Homological Dimensions

This chapter is a natural continuation of Chapter 1 and consists of two long sections. In §4, we study in detail the notion of flat (and faithfully flat) modules, and in §5, we develop the theory of homological dimensions of modules and rings.

The idea of flat and faithfully flat modules plays a special role in many parts of ring theory. On the one hand, flat modules are natural generalizations of projective modules. On the other hand, flat modules are related to injective modules via the formation of character modules. Also, by a theorem of Lazard and Govorov, flat modules are precisely the direct limits of (f.g.) free modules. Thus, in the section on flat modules in this chapter, the theories developed in §1, §2, and §3 for free, projective, and injective modules find their common ground.

Upon developing the theory of flat modules, two other important classes of modules come to the fore. These are the class of *finitely presented* (f.p.) modules, and the class of *coherent* modules. These classes are discussed in some detail in §4. In particular, we shall encounter the class of *left coherent rings* (those rings whose f.g. left ideals are f.p.). By a theorem of S. Chase, these are precisely the rings over which a direct product of flat *right* modules is always flat.

Section 4 concludes with a subsection on *pure exact sequences*. These sequences are intimately related to the notion of flat modules. A highlight in this subsection is the result that pure exact sequences are precisely direct limits of *split* short exact sequences.

We note, in passing, that flat modules are used not only in algebra and algebraic geometry, but also in topology and analysis. For instance, in the homology theory of Banach algebras, the notion of flatness (for bimodules) is related to the notion of "amenability" of such algebras.

In §5, we develop the theory of projective, injective, and flat dimensions of modules via the use of Schanuel's Lemma. By taking suprema of these dimensions over all (right) modules, we arrive at two important homological invariants of a ring, namely, its (right) *global dimension*, and its *weak dimension*. These invariants control the arithmetic of a ring in a rather subtle way. For instance, a famous theorem of Serre says that a commutative noetherian local ring is regular iff it has finite global dimension, which must then be equal to its Krull dimension. An

attempt to "globalize" Serre's result leads to the class of right regular rings (right noetherian rings whose f.g. right modules have finite projective dimensions). These rings are known to be of importance in algebraic K-theory. Due to limitation of space, however, we shall confine ourselves to the study of *commutative* regular rings. For these rings, global dimension agrees with Krull dimension, although these dimensions are not necessarily finite in general.

Most of what we cover in this chapter is standard fare in the theory of rings and modules. To aid the reader's understanding of this material, many interesting examples are included. More module theory of a somewhat different spirit will be presented in Chapter 3.

§4. Flat and Faithfully Flat Modules

§4A. Basic Properties and Flatness Tests

The purpose of §4 is to give an introduction to the theory of flat (and faithfully flat) modules. In *FC*-§24, I have given a brief account of flat modules, with the goal of applying them to the study of perfect and semiperfect rings. Here, we are concerned with flat modules over *arbitrary* rings. For the convenience of the reader, therefore, this section is written independently of *FC*–§24 and does not presuppose a knowledge of perfect and semiperfect rings; any reference to *FC*–§24 will only be of a peripheral nature.

The importance of flat modules can partly be gauged from the fact that "Modules Plats" occupy the very first chapter of Bourbaki's famous treatise "Algèbre Commutative". Fortunately for all algebraists, Bourbaki wrote this chapter without imposing any commutativity conditions on the ground ring. Needless to say, our presentation here is heavily influenced by Bourbaki's. In the course of consulting the literature, however, we were astonished by a nontrivial number of careless statements and wrong assertions about flat modules, even in standard ring theory textbooks. A few of these will be pointed out in the Exercises for this section, as well as in §4I.

(4.0) Definition. A right module P_R is called *flat* (or *R-flat*) if the functor $P \otimes_R -$ is exact on ${}_R\mathfrak{M}$ (the category of left R-modules). Since this functor is in any case right exact, the condition for P to be flat is that, whenever $A \to B$ is injective in ${}_R\mathfrak{M}$, so is $P \otimes_R A \to P \otimes_R B$, in the category of abelian groups.

The term "flat module" is due to J.-P. Serre: a good choice of words since such a module does not "bend" an inclusion map? Just as in the case of projective and injective modules, the notion of flat modules is "functorial" in the following sense.

(4.1) Proposition. *Let $\varphi : R \to S$ be a ring homomorphism, whereby S is viewed as a left R-module. If P_R is R-flat, then the right S-module $P' := P \otimes_R S$ is S-flat.*

Proof. Let $A' \to B'$ be an injection in ${}_S\mathfrak{M}$. We must show that $P' \otimes_S A' \to P' \otimes_S B'$ is an injection of abelian groups. Now

$$P' \otimes_S A' = (P \otimes_R S) \otimes_S A'$$

can be identified with $P \otimes_R A'$ (where A' is viewed as a left R-module via φ) and similarly $P' \otimes_S B'$ can be identified with $P \otimes_R B'$. Since P is R-flat, $P \otimes_R A' \to P \otimes_R B'$ is injective, which gives what we want. □

If an R-module P_R is a direct sum of $\{P_i : i \in I\}$, then the functor $P \otimes_R -$ is the direct sum of the functors $P_i \otimes_R -$, so $P \otimes_R -$ is exact on ${}_R\mathfrak{M}$ iff each $P_i \otimes_R -$ is. We have, therefore:

(4.2) Proposition. *If $P = \bigoplus_i P_i$, then P is flat iff each P_i is flat.*

Since $R \otimes_R A \cong A$ for any left R-module A, the right regular module R_R is flat. Thus, (4.2) implies that any free right module is flat, and, moreover:

(4.3) Proposition. *Any projective (right) R-module is flat.*

The converse of (4.3) is false in general. In fact, we have shown in *FC*–(24.25) that the converse of (4.3) holds precisely when the ring R in question is "right perfect". In any case, it is easy to construct examples of flat modules that are not projective. For instance, let R be a commutative ring and let $P = (S^{-1}R)_R$, where S is a multiplicatively closed set in R and $S^{-1}R$ denotes the localization of R with respect to S. The functor "$P \otimes_R -$" in this case is the localization functor "S^{-1}" which is well-known to be exact. Therefore, *$(S^{-1}R)_R$ is always R-flat.* But, choosing R to be a commutative domain, and $S := R\backslash\{0\}$, $S^{-1}R$ is the quotient field K of R, and K_R is not a projective module unless $R = K$, by (2.18).

(4.4) Proposition. *Let $\{P_i : i \in I\}$ be a direct system of right modules over any ring R, where I is a directed set. If each P_i $(i \in I)$ is flat, then so is the direct limit module $P := \varinjlim P_i$.*

Proof. Let $A \to B$ be an injection in ${}_R\mathfrak{M}$. Then $P_i \otimes_R A \to P_i \otimes_R B$ is an injection for each $i \in I$. It follows easily that

$$\varinjlim\, (P_i \otimes_R A) \longrightarrow \varinjlim\, (P_i \otimes_R B)$$

is also injective. The LHS (resp. RHS) is canonically isomorphic to $P \otimes_R A$ (resp. $P \otimes_R B$). Therefore, $P \otimes_R A \to P \otimes_R B$ is injective, as desired. □

(4.5) Corollary. *If every f.g. submodule of a module P_R is flat, then P_R itself is flat.*

Proof. We can think of P as a direct limit of the direct system of its f.g. submodules. Now apply (4.4). □

(4.6) Corollary. *Let R be a right semihereditary ring. Then any right ideal $\mathfrak{A} \subseteq R$ is flat as a right R-module.*

Proof. Consider any f.g. right ideal $\mathfrak{A}_0 \subseteq \mathfrak{A}$. Since R is right semihereditary, $(\mathfrak{A}_0)_R$ is projective, and hence flat by (4.3). By (4.5), it follows that $\mathfrak{A}_R$ is flat.[31] □

While flat modules are related to projective modules by (4.3), there is also an interesting relationship between flat modules and injective modules, discovered by J. Lambek. This relationship is formulated by using the notion of character modules. For any right R-module P, the *character module* of P is defined to be

$$P' := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z}).$$

This is a *left* R-module via the action $(r, f) \mapsto rf$, where $(rf)(x) = f(xr)$ for $r \in R$, $f \in P'$, and $x \in P$. The abelian group $\mathbb{Q}/\mathbb{Z}$ is used here for two reasons: it is a divisible group and hence an injective $\mathbb{Z}$-module; and it has the following convenient ("cogenerator") property.

(4.7) Lemma. *For any abelian group X with a given nonzero element $x \in X$, there exists a group homomorphism $f : X \to \mathbb{Q}/\mathbb{Z}$ such that $f(x) \neq 0$.*

Proof. Since $\mathbb{Q}/\mathbb{Z}$ contains elements of any given finite order, there clearly exists a homomorphism $f_0 : \mathbb{Z} \cdot x \to \mathbb{Q}/\mathbb{Z}$ with $f_0(x) \neq 0$. By the injectivity of $\mathbb{Q}/\mathbb{Z}$, f_0 can be extended to a homomorphism $f : X \to \mathbb{Q}/\mathbb{Z}$. □

Using this lemma, we can show that the character module formation gives a (contravariant) functor from $\mathfrak{M}_R$ to ${}_R\mathfrak{M}$ with the following exactness property.

(4.8) Proposition. *For right R-modules A, B, C, a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathfrak{M}_R$ is exact iff the induced sequence $C' \xrightarrow{\beta'} B' \xrightarrow{\alpha'} A'$ in ${}_R\mathfrak{M}$ is exact.*

Proof. The "only if" part follows from the injectivity of the abelian group $\mathbb{Q}/\mathbb{Z}$. For the converse, we assume that $\mathrm{im}\,\beta' = \ker \alpha'$, and try to show that $\mathrm{im}\,\alpha = \ker \beta$.

(a) If $\mathrm{im}\,\alpha \subseteq \ker \beta$ does not hold, we would have $\beta\alpha(a) \neq 0$ for some $a \in A$. By (4.7), there exists $f \in C'$ such that

$$0 \neq f(\beta\alpha(a)) = (\alpha'\beta'(f))(a),$$

contradicting $\alpha'\beta' = 0$.

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[31] We shall see later that any *left* ideal in R is also flat; see (4.66).

(b) If $\ker \beta \subseteq \operatorname{im} \alpha$ does not hold, we would have some $b \in B \backslash \operatorname{im} \alpha$ such that $\beta(b) = 0$. Applying (4.7) to $B/\operatorname{im} \alpha$, there exists $f \in B'$ such that $f(\operatorname{im} \alpha) = 0$ and $f(b) \neq 0$. The former implies that $\alpha'(f) = 0$, so $f = \beta'(g)$ for some $g \in C'$. But then

$$f(b) = \beta'(g)(b) = g(\beta(b)) = 0,$$

a contradiction. □

We arrive now at the following remarkable connection between injective modules and flat modules.

(4.9) Theorem (Lambek). *A right R-module P is flat in $\mathfrak{M}_R$ iff its character module P' is injective in ${}_R\mathfrak{M}$.*

Proof. The "only if" part follows from the Injective Producing Lemma (3.5), applied with $S = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. (Of course, (3.5) has to be applied with a left-right switch, since here $P = {}_SP_R$.) For the reader's convenience, we recall this proof. This recall is useful since the proof of the converse will consist of reversing the steps. Assume P is flat, and let $0 \to A \to B$ be exact in ${}_R\mathfrak{M}$. Then $0 \to P \otimes_R A \to P \otimes_R B$ is also exact, and hence

$$\text{(4.10)} \qquad \operatorname{Hom}_{\mathbb{Z}}(P \otimes_R B, \mathbb{Q}/\mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(P \otimes_R A, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

is exact since $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{Z}$-injective. Identifying the two groups above (via canonical isomorphisms) with $\operatorname{Hom}_R(B, \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z}))$ and $\operatorname{Hom}_R(A, \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z}))$, we see that

$$\text{(4.11)} \qquad \operatorname{Hom}_R(B, P') \longrightarrow \operatorname{Hom}_R(A, P') \longrightarrow 0$$

is exact. This shows that P' is an injective left R-module.

Conversely, assume that P' is injective, and let $0 \to A \to B$ be exact in ${}_R\mathfrak{M}$ as before. Then (4.11) is exact, and therefore so is (4.10). Applying (4.8) over the ring $\mathbb{Z}$, it follows that $0 \to P \otimes_R A \to P \otimes_R B$ is also exact. This shows that P is flat in $\mathfrak{M}_R$. □

Combining Lambek's Theorem with Baer's Test for Injectivity, we shall derive the following.

(4.12) Modified Flatness Test. *A right R-module P is flat iff, for any (finitely generated) left ideal $\mathfrak{A} \subseteq R$, the natural map $P \otimes_R \mathfrak{A} \to P\mathfrak{A}$ is an isomorphism (of abelian groups).*

Proof. Note that $P\mathfrak{A}$ is the image of $P \otimes_R \mathfrak{A} \to P \otimes_R R$, if we identify $P \otimes_R R$ with P. Therefore, to say that $P \otimes_R \mathfrak{A} \to P\mathfrak{A}$ is an isomorphism amounts to saying that

$$(*) \qquad 0 \to P \otimes_R \mathfrak{A} \to P \otimes_R R \quad \textit{is exact.}$$

If P is flat, this certainly holds for all left ideals $\mathfrak{A}$. Conversely, if $(*)$ holds for all f.g. left ideals $\mathfrak{A}$, then by taking direct limits, it also holds for all left ideals $\mathfrak{A}$. Therefore, (4.10) and (4.11) are exact for $A = \mathfrak{A}$ and $B = R$. By Baer's Criterion (3.7), this implies that P' is injective in ${}_R\mathfrak{M}$. Now Lambek's Theorem implies that P is flat in $\mathfrak{M}_R$. □

(4.13) Corollary. [32] *Suppose* $0 \to A \to B \to C \to 0$ *is exact in* $\mathfrak{M}_R$. *If* A, C *are flat modules, then so is* B.

Proof. For *any* left ideal $\mathfrak{A}$, we have the following commutative diagram:

$$\begin{array}{ccccccc} A \otimes_R \mathfrak{A} & \xrightarrow{\sigma} & B \otimes_R \mathfrak{A} & \xrightarrow{\tau} & C \otimes_R \mathfrak{A} & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ A\mathfrak{A} & \xrightarrow{\varphi} & B\mathfrak{A} & \xrightarrow{\psi} & C\mathfrak{A} & \longrightarrow & 0 \end{array}$$

where α, γ are isomorphisms, by (4.12). If $x \in \ker \beta$, then $0 = \psi(\beta(x)) = \gamma(\tau(x))$ so $\tau(x) = 0$. Since the top row is exact, we have $x = \sigma(y)$ for some $y \in A \otimes_R \mathfrak{A}$. But then $0 = \beta(\sigma(y)) = \varphi(\alpha(y))$ implies $\alpha(y) = 0$ (since φ is injective). Therefore $y = 0$ and $x = \sigma(y) = 0$. This shows that β is an isomorphism, so B is flat by (4.12). □

Of course, (4.13) is also valid with "flat" replaced by "projective" or "injective". In these cases, the exact sequence actually splits. But in the flat case, it need not.

The Modified Flatness Test has other useful applications. In the next result, we apply it to obtain a criterion for a quotient module of a flat module to be flat.

(4.14) Proposition. *Let* $0 \to K \to F \to P \to 0$ *be exact in* $\mathfrak{M}_R$, *where* F *is flat. Then* P *is flat iff* $K \cap F\mathfrak{A} = K\mathfrak{A}$ *for every (f.g.) left ideal* $\mathfrak{A} \subseteq R$.

Proof. Start with the exact sequence

$$K \otimes_R \mathfrak{A} \to F \otimes_R \mathfrak{A} \to P \otimes_R \mathfrak{A} \to 0. \tag{4.15}$$

Since F is flat, we may identify $F \otimes_R \mathfrak{A}$ with $F\mathfrak{A}$ by (4.12). Therefore, (4.15) induces an isomorphism

$$F\mathfrak{A}/K\mathfrak{A} \longrightarrow P \otimes_R \mathfrak{A}. \tag{4.16}$$

On the other hand, we have an isomorphism

$$F\mathfrak{A}/(K \cap F\mathfrak{A}) \longrightarrow P\mathfrak{A} \tag{4.17}$$

induced by the surjection $F \to P$. If we think of (4.16) and (4.17) as "identifications", the natural surjection $P \otimes_R \mathfrak{A} \to P\mathfrak{A}$ corresponds to the natural

[32] For another proof of this Corollary (and a further refinement), see (4.86)(2).

surjection

$$F\mathfrak{A}/K\mathfrak{A} \longrightarrow F\mathfrak{A}/(K \cap F\mathfrak{A}).$$

By (4.12), P is flat iff this is an isomorphism, for all (f.g.) left ideals $\mathfrak{A} \subseteq R$. The Proposition now follows. □

§4B. *Flatness, Torsion-Freeness, and von Neumann Regularity*

If I_R is any injective right R-module, then, for any $a \in R$ with $\text{ann}_r(a) = 0$, right multiplication by a on I is *surjective* (see (3.17)′). Using (4.14), we can prove the following "dual" of this property for flat modules.

(4.18) Proposition. *Let P be any flat module in $\mathfrak{M}_R$. If an element $a \in R$ has* $\text{ann}_\ell(a) = 0$, *then right multiplication by a on P is injective.*

Proof. Assuming $\text{ann}_\ell(a) = 0$, the map $\mu : R \to R$ given by *right* multiplication by a is an injection in ${}_R\mathfrak{M}$. Since P is flat, $1 \otimes \mu : P \otimes_R R \to P \otimes_R R$ is also an injection. But, after identifying $P \otimes_R R$ with P as usual, $1 \otimes \mu$ is just right multiplication by a on P. This gives the desired conclusion. □

If a right module P_R has the property in the conclusion of (4.18), let us say that P is torsion-free. (Clearly, this definition of torsion-freeness generalizes the earlier one given for modules over commutative domains in §2.) Thus, (4.18) says that *any flat module in $\mathfrak{M}_R$ is torsion-free.* The converse is not true in general, as the following easy example shows.

(4.19) Example. Let $R = k[x, y]$, where k is any commutative domain. Then *the torsion-free R-module $\mathfrak{A} = (x, y) = xR + yR$ is not flat.* The quickest way to see this is to use (4.1) with $S = R/(x)$, which we can identify with $k[y]$. If $\mathfrak{A}$ is R-flat, then by (4.1) applied to the quotient map $R \to S$,

$$\mathfrak{A} \otimes_R S = \mathfrak{A} \otimes_R (R/(x)) \cong \mathfrak{A}/\mathfrak{A}x = (x, y)/(x^2, yx)$$

is S-flat, and hence torsion-free as an S-module. But this is not the case since $0 \neq y \in S$ annihilates the nonzero element $x = x \otimes 1 \in \mathfrak{A} \otimes_R S$. Another way to show that $\mathfrak{A}$ is not R-flat is to use (4.12): we shall show below that the natural map $f : \mathfrak{A} \otimes_R \mathfrak{A} \to \mathfrak{A}^2$ is *not* injective. Since

$$f(x \otimes y - y \otimes x) = xy - yx = 0,$$

we are done if we can show that $x \otimes y \neq y \otimes x$ in $\mathfrak{A} \otimes_R \mathfrak{A}$. Consider the natural map

$$g : \mathfrak{A} \otimes_R \mathfrak{A} \longrightarrow (\mathfrak{A}/\mathfrak{A}^2) \otimes_R (\mathfrak{A}/\mathfrak{A}^2).$$

We can identify $(\mathfrak{A}/\mathfrak{A}^2)_R$ with V_R, where $V := k\overline{x} \oplus k\overline{y}$ is acted on trivially by $\mathfrak{A}$. With this identification,

$$(\mathfrak{A}/\mathfrak{A}^2) \otimes_R (\mathfrak{A}/\mathfrak{A}^2) = V \otimes_R V = V \otimes_k V$$
$$= k(\overline{x} \otimes \overline{x}) \oplus k(\overline{x} \otimes \overline{y}) \oplus k(\overline{y} \otimes \overline{x}) \oplus k(\overline{y} \otimes \overline{y}).$$

Hence $g(x \otimes y) \neq g(y \otimes x)$, which shows that $x \otimes y \neq y \otimes x$ in $\mathfrak{A} \otimes_R \mathfrak{A}$.

Over certain rings, torsion-free right modules can be shown to be flat. For instance, we have:

(4.20) Proposition. *Let R be a Prüfer domain (see* (2.28)*). Then a right R-module P is flat if (and only if) it is torsion-free.*

Proof. It suffices to prove the "if" part, so assume P is torsion-free. By (4.5), we are done if we can show that any f.g. submodule $P_0 \subseteq P$ is flat. Since P_0 is torsion-free, (2.31) implies that P_0 is projective, so by (4.3), P_0 is indeed flat. □

The Proposition applies, in particular, to all Dedekind domains. For instance, applying it over $\mathbb{Z}$, we see that *any torsion-free abelian group* (e.g., $P = \mathbb{Z} \times \mathbb{Z} \times \dots$) *is $\mathbb{Z}$-flat.* (Recall, however, that P is not $\mathbb{Z}$-projective, by (2.8).)

More examples of flat modules are given by the following interesting result, which may be viewed as the "flat analogue" of similar earlier results on projectives and injectives (cf. *FC*–(2.8), (2.9)).

(4.21) Theorem. *For any ring R, the following are equivalent:*

(1) *R is von Neumann regular;*
(2) *any right R-module P is flat;*
(3) *any cyclic right R-module is flat.*

Proof. (1)⟹(2). Let $\mathfrak{A}$ be any f.g. left ideal of R. Then $\mathfrak{A} = Re$ for some $e = e^2 \in R$ (by *FC*–(4.23)), and so the inclusion map $i : \mathfrak{A} \to R$ is a split injection in ${}_R\mathfrak{M}$. It follows that $1 \otimes i : P \otimes_R \mathfrak{A} \to P \otimes_R R$ is also a (split) injection. By (4.12), this implies that P is flat.

(2)⟹(3) is a tautology.

(3)⟹(1). For any $a \in R$, consider the exact sequence

$$0 \longrightarrow aR \longrightarrow R \longrightarrow R/aR \longrightarrow 0.$$

Since R/aR is flat by (3) and R is flat by (4.3), (4.14) implies that $aR \cap Ra \subseteq aRa$. In particular, we have $a \in aRa$, as desired. □

It follows immediately from (4.9) and (4.21) that

(4.22) Corollary. *For any right module P over a von Neumann regular ring, the character module P' is an injective left R-module.*

§4C. More Flatness Tests

Developing the theme of (4.14) a bit further, we next prove the following theorem of O. Villamayor.

(4.23) Theorem. *Let $0 \to K \to F \to P \to 0$ be exact in $\mathfrak{M}_R$, where F is a free module with basis $\{e_i : i \in I\}$. Then the following are equivalent:*

(1) *P is flat.*
(2) *For any $c \in K$, there exists $\theta \in \mathrm{Hom}_R(F, K)$ with $\theta(c) = c$.*
(3) *For any $c_1, \ldots, c_n \in K$ $(n \in \mathbb{N})$, there exists $\theta \in \mathrm{Hom}_R(F, K)$ with $\theta(c_i) = c_i$ for all i.*

Proof. (2)$\Longrightarrow$(1). We shall check that $K \cap F\mathfrak{A} \subseteq K\mathfrak{A}$, where $\mathfrak{A} \subseteq R$ is any left ideal. For $c \in K \cap F\mathfrak{A}$, write $c = e_{i_1}r_1 + \cdots + e_{i_m}r_m$ $(r_j \in \mathfrak{A})$. Take $\theta \in \mathrm{Hom}_R(F, K)$ with $\theta(c) = c$. Then

$$c = \theta(e_{i_1}r_1 + \cdots + e_{i_m}r_m) = \theta(e_{i_1})r_1 + \cdots + \theta(e_{i_m})r_m \in K\mathfrak{A},$$

as desired.

(1)$\Longrightarrow$(2). Write the given $c \in K$ as $e_{i_1}r_1 + \cdots + e_{i_m}r_m$ $(r_j \in R)$, and let $\mathfrak{A} = \sum_j Rr_j$. Since P is flat, we have $c \in K \cap F\mathfrak{A} = K\mathfrak{A}$ by (4.14). Therefore, $c = \sum_\alpha c_\alpha s_\alpha$ for suitable $c_\alpha \in K$ and $s_\alpha \in \mathfrak{A}$. Writing further $s_\alpha = \sum_j t_{\alpha j} r_j$ $(t_{\alpha j} \in R)$, we have then

$$(4.23') \qquad c = \sum_\alpha c_\alpha \sum_j t_{\alpha j} r_j = \sum_j \Big(\sum_\alpha c_\alpha t_{\alpha j}\Big) r_j.$$

Defining $\theta \in \mathrm{Hom}_R(F, K)$ by sending e_{i_j} to $c_j' := \sum_\alpha c_\alpha t_{\alpha j} \in K$ (for $1 \le j \le m$) and sending all other basis elements to zero (for instance), we have

$$\theta(c) = \theta\Big(\sum_j e_{i_j} r_j\Big) = \sum_j c_j' r_j = c \quad \text{by (4.23')}.$$

(3)$\Longrightarrow$(2) follows by taking $n = 1$.

(2)$\Longrightarrow$(3). We prove (3) by induction on n, the case $n = 1$ being covered by (2). For $n > 1$, find first $\theta_n \in \mathrm{Hom}_R(F, K)$ fixing c_n, and define $c_i' = c_i - \theta_n(c_i) \in K$ for $i \le n-1$. By the inductive hypothesis, there exists $\theta' \in \mathrm{Hom}_R(F, K)$ fixing all c_i'. Now define $\theta := \theta' + \theta_n - \theta'\theta_n \in \mathrm{Hom}_R(F, K)$. Then

$$\begin{aligned} \theta(c_n) &= \theta'(c_n) + c_n - \theta'(c_n) = c_n, \quad \text{and} \\ \theta(c_i) &= \theta'(c_i) + (c_i - c_i') - \theta'(c_i - c_i') \\ &= c_i - c_i' + \theta'(c_i') = c_i \end{aligned}$$

for $i \leq n-1$, as desired. □

As a consequence of (4.23), we shall present some other characterizations of flat modules in terms of linear relations. The exposition here follows closely [Chase: 60].

(4.24) Theorem. *(Equational Criteria for Flatness) For any right R-module P, the following are equivalent:*

(1) *P is R-flat.*
(2) *Given any linear relation $\sum_j a_j r_j = 0$ $(a_j \in P,\ r_j \in R,\ 1 \leq j \leq n)$, there exist*

$$b_i \in P \ (1 \leq i \leq m) \ \text{ and } \ s_{ij} \in R \ (1 \leq i \leq m,\ 1 \leq j \leq n)$$

such that $a_j = \sum_i b_i s_{ij}$ for all j, and $\sum_k s_{ik} r_k = 0$ for all i.
(3) *Given any linear relations $\sum_j a_j r_{j\ell} = 0$ $(a_j \in P,\ r_{j\ell} \in R,\ 1 \leq j \leq n,\ 1 \leq \ell \leq p)$, there exist*

$$b_i \in P \ (1 \leq i \leq m) \ \text{ and } \ s_{ij} \in R \ (1 \leq i \leq m,\ 1 \leq j \leq n)$$

such that $a_j = \sum_i b_i s_{ij}$ for all j, and $\sum_j s_{ij} r_{j\ell} = 0$ for all i and all ℓ.

Remarks. (3) may be expressed *formally* in matrix notation as follows. Let

$$\alpha = (a_1, \ldots, a_n) \in P^n,$$

and $\rho = (r_{j\ell})$ (an $n \times p$ matrix over R); if $\alpha\rho = 0$, then $\alpha = \beta\sigma$ for some $\beta = (b_1, \ldots, b_m) \in P^m$ and some $m \times n$ matrix σ over R, such that $\sigma\rho = 0$. (2) is the special case of (3) when $p = 1$. In a manner of speaking, (2) and (3) express the fact that *linear relations in P are consequences of linear relations in R*. In the special case when P is a ring containing R as a subring, the condition for P_R to be flat boils down to the following: any solution over P for a homogeneous system of linear equations defined over R is a P-combination of solutions of the same system over R.

Proof. (3)⟹(2) is trivial, as we have noted above.

(2)⟹(1). Fix an epimorphism $f: F \to P$, where F is a suitable free R-module. Let $K = \ker(f)$. We shall show that P is flat by checking (2) in (4.23). Given any $c \in K$, write $c = e_1 r_1 + \cdots + e_n r_n$, where $r_j \in R$ and $e_1, \ldots, e_n$ are part of a basis of F. Let $a_j = f(e_j)$. Then $\sum_j a_j r_j = 0$ in P, and we can find the $\{b_i\}$, $\{s_{ij}\}$ as in (2). Fixing $z_i \in F$ such that $f(z_i) = b_i$, we can define $\theta: F \to F$ by sending each e_j $(1 \leq j \leq n)$ to $e_j - \sum_i z_i s_{ij}$, and sending all other basis elements to zero. Since

$$f\big(e_j - \sum_i z_i s_{ij}\big) = a_j - \sum_i b_i s_{ij} = 0$$

for all $j \le n$, we have $\theta(F) \subseteq K$, and finally:

$$\theta(c) = \theta\Big(\sum_k e_k r_k\Big) = \sum_k \Big(e_k - \sum_i z_i s_{ik}\Big) r_k$$
$$= \sum_k e_k r_k - \sum_i z_i \Big(\sum_k s_{ik} r_k\Big) = c.$$

(1)$\Longrightarrow$(3). Fix $f : F \to P$ and $K = \ker(f)$ as above. Choose $x_j \in F$ such that $f(x_j) = a_j$, and let $y_\ell = \sum_j x_j r_{j\ell}$. Then $f(y_\ell) = \sum_j a_j r_{j\ell} = 0$, so $y_\ell \in K$. Since P is flat, (4.23) implies that there exists $\theta \in \mathrm{Hom}_R(F, K)$ fixing each y_ℓ. Write $x_j - \theta(x_j) = \sum_{i=1}^m e_i s_{ij}$, where $s_{ij} \in R$, and $e_1, \ldots, e_m$ are part of a basis of F. Then, for $b_i := f(e_i)$, we have

$$a_j = f(x_j) = f(x_j - \theta(x_j)) = f\Big(\sum_i e_i s_{ij}\Big) = \sum_i b_i s_{ij}.$$

Finally, since

$$0 = y_\ell - \theta(y_\ell) = \sum_j \big(x_j - \theta(x_j)\big) r_{j\ell}$$
$$= \sum_j \sum_i e_i s_{ij} r_{j\ell} = \sum_i e_i \Big(\sum_j s_{ij} r_{j\ell}\Big),$$

we have $\sum_j s_{ij} r_{j\ell} = 0$ for all i and ℓ, as desired. □

§4D. *Finitely Presented (f.p.) Modules*

Before we give further applications of (4.23) and (4.24), we need to introduce a couple more definitions:

(4.25) Definitions. (a) A module P_R is said to be *finitely related* (abbreviated f.r.) if there exists an exact sequence $0 \to K \to F \to P \to 0$ in $\mathfrak{M}_R$ where F is free (of arbitrary rank) and K is f.g. (finitely generated).

(b) A module P_R is said to be *finitely presented* (abbreviated f.p.) if there exists an exact sequence $0 \to K \to F \to P \to 0$ in $\mathfrak{M}_R$ where F is free of finite rank, and K is f.g. (or equivalently, there exists an exact sequence $R^m \to R^n \to P \to 0$ with $m, n \in \mathbb{N}$).

A word of caution is necessary here. The definition of "finitely presented" above is universally accepted, but in some books, "finitely related" is taken to mean the same thing as "finitely presented". We believe this choice is unwise, and that "finitely related" should be defined as in (a) above (meaning that P can be generated with a certain set of generators subject to a *finite number of relations*). Our terminology here follows that of P. M. Cohn [91].

We record in the following a few facts relating the notions introduced in Def. (4.25). For the proofs of these, we shall need Schanuel's Lemma, which will

appear, for expository reasons, in the next section. We suggest, therefore, that the reader look up the statement of Schanuel's Lemma (5.1) at this point, in order to apply it to the proof of the proposition below.

(4.26) Proposition. (a) *A module P_R is f.p. iff it is both f.g. and f.r.*
(b) *Let P_R be f.p., and $\beta : Q \to P$ be an epimorphism. If Q is f.g., then so is* $\ker(\beta)$.
(c) *A module P_R is f.r. iff it is a direct sum of a free module (of arbitrary rank) and a f.p. module.*

Proof. (a) The "only if" part is clear. For the "if" part, assume P is both f.g. and f.r. Then, by definition, we have exact sequences

$$(4.27) \qquad 0 \to L \to R^k \to P \to 0 \quad \text{and} \quad 0 \to K \to F \to P \to 0,$$

where $k \in \mathbb{N}$, F is free, and K is f.g. Since R^k and F are both *projective*, Schanuel's Lemma (5.1) gives an R-isomorphism $L \oplus F \cong K \oplus R^k$. The *RHS* is f.g., so L is also f.g., and the first sequence in (4.27) shows that P is f.p.

(b) Fix an epimorphism $\alpha : R^k \to Q$, where $k < \infty$. We have a composite epimorphism $\beta\alpha : R^k \to P$, inducing an epimorphism $\ker(\beta\alpha) \to \ker(\beta)$. By the work done in (a), $\ker(\beta\alpha)$ is f.g., so $\ker(\beta)$ is also f.g.

(c) The "if" part is clear. For the "only if" part, let P be f.r., say, with $0 \to K \to F \to P \to 0$, where F is free, and K is f.g. Working with a finite number of generators for K, we can find a decomposition $F = F_1 \oplus F_2$ where F_1 is free, and F_2 is f.g. free containing K. But then $P \cong F/K \cong F_1 \oplus (F_2/K)$, with F_1 free and F_2/K f.p., as desired. □

(4.28) Remark. It is possible to prove the "if" part of (a) without using Schanuel's Lemma, as follows. Let P be f.r. By (c), $P = F_1 \oplus P_1$, where F_1 is free and P_1 is f.p. If P is also f.g., then $F_1 \cong R^n$ for some $n < \infty$. It follows easily that P is f.p.

In general, "f.g." and "f.r." are each weaker than "f.p." It would be futile to look for rings whose f.r. (right) modules are all f.p.: only the zero ring has this property. (Why?) It is, however, not futile to look for rings whose f.g. (right) modules are all f.p. We have, in fact, the following clean-cut result.

(4.29) Proposition. *A ring R is right noetherian iff all f.g. (resp. cyclic) right R-modules are f.p.*

Proof. First assume R is right noetherian. Let P be a f.g. right R-module, and fix a surjection $f : R^k \to P$ ($k \in \mathbb{N}$). By *FC*–(1.21), R^k is a noetherian module, so $\ker f$ is also noetherian and hence f.g. This shows that P is f.p. Conversely, assume all f.g. (or just cyclic) right R-modules are f.p. Let $\mathfrak{A} \subseteq R$ be any right

ideal, and consider the exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow R \longrightarrow R/\mathfrak{A} \longrightarrow 0 \quad \text{in } \mathfrak{M}_R.$$

Since $R/\mathfrak{A}$ is cyclic and hence f.p., (4.26)(b) implies that $\mathfrak{A}_R$ is f.g. This shows that the ring R is right noetherian. □

We now return to (4.23), (4.24), and give some further applications. First we have:

(4.30) Theorem. *Let P_R be a f.r. module over any ring R. Then P is flat iff it is projective.*

Proof. It suffices to prove the "only if" part, so assume P is flat. Fix an exact sequence

$$(4.31) \qquad 0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$$

as in (4.25)(a), where F is R-free (of any rank), and K is f.g., say, by $c_1, \ldots, c_n$. By (4.23), there exists a $\theta \in \operatorname{Hom}_R(F, K)$ with $\theta(c_i) = c_i$ for all i. Therefore, θ is the identity on K, and splits the exact sequence (4.31). This then implies that P_R is a projective module. □

Next we shall give a reformulation of the Equational Criteria for flat modules in (4.24), using the notion of f.p. modules.

(4.32) Theorem. *A module P_R is flat, iff for any R-homomorphism $\lambda : M \to P$ where M is any f.p. R-module, there exist R-homomorphisms $M \xrightarrow{\nu} R^m \xrightarrow{\mu} P$ (for some $m < \infty$) such that $\lambda = \mu \circ \nu$ (i.e., iff any λ factors through a f.g. free module).*

Proof. For the "only if" part, assume P is flat and let λ be given. The idea of M being f.p. is that M can be generated by a finite number of elements, say, $x_1, \ldots, x_n$, which are subject to a finite number of relations, say, $\sum_j x_j r_{j\ell} = 0$ $(1 \le \ell \le p)$. For $a_j = \lambda(x_j)$, we have then $\sum_j a_j r_{j\ell} = 0$. Since P is flat, we can find the $b_i \in P$ $(1 \le i \le m)$ and $s_{ij} \in R$ $(1 \le i \le m, 1 \le j \le n)$ as in (4.24)(3). Now let $e_1, \ldots, e_m$ be the standard basis of R^m and define $\mu : R^m \to P$ by $\mu(e_i) = b_i$, and $\nu : M \to R^m$ by $\nu(x_j) = \sum_i e_i s_{ij}$. The latter is well-defined, since

$$\nu\Big(\sum_j x_j r_{j\ell}\Big) = \sum_j \sum_i e_i s_{ij} r_{j\ell} = \sum_i e_i \sum_j s_{ij} r_{j\ell} = 0.$$

(This shows that the necessary relations on the x_j 's are "respected" by ν.) Now

$$\mu\nu(x_j) = \mu\Big(\sum_i e_i s_{ij}\Big) = \sum_i b_i s_{ij} = a_j = \lambda(x_j)$$

for all j, so $\mu\nu = \lambda$ on M. The "if" part is proved similarly. □

(4.33) Corollary. *A module P_R is flat iff, for any R-epimorphism $\varphi : Q \to P$ (where Q is any R-module) and any f.p. R-module M, any homomorphism $\lambda : M \to P$ can be "lifted" to some $\psi : M \to Q$.*

Proof. First assume P is flat, and consider φ, λ as above. By (4.32), we can "factor" λ into $M \xrightarrow{\nu} R^m \xrightarrow{\mu} P$ for some $m < \infty$. Since R^m is free, μ can be "lifted" to some $\delta : R^m \to Q$. Taking $\psi = \delta\nu : M \to Q$ we have $\varphi\psi = (\varphi\delta)\nu = \mu\nu = \lambda$, as desired. Conversely, assume the lifting condition. We shall check that P is flat by applying (the "if" part of) (4.32). Consider $\lambda : M \to P$, where M is any f.p. R-module. Fix an epimorphism $\varphi : F \to M$, where F is a suitable (possibly not f.g.) free module. By assumption, λ can be "lifted" to some $\nu : M \to F$. Since M is f.g., $\nu(M)$ is contained in *some* f.g. free submodule F_0 of F. Taking $\mu = \varphi | F_0$, we obtain a factorization $M \xrightarrow{\nu} F_0 \xrightarrow{\mu} P$ for λ, as desired. □

We are now in a position to prove the following remarkable characterization theorem for flat modules.

(4.34) Theorem (Lazard, Govorov). *A right module P_R is flat iff it is a direct limit of f.g. free modules.*

Proof. By (4.3) and (4.4), any direct limit of projective modules is flat. Thus, it suffices to prove the "only if" part of the theorem. For any module P, let F be the free module with basis $\{e_{(p,n)} : (p,n) \in P \times \mathbb{N}\}$, and let $\varphi : F \to P$ be defined by $\varphi(e_{(p,n)}) = p$. We shall define a directed set I as follows. An element $\alpha \in I$ is a pair $\alpha = (L_\alpha, K_\alpha)$, where L_α is a *finite* subset of $P \times \mathbb{N}$, and K_α is a f.g. submodule of $\ker \varphi$ lying in F_α, the free submodule of F with basis $\{e_{(p,n)} : (p,n) \in L_\alpha\}$. For $\alpha, \beta \in I$, we define $\alpha \le \beta$ if $L_\alpha \subseteq L_\beta$ and $K_\alpha \subseteq K_\beta$. Let P_α be the *finitely presented* module F_α / K_α. For $\alpha \le \beta$, we have a natural map $P_\alpha \to P_\beta$, so we get a direct system $\{P_\alpha : \alpha \in I\}$. The natural maps $P_\alpha \to P$ induce $\varinjlim P_\alpha \to P$, which is easily seen to be an isomorphism.[33] The main point of the proof is to show that, *if P is flat*, the set

(4.35) $$I_0 := \{\beta \in I : P_\beta \text{ is (f.g.) free}\} \textit{ is cofinal in } I,$$

for then P is the direct limit of the f.g. free modules $\{P_\beta : \beta \in I_0\}$. To prove (4.35), let $\alpha \in I$. By (4.32), $P_\alpha \to P$ has a factorization $P_\alpha \xrightarrow{\nu} R^m \xrightarrow{\mu} P$ for some $m < \infty$. We define a new "index" $\beta = (L_\beta, K_\beta) \in I$ as follows. Let $\{e_1, \ldots, e_m\}$ be a basis on R^m and let $p_i = \mu(e_i) \in P$. We take

$$L_\beta := L_\alpha \cup \{(p_1, n_1), \ \ldots \ , (p_m, n_m)\} \subseteq P \times \mathbb{N},$$

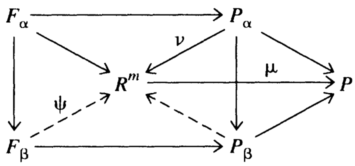
[33]The argument given so far shows that any module P is a direct limit of f.p. modules. For this, of course, we could have used $\{e_p : p \in P\}$ for the basis of F. The choice of $\{e_{(p,n)} : (p,n) \in P \times \mathbb{N}\}$ will be justified a bit later in the proof.

where $n_1, \ldots, n_m$ are chosen to be distinct and such that $(p_i, n_i) \notin L_\alpha$. Then define $\psi : F_\beta \to R^m$ by

$$\psi(p_i, n_i) = e_i \quad \text{and} \quad \psi(p, n) = \nu(e_{(p,n)} + K_\alpha) \ \text{ for } \ (p, n) \in L_\alpha.$$

Clearly, $\mu\psi = \varphi | F_\beta$, so ker $\psi \subseteq$ ker φ. Since ψ is an *epimorphism* onto R^m, it splits. Therefore, ker ψ is f.g., and it makes sense to define $K_\beta =$ ker ψ.

(4.36)



From the definition of ψ, we have $K_\alpha \subseteq K_\beta$, so $\alpha \le \beta$. Finally, $\beta \in I_0$, since $P_\beta = F_\beta / K_\beta \cong R^m$. □

§4E. *Finitely Generated Flat Modules*

We have shown earlier (in (4.30)) that any f.r. flat module is projective. This gives rise to the question: *is every f.g. flat module also projective*? The following example shows that the answer is "no" in general; see also Exercise 17.

(4.37) Example. Let R be any nonsemisimple von Neumann regular ring. Since R_R is not a semisimple module, there exists a right ideal $\mathfrak{A}$ that is not a direct summand of R_R. By (4.21), $P = (R/\mathfrak{A})_R$ is flat (and cyclic). But P is not projective, for otherwise $0 \to \mathfrak{A} \to R \to P \to 0$ would split in $\mathfrak{M}_R$.

Over certain rings, however, f.g. flat modules can be shown to be projective. We collect some results of this nature in the following theorem.

(4.38) Theorem. *Assume that the ring R satisfies one of the following conditions:*

(1) *R is a right noetherian ring.*
(2) *R is a local ring (see FC–§19).*
(3) *R is a domain satisfying the strong rank condition; that is, for any n, any set of $n+1$ elements in $(R^n)_R$ is linearly dependent.*

Then, any f.g. flat module P_R is projective.

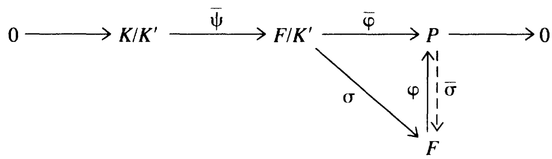
Proof. Under assumption (1), the assertion follows from (4.29) and (4.30). *Next, assume R is local,* and let $\mathfrak{m}$ be the maximal ideal of R. Choose $a_1, \ldots, a_n \in P$ such that $\overline{a}_1, \ldots, \overline{a}_n$ form a basis of $P/P\mathfrak{m}$ as a vector space over the division ring $R/\mathfrak{m}$. Then $P = P\mathfrak{m} + \sum a_i R$, and Nakayama's Lemma (*FC*–(4.22)) implies that $P = \sum a_i R$. Let F be a free (right) R-module with basis $e_1, \ldots, e_n$ and let $\varphi : F \to P$ be the epimorphism defined by $\varphi(e_i) = a_i$. If $\sum e_i r_i \in K :=$ ker φ, then $\sum \overline{a}_i r_i = 0 \in P/P\mathfrak{m}$ and hence all $r_i \in \mathfrak{m}$. This shows that $K \subseteq F\mathfrak{m}$. Let

$\mathfrak{A} = \sum Rr_i$. By (4.14),

(4.39) $$\sum e_i r_i \in K \cap F\mathfrak{A} = K\mathfrak{A} \subseteq F\mathfrak{m}\mathfrak{A}.$$

This shows that $\mathfrak{A} = \mathfrak{m}\mathfrak{A}$. Since ${}_R\mathfrak{A}$ is f.g., Nakayama's Lemma implies that $\mathfrak{A} = 0$. Therefore $K = 0$ and we have $P \cong F$.

Finally, assume that R is as in (3). Fix a short exact sequence $0 \to K \xrightarrow{\psi} F \xrightarrow{\varphi} P \to 0$, where $F \cong R^n$ $(n < \infty)$. Let m be the largest integer such that K contains a submodule $K' \cong R^m$. (Such an integer $m \geq 0$ exists by (3).) Then K/K' is a torsion module, in the sense that every element of K/K' is killed by some nonzero element of R. Consider the commutative diagram:



Here the existence of σ is given by (4.33) (noting that P is flat and F/K' is f.p.) But then $\sigma\overline{\psi} = 0$, since K/K' is torsion and F is torsion-free. Therefore, σ induces $\overline{\sigma} : P \to F$ such that $\sigma = \overline{\sigma}\,\overline{\varphi}$. Now $\varphi\overline{\sigma}\,\overline{\varphi} = \varphi\sigma = \overline{\varphi}$, so $\varphi\overline{\sigma} = 1_P$ (since $\overline{\varphi}$ is an epimorphism). Thus, φ splits, and this shows that P is projective. □

(4.40) Remark. The theorem above is folklore in Case (1), and an observation of Endo in Case (2). Case (3) covers in particular the case of commutative domains; the theorem in this latter case is a result of Cartier. In general, the domains satisfying the strong rank condition turn out to be precisely the right Ore domains to be studied later in §10 (see Exercise (10.21)). There are other classes of rings for which f.g. right flat modules are projective, e.g. semiperfect rings (generalizing local rings); see Exercise 21.

§4F. *Direct Products of Flat Modules*

The next topic of our discussion is the direct product of flat modules. In general, if P_i $(i \in I)$ are flat right R-modules, $P = \prod_i P_i$ need not be flat. But S. U. Chase has determined the rings R for which $P = \prod_i P_i$ is always flat (for arbitrary flat P_i 's): these rings are the *left coherent rings* to be defined below (note the switch from "right" to "left"). Because of its completeness and elegance, Chase's result has become a fixture in the modern treatment of flat modules. Our exposition of Chase's result below follows Goodearl [76].

We begin with a general discussion of direct products and tensor products. Let B_i $(i \in I)$ be right R-modules and A be a left R-module. We have a natural group homomorphism:

(4.41) $$\varepsilon : \left(\prod_i B_i\right) \otimes_R A \longrightarrow \prod_i (B_i \otimes_R A)$$

defined on the generators of the domain by

$$(\varepsilon(b \otimes a))_i = b_i \otimes a \qquad \big(b = (b_i)_{i \in I} \in \prod_i B_i\,;\;\; a \in A\big).$$

In the special case when each $B_i = R_R$, we can identify $B_i \otimes_R A$ with A as usual, and get a map

$$\delta:\; R^I \otimes_R A \longrightarrow A^I \tag{4.42}$$

defined by $b \otimes a \mapsto (b_i a)_{i \in I}$, for $b \in R^I$. Here, A^I denotes $\prod_{i \in I} A$, whose elements are "functions" from I to A.

In general, ε is not an isomorphism. For instance, for $R = \mathbb{Z}$, take $B_i = \mathbb{Z}/i\,\mathbb{Z}$ ($i \in \mathbb{N}$) and $A = \mathbb{Q}$. Then $\prod_i (B_i \otimes_R A) = (0)$ since B_i is torsion and A is divisible. However, we can find an embedding of $\mathbb{Z}$ into $\prod_i B_i$, so there is also an embedding of $\mathbb{Q}$ into $\big(\prod_i B_i\big) \otimes_{\mathbb{Z}} \mathbb{Q}$ (since $\mathbb{Q}$ is $\mathbb{Z}$-flat). Thus, the map ε has nonzero domain and zero range, so it is not injective. On the other hand, if we choose $B_i = \mathbb{Z}$ ($i \in \mathbb{N}$) and $A = \mathbb{Q}$, the map $\varepsilon : \big(\prod_i \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \prod_i \mathbb{Q}$ is not surjective, since, for any $(q_i)_{i \in I}$ ($q_i \in \mathbb{Q}$) to lie in $\operatorname{im}(\varepsilon)$, all q_i must lie in $\frac{1}{n}\mathbb{Z}$ for some "common denominator" n.

For a given left R-module A, the next two propositions determine exactly what is needed to guarantee that ε be surjective or bijective. (The corresponding condition for ε to be injective for all $\{B_i\}$ is more subtle, and will be left out.)

(4.43) Proposition. *For any left R-module, the following are equivalent:*

(1) *ε is surjective for all families of right R-modules $\{B_i\}$.*
(2) *δ is surjective for all indexing sets I.*
(3) *A is a f.g. R-module.*

Proof. (3)$\Longrightarrow$(1). Note that ε is an isomorphism in the special case when $A = R^n$. If A is f.g., fix an epimorphism $R^n \to A$. Comparing the ε-maps for R^n and for A in the form of a commutative diagram, we see quickly that (4.41) is surjective for A. (Cf. Exercise 10.)

(1)$\Longrightarrow$(2) is obvious, since δ is a special case of ε.

(2)$\Longrightarrow$(3). We shall only assume (2) for $I = A$, namely, that $\delta : R^A \otimes_R A \to A^A$ is surjective. In fact, we need only assume that $t \in \operatorname{im}(\delta)$, where t is the special element in A^A given by $t_a = a$ ($\forall a \in A$). Writing $t = \delta\big(\sum_{j=1}^n b_j \otimes a_j\big)$ for $b_j \in R^A$ and $a_j \in A$, we have for every $a \in A$:

$$a = t_a = \sum_j \big(\delta(b_j \otimes a_j)\big)_a = \sum_j (b_j)_a\, a_j,$$

so $A = \sum_{j=1}^n R a_j$. □

(4.44) Proposition. *For any left R-module A, the following are equivalent:*

(1) *ε is bijective for all families of right R-modules $\{B_i\}$.*
(2) *δ is bijective for all indexing sets I.*

(3) *A is a f.p. R-module.*

Proof. (3)$\Longrightarrow$(1). Fix a presentation $R^m \to R^n \to A \to 0$ in ${}_R\mathfrak{M}$. The conclusion (1) follows easily from the commutative diagram relating the ε-maps for R^m, R^n, and A, noting that (4.41) is an isomorphism for $A = R^m$, R^n and that the tensor product functor is right exact. (Cf. Exercise 10.)

(1)$\Longrightarrow$(2) is obvious as before.

(2)$\Longrightarrow$(3). Assuming (2), we know already from (4.43) that A is f.g. Fix a short exact sequence $0 \to K \to F \to A \to 0$ in ${}_R\mathfrak{M}$, where F is f.g. free. For any indexing set I, consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & R^I \otimes_R K & \longrightarrow & R^I \otimes_R F & \longrightarrow & R^I \otimes_R A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_K} & & \downarrow{\scriptstyle \delta_F} & & \downarrow{\scriptstyle \delta_A} & & \\ 0 & \longrightarrow & K^I & \longrightarrow & F^I & \longrightarrow & A^I & \longrightarrow & 0 \end{array}$$

whose rows are both exact. By assumption δ_A is bijective, and by (4.43) δ_F is surjective. An easy diagram chase shows that δ_K is also surjective. Since this holds for all I, (4.43) implies that K is f.g., so A is f.p. □

In preparation for Chase's Theorem, we shall now introduce the notion of a left coherent ring.

(4.45) Definition. A ring R is said to be *left coherent* if every f.g. left ideal of R is f.p. (as a left R-module). Right coherent rings are defined similarly, and, as usual, we say that R is *coherent* if it is both left coherent and right coherent.

(4.46) Examples.

(a) *If R is left noetherian, then it is left coherent.* This follows from (the left analogue of) (4.29).

(b) *If R is left semihereditary, then it is left coherent.* In fact, if $\mathfrak{A}$ is a f.g. left ideal, then ${}_R\mathfrak{A}$ is projective, and hence f.p. by Exercise 1(a). In particular, any (commutative) valuation ring is coherent, and any von Neumann regular ring is coherent.

(c) Given any cyclic left module $R \cdot a$, we have an exact sequence

$$0 \longrightarrow \text{ann}(a) \longrightarrow R \xrightarrow{\varphi} R \cdot a \longrightarrow 0 \qquad (\varphi(r) = ra).$$

Therefore, $R \cdot a$ is f.p. iff $\text{ann}(a)$ is f.g. as a left ideal. In particular, if R is left coherent, $\text{ann}_\ell(a)$ is a f.g. left ideal for any $a \in R$.

(d) Let us construct a ring R with an element $y \in R$ such that $\text{ann}_\ell(y)$ is not f.g. as a left ideal. Such an R will be an example of a non left coherent ring. Define R to be the $\mathbb{Q}$-algebra with generators y, x_1, x_2, ... and relations $x_i y = 0$ for all i. We can represent R as a direct sum $\bigoplus_{j=0}^{\infty} y^j \mathbb{Q}\langle X \rangle$, where $X = \{x_1, x_2, \dots\}$. Using this representation, we see easily that $\text{ann}_\ell(y) = \sum_i Rx_i$. This left ideal

is not f.g., since its image in $\mathbb{Q}\langle X\rangle$ (by specializing y to 0) is already not f.g. Therefore, the principal left ideal Ry is not f.p., and *the ring R is not left coherent.* For a commutative example, we can simply add the relations $yx_i = 0$, $x_ix_j = x_jx_i$ to R. The proof that the (commutative) quotient ring is not coherent is the same as above. (For another commutative example, see Exercise 17.)

(e) We construct here a right coherent (in fact right artinian) ring that is not left coherent. This example is taken from Bourbaki's *Algèbre*. Take K to be a field with a subfield L such that $\dim_L K = \infty$, and that there exists a field isomorphism $\varphi : K \to L$. (For instance, $K = \mathbb{Q}(x_1, x_2, x_3, \ldots)$, $L = \mathbb{Q}(x_2, x_3, \ldots)$.) We define a ring R by taking $R = K \times K$, with multiplication

$$(x,\ y)(x',\ y') = (xx',\ \varphi(x)y' + yx') \qquad (x,\ y,\ x',\ y' \in K).$$

This ring is simply a "trivial extension" of K by K in the sense of (2.22)(A), with K viewed as a (K, K)-bimodule where the left K-action on K is given by $x \cdot y' = \varphi(x)y'$. The identity element for R is (1,0). Let $a = (0,\ 1)$. Since $(x,\ y)a = (0, \varphi(x))$, we see that $\text{ann}_\ell(a) = (0,\ K)$. For any $z \in K$,

$$R \cdot (0,\ z) = \{(0,\ \varphi(x)z) :\ x \in K\} \cong Lz.$$

Since $\dim_L K = \infty$, it follows that the left ideal $\text{ann}_\ell(a) \subseteq R$ is not f.g. In particular, *R is not a left coherent ring.* On the other hand, it is easy to see[34] that R has exactly three *right* ideals, (0), R, and $(0,\ K)$. Therefore, R is right noetherian, right artinian, and in particular right coherent.

More examples of left/right coherent rings will be given later; see (4.62). Without further ado, we now give Chase's result.

(4.47) Theorem (Chase). *For any ring R, the following are equivalent:*

(1) *Any direct product of flat right R-modules is flat.*
(2) *For any indexing set I, $(R^I)_R$ is flat.*
(3) *The ring R is left coherent.*

Proof. (1)$\Longrightarrow$(2) is clear, since R_R is flat.

(2)$\Longrightarrow$(3). Let A be any f.g. left ideal in R. To show that A is f.p., it suffices (by (4.44)) to show that the map $\delta : R^I \otimes_R A \to A^I$ in (4.42) is bijective, for any indexing set I. By (4.43), we know already that δ is surjective, so we need only show that δ is *injective.* Consider the following factorization of δ:

(4.48) $$R^I \otimes_R A \longrightarrow R^I A \hookrightarrow A^I.$$

Since $(R^I)_R$ is flat by assumption, the first map is injective by (4.12). Therefore, the composition δ in (4.48) is also injective.

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[34]For $(x,\ y) \neq 0$, a direct calculation shows that $(x,\ y) \cdot R$ is R if $x \neq 0$, and is $(0,\ K)$ if $x = 0$.

(3)$\Longrightarrow$(1). Let $\{B_i : i \in I\}$ be flat right R-modules. To show that $\prod_i B_i$ is also flat, it suffices by (4.12) (the Modified Flatness Test), to show that the natural surjective map $\alpha : \left(\prod_i B_i\right) \otimes_R \mathfrak{A} \to \left(\prod_i B_i\right) \mathfrak{A}$ is injective for any f.g. left ideal $\mathfrak{A} \subseteq R$. Consider the following commutative diagram:

(4.49)
$$\begin{array}{ccc} \left(\prod_i B_i\right) \otimes_R \mathfrak{A} & \xrightarrow{\alpha} & \left(\prod_i B_i\right)\mathfrak{A} \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow \\ \prod_i \left(B_i \otimes_R \mathfrak{A}\right) & \xrightarrow{\beta} & \prod_i \left(B_i \mathfrak{A}\right) \end{array}$$

Since R is left coherent, ${}_R\mathfrak{A}$ is f.p., so ε is bijective by (4.44). By (4.12) again, β is also bijective. It follows from (4.49) that α is injective, as desired. □

Recalling (4.46)(a,b), we have the following immediate application of (4.47):

(4.50) Corollary. *Let R be either left noetherian or left semihereditary. Then any direct product of flat right R-modules is flat.*

§4G. Coherent Modules and Coherent Rings

The notion of "left coherence" for rings can be generalized to left modules over an arbitrary ring as follows.

(4.51) Definition. A f.g. left module ${}_RA$ is said to be *coherent* if every f.g. submodule of A is f.p. (Thus, a ring R is left coherent iff the left regular module ${}_RR$ is coherent.)

With this definition, we have the following supplement to Chase's Theorem.

(4.52) Corollary. *A ring R is left coherent iff any f.p. left R-module A is coherent.*

Proof. The "if" part is clear by taking A to be ${}_RR$. For the "only if" part, assume that R is left coherent. Let A be any f.p. left R-module, and let $A' \subseteq A$ be any f.g. submodule. For any indexing set I, we have the following commutative diagram:

(4.53)
$$\begin{array}{ccc} R^I \otimes_R A' & \xrightarrow{\delta_{A'}} & A'^I \\ {\scriptstyle \gamma}\downarrow & & \downarrow \\ R^I \otimes_R A & \xrightarrow{\delta_A} & A^I \end{array}$$

By (4.44), δ_A is bijective. By (4.47), $(R^I)_R$ is flat, so γ is injective. It follows that $\delta_{A'}$ is injective. By (4.43), $\delta_{A'}$ is also surjective. Therefore, $\delta_{A'}$ is *bijective*, and by (4.44) again, A' is f.p. This shows that the left R-module A is coherent. □

We want to offer a couple of interesting characterizations for coherent modules. To do this, we need the following two elementary lemmas on f.p. modules.

(4.54) Lemma. *Let* $0 \to M' \to M \to M'' \to 0$ *be exact in* ${}_R\mathfrak{M}$.

(1) *If* M *is f.p. and* M' *is f.g., then* M'' *is f.p.*
(2) *If* M' *and* M'' *are f.p., then* M *is f.p.*
(3) *A direct sum* $M_1 \oplus \cdots \oplus M_n$ *is f.p. iff each* M_i *is f.p.*

The proof of this Lemma is easy, and is left as an exercise (see Exercise 8).

(4.55) Lemma. *Let* A, B *be submodules of a left* R*-module* M.

(1) *Assume that* A, B *are f.p. Then* $A + B$ *is f.p. iff* $A \cap B$ *is f.g.*
(2) *Assume that* A *is f.p. and* $B = Rb$ *for some* $b \in M$. *Then* $A + B$ *is f.p. iff* $Ab^{-1} := \{r \in R : rb \in A\}$ *is a f.g. left ideal in* R.[35]

Proof. (1) Consider the exact sequence

$$(4.56) \qquad 0 \longrightarrow A \cap B \xrightarrow{\varphi} A \oplus B \xrightarrow{\psi} A + B \longrightarrow 0,$$

where $\varphi(a) = (a, a)$, $\psi(a, b) = a - b$. Here $A \oplus B$ is f.p. by (4.54)(3). If $A \cap B$ is f.g., $A + B$ is f.p. by (4.56) and (4.54)(1). Conversely, if $A + B$ is f.p., $A \cap B$ is f.g. by (4.56) and (4.26)(b).

(2) Consider the exact sequence

$$(4.57) \qquad 0 \longrightarrow A \longrightarrow A + B \longrightarrow R\bar{b} \longrightarrow 0,$$

where we have identified $(A + B)/A$ with $B/A \cap B = R\bar{b}$. Note that $\operatorname{ann}(\bar{b})$ is exactly Ab^{-1}. If this is f.g., then $R\bar{b}$ is f.p. by (4.46)(c), so $A + B$ is f.p. by (4.57) and (4.54)(2). Conversely, if $A + B$ is f.p., (4.57) and (4.54)(1) imply that $R\bar{b}$ is f.p., and (4.46)(c) implies that $\operatorname{ann}(\bar{b}) = Ab^{-1}$ is f.g. □

(4.58) Theorem. *For any f.g. left* R*-module* M, *the following are equivalent:*

(1) M *is coherent.*
(2) *For any* $b \in M$ *and any f.g. submodule* $A \subseteq M$, Ab^{-1} *is a f.g. left ideal.*
(3) $\operatorname{ann}(b)$ *is a f.g. left ideal for any* $b \in M$, *and the intersection of any two f.g. submodules of* M *is f.g.*

Proof. (1)⟹(3). The cyclic module $R \cdot a$ must be f.p., so $\operatorname{ann}(a)$ is a f.g. left ideal in R by (4.46)(c). Let A, B be f.g. submodules of M. Then A, B and $A + B$ are f.p., so by (4.55)(1), $A \cap B$ is f.g.

[35] Ab^{-1} is a somewhat informal notation, since b^{-1} itself is undefined.

(3)$\Longrightarrow$(1). We shall show, by induction on n, that any submodule

$$Ra_1 + \cdots + Ra_n + Rb \subseteq M$$

is f.p. If $n = 0$, this follows from (4.46)(c). For $n \geq 1$, let

$$A = Ra_1 + \cdots + Ra_n \quad \text{and} \quad B = Rb.$$

Then B is f.p. as above, and A is f.p. by the inductive hypothesis. By assumption $A \cap B$ is f.g., so (4.55)(1) implies that $A + B$ is f.p.

(1)$\Longrightarrow$(2). Here, the f.g. submodules A and $A + Rb$ must be f.p., so by (4.55)(2), Ab^{-1} is a f.g. left ideal.

(2)$\Longrightarrow$(1). The argument here is the same as in (3)$\Longrightarrow$(1). Just use (4.55)(2) instead of (4.55)(1), and note that $\text{ann}(b) = (0)b^{-1}$. □

(4.59) Corollary. *Let R be a domain. Then a f.g. torsion-free left R-module M is coherent iff the intersection of any two f.g. submodules of M is f.g.*

Theorem 4.58 is due to S. Chase in the case when M is the left regular module $_RR$. In this case, we get the following nice characterization of left coherent rings in terms of purely ideal-theoretic conditions.

(4.60) Corollary (Chase). *For any ring R, the following are equivalent:*

(1) *R is a left coherent ring;*

(2) *for any $b \in R$ and any f.g. left ideal $\mathfrak{A} \subseteq R$,*

$$\mathfrak{A}b^{-1} := \{r \in R : rb \in \mathfrak{A}\}$$

is a f.g. left ideal;

(3) $\text{ann}_\ell(b)$ *is a f.g. left ideal for any $b \in R$, and the intersection of any two f.g. left ideals in R is f.g.*

In particular, a domain R is left coherent iff the intersection of any two f.g. left ideals in R is f.g.

At this point, it behooves us to mention some more examples of left coherent rings.

(4.61) Examples.

(a) Let K be any field, and X be any (possibly infinite) set of commuting indeterminates. *Then the polynomial ring $R = K[X]$ is (left) coherent.* To see this, consider any f.g. (left) ideal $J = \sum_{i=1}^n Rf_i$. Choose a subset $\{x_1, \ldots, x_m\} \subseteq X$ such that all f_i lie in $S := K[x_1, \ldots, x_m]$. Then, for $\mathfrak{A} = \sum_{i=1}^n Sf_i$, we have $J = R\mathfrak{A}$. Since S is a noetherian ring (by the Hilbert Basis Theorem), we have a finite presentation $S^p \to S^q \to \mathfrak{A} \to 0$ in $_S\mathfrak{M}$. Tensoring this with R, we have an exact sequence

$$R^p \longrightarrow R^q \longrightarrow R \otimes_S \mathfrak{A} \longrightarrow 0 \quad \text{in } {}_R\mathfrak{M}. \tag{4.62}$$

Now R is a polynomial ring over S, so R_S is a free (and hence flat) module. By (4.12), we can identify $R \otimes_S \mathfrak{A}$ with $R\mathfrak{A} = J$, so (4.62) gives a finite presentation for J. The same argument works over any left noetherian ring K, for, by a more general form of the Hilbert Basis Theorem, K being left noetherian implies that $K[x_1, \ldots, x_m]$ is also left noetherian.

As a note of caution, however, we should point out that, in general, the left coherence of a ring K *does not* imply that of $K[x]$ for one variable x. In fact, if K is a countable direct product of the polynomial ring $\mathbb{Q}[y, z]$, then K is coherent but $K[x]$ is not coherent, according to a result of Soublin [70]. (In the positive direction, Nagata has shown that if K is a commutative *semihereditary* ring, then indeed $K[x_1, \ldots, x_n]$ is coherent for any n.)

(b) By what we said in (a) above, $\mathbb{Z}[X]$ is a coherent ring for any set of commuting variables X. Since any commutative ring is isomorphic to a quotient ring of some $\mathbb{Z}[X]$, and not every commutative ring is coherent (cf. last part of (4.46)(d)), we see that *a quotient ring of a coherent ring need not be coherent.*

(c) Let $\mathfrak{A}$ be an ideal in R which is f.g. as a left ideal. *If R is left coherent, then so is the quotient ring $\overline{R} := R/\mathfrak{A}$.* To see this, note that any f.g. left ideal of $\overline{R}$ has the form $\mathfrak{B}/\mathfrak{A}$, where $\mathfrak{B}$ is a f.g. left ideal of R. (This uses the fact ${}_R\mathfrak{A}$ is f.g.) Taking a finite presentation $R^m \to R^n \to \mathfrak{B} \to 0$ in ${}_R\mathfrak{M}$ and tensoring it with $R/\mathfrak{A}$, we get an exact sequence $\overline{R}^m \to \overline{R}^n \to \mathfrak{B}/\mathfrak{A}\mathfrak{B} \to 0$ in ${}_{\overline{R}}\mathfrak{M}$, so $\mathfrak{B}/\mathfrak{A}\mathfrak{B}$ is a f.p. left $\overline{R}$-module. Viewing

$$(4.63) \qquad 0 \longrightarrow \mathfrak{A}/\mathfrak{A}\mathfrak{B} \longrightarrow \mathfrak{B}/\mathfrak{A}\mathfrak{B} \longrightarrow \mathfrak{B}/\mathfrak{A} \longrightarrow 0$$

as an exact sequence in ${}_{\overline{R}}\mathfrak{M}$ and using the fact that $\mathfrak{A}/\mathfrak{A}\mathfrak{B}$ is f.g., we see (from (4.54)(1)) that $\mathfrak{B}/\mathfrak{A}$ is f.p. in ${}_{\overline{R}}\mathfrak{M}$. Thus, the ring $\overline{R}$ is left coherent. However, in general, *the left coherence of $\overline{R}$ need not imply the left coherence of R.* For instance, the commutative ring

$$R = \mathbb{Q}[y, x_1, x_2, \ldots]/(x_1 y, x_2 y, \ldots)$$

is not coherent by (the last part of) (4.46)(d); but for the principal ideal $\mathfrak{A} = Ry$, the quotient ring

$$\overline{R} = R/\mathfrak{A} \cong \mathbb{Q}[x_1, x_2, \ldots]$$

is coherent, by (a) above.

(d) Here, we modify the construction in (4.46)(d) to get an example of a noncommutative non-noetherian coherent ring. This example does not seem to have been pointed out before in the literature. Let R be the $\mathbb{Q}$-algebra with generators x, y and relation $xy = 0$, so that $R = \bigoplus_{i=0}^{\infty} y^i \mathbb{Q}[x]$. *We claim that R is left coherent.* To see this, first note that $yR \left(= \bigoplus_{i=1}^{\infty} y^i \mathbb{Q}[x]\right)$ and Rx are both ideals of R. Let A be the quotient ring R/Rx. As a left R-module, A is f.p. Let $\mathfrak{A}$ be any f.g. left ideal of R. To show that it is f.p., let us first treat the important special case $\mathfrak{A} \subseteq yR$. In this case, $x\mathfrak{A} = 0$, so $\mathfrak{A}$ may be viewed as a (f.g.) left module over A. Since $A \cong \mathbb{Q}[y]$ is a noetherian ring, there exists a finite presentation $A^m \to A^n \to \mathfrak{A} \to 0$ in ${}_A\mathfrak{M}$. But A^n and A^m are f.p. in ${}_R\mathfrak{M}$ (since A itself

is), so $\mathfrak{A}$ is also f.p. in ${}_R\mathfrak{M}$ (see (1) and (3) in (4.54)). It remains to treat the case when $\mathfrak{A}$ is *not* contained in yR. Let us write $\overline{R}$ for the quotient ring R/yR, which we identify with $\mathbb{Q}[x]$. Since this is a PID, $\overline{\mathfrak{A}} = \mathbb{Q}[x]f_0$ for some polynomial $f_0(x) \neq 0$. Therefore, $\mathfrak{A}$ contains an element $\alpha = f_0 + yr_0$ $(r_0 \in R)$. Let $\mathfrak{A}$ be generated by

$$\alpha_i = f_i + yr_i \qquad (f_i \in \mathbb{Q}[x],\ \ r_i \in R,\ \ 1 \leq i \leq n).$$

Then $f_i = h_i f_0$ for suitable $h_i \in \mathbb{Q}[x]$, and we have

$$\alpha_i = h_i(\alpha - yr_0) + yr_i = h_i\alpha + yr_i'$$

for some $r_i' \in R$. Therefore, $\mathfrak{A} = R\alpha + \mathfrak{A}_0$, where $\mathfrak{A}_0$ is the f.g. left ideal $\sum Ryr_i'$ in yR. Since $\alpha = f_0 + yr_0$ with $f_0 \neq 0$, we see easily that $\text{ann}_\ell(\alpha) = 0$, so $R\alpha \cong {}_RR$; in particular, it is f.p. On the other hand, $\mathfrak{A}_0$ and hence $\mathfrak{A}_0 \cap R\alpha$ are f.g. over the noetherian ring A. By the case we have already treated, $\mathfrak{A}_0$ and $\mathfrak{A}_0 \cap R\alpha$ are both f.p. in ${}_R\mathfrak{M}$. By (4.55)(1), it follows that $\mathfrak{A} = R\alpha + \mathfrak{A}_0$ is f.p. as well. This completes the proof that R is left coherent, and by left-right symmetry, R is right coherent as well. To produce a "less symmetric" example, take $S = R/y^2R = \mathbb{Q}\langle x, y\rangle$ with relations $xy = y^2 = 0$ (cf. *FC*–(1.26)). As in *FC*–(1.26), S is right noetherian and not left noetherian. By a simple modification of the argument above, we can show that S is nevertheless left coherent.[36]

§4H. *Semihereditary Rings Revisited*

As a byproduct of his investigations on direct products of flat modules, Chase has also obtained an interesting characterization of left semihereditary rings. This characterization is in terms of the notion of "torsionless" modules due to H. Bass.

(4.64) Definition. A right module B over R is said to be *torsionless* if B can be embedded into some direct product $(R^I)_R$.

(4.65) Remarks.

(a) Clearly, B_R is torsionless iff, for any $b \neq 0$ in B, there exists a functional $f \in B^* = \text{Hom}_R(B, R)$ such that $f(b) \neq 0$. Thus, B is torsionless iff the natural map $i : B \to B^{**}$ is injective. (Because of this characterization, torsionless modules are sometimes called *semi-reflexive*; recall that a module B is *reflexive* if $i : B \to B^{**}$ is an isomorphism.)

(b) Any submodule of a (right) free module is torsionless. Thus, any projective right module as well as any right ideal is torsionless.

(c) Clearly, any submodule of a torsionless module is torsionless, and any direct product (or direct sum) of torsionless modules is torsionless.

(d) Let R be any domain. Then any torsionless module B_R is torsion-free.

[36]It would have been nice if we could get this conclusion from (c). Unfortunately, (c) does not apply here since y^2R is not f.g. as a *left* ideal.

(e) Let R be a commutative domain, and B be a f.g. right R-module. If B is torsion-free, then B can be embedded into some R^m (by the paragraph preceding (2.31)), so B is torsionless. But in general, *a torsion-free module need not be torsionless.* For instance, over $R = \mathbb{Z}$, the module $B = \mathbb{Q}$ is torsion-free, but not torsionless.

(f) For any left module ${}_RA$, A^* has a natural structure as a right R-module. It can be shown that the natural map $A^* \to A^{***}$ is always a split monomorphism. In particular, *A^* is always a torsionless right R-module.* (For a more general perspective, see (19.38).)

Before we state Chase's characterization of left semihereditary rings, it is useful to first formulate the following preliminary result.

(4.66) Lemma. *For any ring R, the following are equivalent:*[37]

(1) *All right ideals of R are flat.*
(2) *All left ideals of R are flat.*
(3) *Submodules of flat right R-modules are flat.*
(4) *Submodules of flat left R-modules are flat.*

Proof. (1)$\Longleftrightarrow$(2). By (4.12) (and its analogue for left modules), we see that each of (1), (2) amounts to the condition that $\mathfrak{B} \otimes_R \mathfrak{A} \to \mathfrak{B}\,\mathfrak{A}$ is an isomorphism for all right ideals $\mathfrak{B}$ and left ideals $\mathfrak{A}$. We finish by showing (1)$\Longrightarrow$(3). (This will give (1)$\Longleftrightarrow$(3), and (2)$\Longleftrightarrow$(4) is similar.) Assume (1).

Step 1. *All submodules $M \subseteq (R^n)_R$ are flat.* We induct on n, the case $n = 1$ being covered by (1). Let π be the projection $R^n \to R$ given by the "first coordinate". We identify $\ker \pi$ with R^{n-1}, and get an exact sequence

$$0 \longrightarrow M \cap R^{n-1} \longrightarrow M \longrightarrow \pi(M) \longrightarrow 0.$$

Now apply (1) (to $\pi(M)$), the inductive hypothesis, and (4.13).

Step 2. *Any submodule M of a free R-module F_R is flat.* By (4.5), it suffices to show that any f.g. submodule $M_0 \subseteq M$ is flat. There exists a f.g. free submodule $F_0 \subseteq F$ such that $M_0 \subseteq F_0$. By Step 1, M_0 is indeed flat.

Step 3. *Any submodule N of a flat module P_R is flat.* Fix a short exact sequence $0 \longrightarrow K \longrightarrow F \xrightarrow{\varphi} P \longrightarrow 0$ where F is free. By Step 2, $M := \varphi^{-1}(N)$ is flat, and by (4.14), $K \cap F\mathfrak{A} = K\mathfrak{A}$ for any left ideal $\mathfrak{A} \subseteq R$. But then we also have

$$K \cap M\mathfrak{A} \subseteq K \cap F\mathfrak{A} \subseteq K\mathfrak{A},$$

[37] As it turns out, the rings R satisfying the conditions (1), (2), (3), (4) are precisely those with (left/right) weak dimension ≤ 1. See (5.52).

so $K \cap M\mathfrak{A} = K\mathfrak{A}$. Applying (4.14) now to $0 \to K \to M \xrightarrow{\varphi} N \to 0$, we conclude that N is flat. □

(4.67) Theorem (Chase). *For any ring R, the following are equivalent:*

(A) *R is left semihereditary.*
(B) *All torsionless right R-modules are flat.*
(C) *R is left coherent, and satisfies one (and hence all) of the conditions* (1), (2), (3), (4) *in* (4.66).

Proof. (A)$\Longrightarrow$(C) follows from (4.46)(b) and (the left analogue of) (4.6).

(C)$\Longrightarrow$(B). For any set I, $(R^I)_R$ is flat by (4.47), so by (3), any submodule of $(R^I)_R$ is flat. This gives (B).

(B)$\Longrightarrow$(A). From (B), it follows that $(R^I)_R$ is flat (for any I), so by (4.47) again, R is left coherent. Let $\mathfrak{B} \subseteq R$ be any f.g. left ideal. Then $\mathfrak{B}$ is f.p., and flat (since $\mathfrak{B}$ is torsionless). By (4.30), ${}_R\mathfrak{B}$ is projective, so we have proved (A). □

(4.68) Remark. In general, the conditions (A), (B) in (4.67) are stronger than the conditions (1), (2), (3), (4) in (4.66). In fact, take a ring R that is right semihereditary but not left semihereditary. (See (2.34): take R to be the opposite ring of the ring T constructed there.) By (4.6), R satisfies (1) (and hence (2), (3), (4)), but R does not satisfy (A) by choice. Being right semihereditary, R is also right coherent, but by (4.67), R cannot be left coherent.

In the case of commutative domains, it turns out that there is no difference among (A), (B), and (1), (2), (3), (4) after all. This is part of the following characterization theorem for Prüfer domains, which, incidentally, provides a converse to (4.20) (cf. also (2.31)).

(4.69) Theorem. *For any commutative domain R, the following are equivalent:*

(A) *R is a Prüfer domain.*
(B) *All torsionless (right) R-modules are flat.*
(D) *All torsionfree (right) R-modules are flat.*
(1) *All ideals are flat.*
(3) *Submodules of flat modules are flat.*
(5) *A, B torsionfree in $\mathfrak{M}_R \Longrightarrow A \otimes_R B$ is torsionfree.*
(6) *$\mathfrak{A}$, $\mathfrak{B}$ ideals in $R \Longrightarrow \mathfrak{B} \otimes_R \mathfrak{A}$ is torsionfree.*

Proof. We have (D)$\Longrightarrow$(B)$\Longrightarrow$(A)$\Longrightarrow$(1)$\Longrightarrow$(3), where each implication is either already known or else a tautology. To close this cycle, let us prove (3)$\Longrightarrow$(D). Consider any torsionfree R-module P. Any f.g. submodule P_0 of P can be embedded in some R^n (see the paragraph preceding (2.31)), so by (3), P_0 is flat. Since P is a direct limit of such P_0's, P is also flat.

For the rest of the proof (and for the statements (5), (6)), we use the fact that, over a commutative ring R, the tensor product of two R-modules is an R-module (in a natural way). Since the tensor product of two flat modules is flat (Exercise 18), and flat modules are torsionfree, we have (D)$\Longrightarrow$(5). Clearly, (5)$\Longrightarrow$(6), *so we can finish by showing* (6)$\Longrightarrow$(1). Let $\mathfrak{B}$ be any nonzero ideal. To show that $\mathfrak{B}$ is flat, it suffices to show that, for any ideal $\mathfrak{A}$, the multiplication map $\theta : \mathfrak{B} \otimes_R \mathfrak{A} \to \mathfrak{B}\mathfrak{A}$ is injective (cf. (4.12)). Let $x = \sum b_i \otimes a_i \in \ker(\theta)$, so $\sum b_i a_i = 0$. For $b \neq 0$ in $\mathfrak{B}$, we have then

$$bx = \sum bb_i \otimes a_i = b \otimes \sum b_i a_i = 0.$$

Since (by (6)) $\mathfrak{B} \otimes_R \mathfrak{A}$ is torsionfree, we have $x = 0$, so θ is injective, as desired. □

There are many more known criteria for a commutative domain R to be Prüfer. For instance, Bourbaki's "Commutative Algebra" (pp. 558-559, Addison-Wesley, 1972) listed a total of 14 characterizations for Prüfer domains (which did not include all of ours). Some of these, for example, are formulated in terms of localizations, Chinese Remainder Theorem, the distributive laws for ideals, and so forth. They are all of interest one way or another. In Theorem (4.69), we have settled with the five characterizations germane to this section, in terms of flatness, tensor products, and torsion-free modules.

In concluding our exposition of Chase's work on direct products and flat modules, we should point out that Chase has also characterized rings R over which any direct product of projective right R-modules is projective: these turn out to be precisely the rings that are left coherent and right perfect (and in the commutative case, these are just the artinian rings; see Exercise 22). However, we do not want to assume the material on right perfect rings from *FC*–§24, so we shall not present this theorem here.

§4I. Faithfully Flat Modules

In this subsection, we give a short introduction to faithfully flat modules. These modules are useful in ring theory and algebraic geometry for making certain "descent" arguments.

(4.70) Theorem. *For any right module P over a ring R, the following are equivalent:*

(1) *A sequence $M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$ in ${}_R\mathfrak{M}$ is exact iff*

$$P \otimes_R M' \to P \otimes_R M \to P \otimes_R M''$$

is exact.

(2) *P is flat, and for any left R-module M, $P \otimes_R M = 0 \Longrightarrow M = 0$.*

(3) *P is flat, and a homomorphism $M' \xrightarrow{\varphi} M''$ in ${}_R\mathfrak{M}$ is zero if the induced homomorphism $P \otimes_R M' \to P \otimes_R M''$ is zero.*

If any of these conditions holds, we say that P_R is faithfully flat.

Proof. (1)$\Longrightarrow$(2). The "only if" part of (1) says that "$P \otimes_R -$" is exact on ${}_R\mathfrak{M}$, so P is flat. Next, assume $P \otimes_R M = 0$, where M is a left R-module. Then

$$P \otimes_R (0) \longrightarrow P \otimes_R M \longrightarrow P \otimes_R (0)$$

is exact, so the "if" part of (1) implies that $0 \to M \to 0$ is exact; that is, $M = 0$.
(2)$\Longrightarrow$(3). Let $M = \varphi(M')$, and assume $1_P \otimes \varphi : P \otimes_R M' \to P \otimes_R M''$ is zero. Then the composition of

$$P \otimes_R M' \longrightarrow P \otimes_R M \longrightarrow P \otimes_R M''$$

is zero. Here the first map is onto, and the second map is one-one (since P_R is flat). Therefore, $P \otimes_R M$ must be (0), and (2) implies that $M = 0$; that is, $\varphi = 0$.
(3)$\Longrightarrow$(1). The "only if" part of (1) follows from the flatness of P_R, so it suffices to show the "if" part. Assume $P \otimes_R M' \to P \otimes_R M \to P \otimes_R M''$ is exact. By (3), $M' \xrightarrow{\psi\varphi} M''$ is zero. Writing $I = \text{im}(\varphi)$, $K = \ker(\psi)$, we have then $I \subseteq K$. Since $0 \to K \to M \to M''$ is exact, so is

$$0 \to P \otimes_R K \to P \otimes_R M \xrightarrow{1\otimes\psi} P \otimes_R M'';$$

and hence

$$P \otimes_R K = \ker(1 \otimes \psi) = \text{im}(1 \otimes \varphi) = P \otimes_R I \subseteq P \otimes_R M.$$

Now the natural map $\pi : K \to K/I$ induces

$$P \otimes_R K \to P \otimes_R (K/I) = (P \otimes_R K)/P \otimes_R I = 0.$$

By (3), $\pi = 0$; that is, $K = I$, so $M' \to M \to M''$ is exact. □

The next result gives a useful way to check that a flat module is faithfully flat, using the maximal left ideals of R.

(4.71) Proposition. *A flat right module P_R is faithfully flat iff $P\mathfrak{m} \neq P$ for any maximal left ideal $\mathfrak{m}$ of R.*

Proof. Assume P is faithfully flat, and let $\mathfrak{m}$ be any maximal left ideal. Since $R/\mathfrak{m} \neq 0$ in ${}_R\mathfrak{M}$, we have

$$0 \neq P \otimes_R (R/\mathfrak{m}) = P/P\mathfrak{m}, \quad \text{so } P \neq P\mathfrak{m}.$$

Now assume $P \neq P\mathfrak{m}$ for any maximal left ideal $\mathfrak{m}$, and let M be any nonzero left R-module. Fix a nonzero element $x \in M$. Then $R \cdot x \cong R/\mathfrak{A}$ for some left ideal $\mathfrak{A} \subseteq R$. Since $\mathfrak{A}$ is contained in some maximal left ideal, $P\mathfrak{A} \neq P$. By the flatness of P_R, $P \otimes_R M$ contains

$$P \otimes_R Rx \cong P \otimes_R (R/\mathfrak{A}) \cong P/P\mathfrak{A} \neq 0,$$

so $P \otimes_R M \neq 0$. By (4.70), P is faithfully flat. □

(4.72) Examples.

(1) Let R be any commutative ring. For any prime ideal $\mathfrak{p} \subset R$, the localization $R_\mathfrak{p}$ is R-flat. By (4.2), $P := \bigoplus_\mathfrak{p} R_\mathfrak{p}$ is also R-flat, where $\mathfrak{p}$ ranges over all prime (or maximal) ideals. We claim that P *is faithfully flat.* Indeed, if M is any R-module such that $P \otimes_R M = 0$, then

$$0 = \left(\bigoplus_\mathfrak{p} R_\mathfrak{p}\right) \otimes_R M \cong \bigoplus_\mathfrak{p} (R_\mathfrak{p} \otimes_R M) \cong \bigoplus_\mathfrak{p} M_\mathfrak{p} ,$$

so each localization $M_\mathfrak{p} = 0$. It is well known that this implies that $M = 0$, so by (4.70), P_R is faithfully flat.

(2) Over $R = \mathbb{Z}$, the module $\mathbb{Q}$ is flat, but not faithfully flat, since $\mathbb{Q} \otimes_\mathbb{Z} (\mathbb{Z}/2\mathbb{Z}) = 0$. In general, *a $\mathbb{Z}$-module P is faithfully flat iff P is torsionfree and $Pp \neq P$ for any prime number p* (use (4.71), and the paragraph after (4.20)). A similar statement holds over any (commutative) PID.

(3) Over any ring R, a nonzero free module F_R is always faithfully flat. In particular, any polynomial ring $R[X]$ (and any free R-ring $R\langle X \rangle$) is faithfully flat as right R-module.

(4) Over any ring R, if P_R is flat and P'_R is faithfully flat, then $P \oplus P'$ is faithfully flat; in particular, $P \oplus R_R$ is always faithfully flat.

(5) Let $P \neq (0)$ be a f.g. flat right module over a local ring $(R, \mathfrak{m})$. By Nakayama's Lemma (*FC*–(4.22)), $P\mathfrak{m} \neq P$. Since $\mathfrak{m}$ is the *only* maximal left ideal in R, (4.71) implies that P_R is faithfully flat.

(6) Let R be any simple artinian ring. Then any nonzero module P_R is faithfully flat. It suffices to show this in the case when P_R is simple. For such P, we have $R_R \cong P^n$ for some n. If M is a left R-module such that $P \otimes_R M = 0$, then

$$M \cong R \otimes_R M \cong (P \otimes_R M)^n = 0.$$

Since P_R is projective, it follows that P_R is faithfully flat.

(7) Let $\varphi : R \to S$ be a ring homomorphism, whereby we can view S as a left R-module. If P_R is faithfully flat over R, then $P \otimes_R S$ is faithfully flat over S. Indeed, by (4.1), $P \otimes_R S$ is S-flat, and, for any left S-module $A \neq 0$,

$$(P \otimes_R S) \otimes_S A \cong P \otimes_R A \neq 0.$$

(4.73) Proposition. *A faithfully flat module P_R is both faithful and flat.*

Proof. Suppose $Pr_0 = 0$ for some $r_0 \in R$. Consider the left R-module homomorphism $\varphi : R \to R$ defined by $\varphi(r) = rr_0$. The induced map $1 \otimes \varphi : P \otimes_R R \to P \otimes_R R$ is zero, since

$$(1 \otimes \varphi)(p \otimes r) = p \otimes rr_0 = (pr)r_0 \otimes 1 = 0$$

for every $p \in P$, $r \in R$. Therefore, $\varphi = 0$ by (4.70), so $r_0 = \varphi(1) = 0$. This shows that P_R is faithful. ☐

The example of $\mathbb{Q}$ over the ring $\mathbb{Z}$ in (4.72)(2) shows that a module may be faithful and flat, but not faithfully flat! Of course, $\mathbb{Q}$ is not $\mathbb{Z}$-projective. In general, *even a faithful projective right module P_R need not be faithfully flat.* For instance, let R be the direct product $\mathbb{Z} \times \mathbb{Z} \times \cdots$, and let P be the ideal $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ in R. Clearly P_R is faithful, and it is projective by (2.12C). However, we have $P^2 = P$, so for any maximal ideal $\mathfrak{m}$ of R containing P, we have $P\mathfrak{m} = P$. By (4.71), P_R is *not* faithfully flat.

Let $\varphi : R \to S$ be a ring homomorphism. Then S can be viewed as a right R-module via φ. It is of importance to understand the situation when S_R is a faithfully flat R-module. The following result gives a few characterizations for this condition.

(4.74) Theorem. *For $\varphi : R \to S$ as above, the following are equivalent:*

(1) *S_R is faithfully flat.*
(2) *S_R is flat, and for any left ideal $\mathfrak{A} \subseteq R$, $\varphi^{-1}(S\mathfrak{A}) = \mathfrak{A}$.*
(3) *S_R is flat, and for any maximal left ideal $\mathfrak{m} \subset R$, there exists a maximal left ideal $\mathfrak{m}' \subset S$ such that $\mathfrak{m} = \varphi^{-1}(\mathfrak{m}')$.*
(4) *φ is injective, and the right R-module $(S/\varphi(R))_R$ is flat.*

In case R and S are commutative rings, these are also equivalent to

(5) *S_R is flat, and for any prime ideal $\mathfrak{p} \subset R$, there exists a prime ideal $\mathfrak{p}' \subset S$ such that $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}')$.*

Proof. (1)$\Longrightarrow$(2). We need to show that $\overline{\varphi} : R/\mathfrak{A} \to S/S\mathfrak{A}$ is injective. Since S_R is faithfully flat, it suffices to show that

$$(4.75) \qquad 1 \otimes \overline{\varphi} : S \otimes_R (R/\mathfrak{A}) \longrightarrow S \otimes_R (S/S\mathfrak{A})$$

is injective. As usual, we can identify $S \otimes_R (R/\mathfrak{A})$ with $S/S\mathfrak{A}$. The map

$$S/S\mathfrak{A} \longrightarrow S \otimes_R (S/S\mathfrak{A})$$

is in fact a split injection, since we can define a left-inverse by sending $s \otimes (s' + S\mathfrak{A})$ to $ss' + S\mathfrak{A}$.

(2)$\Longrightarrow$(3). By (2), we have $S\mathfrak{m} \neq S$, so $S\mathfrak{m}$ is contained in a maximal left ideal $\mathfrak{m}'$ of S. Clearly, $\varphi^{-1}(\mathfrak{m}') = \mathfrak{m}$.

(3)$\Longrightarrow$(1). Let $\mathfrak{m}$ be any maximal left ideal of R. Choose $\mathfrak{m}'$ as in (3). Then $S\mathfrak{m} \subseteq \mathfrak{m}'$ implies that $S\mathfrak{m} \neq S$. By (4.71), S_R is faithfully flat.

(1)$\Longrightarrow$(4). Since S_R is faithful by (4.73), φ is clearly injective. Let us identify R with $\varphi(R)$. Using (1)$\Longrightarrow$(2), we have $R \cap S\mathfrak{A} = \mathfrak{A}$ for any left ideal $\mathfrak{A} \subseteq R$. Applying (4.14) to

$$(4.76) \qquad 0 \longrightarrow R \longrightarrow S \longrightarrow S/R \longrightarrow 0 \quad \text{in } \mathfrak{M}_R,$$

it follows that $(S/R)_R$ is flat.

(4)$\Longrightarrow$(2). In (4.76), R_R and $(S/R)_R$ are flat, so S_R is flat by (4.13). The second statement in (2) now follows from (4.14).

Finally, assume that R and S are both commutative.

(5)$\Longrightarrow$(3). Let $\mathfrak{m}$ be any maximal ideal in R. By (5), $\mathfrak{m} = \varphi^{-1}(\mathfrak{p}')$ for a suitable prime ideal $\mathfrak{p}' \subset S$. For any maximal ideal $\mathfrak{m}'$ of S containing $\mathfrak{p}'$, we have clearly $\varphi^{-1}(\mathfrak{m}') = \mathfrak{m}$.

(2)$\Longrightarrow$(5). Let $\mathfrak{p}$ be any prime ideal in R. By (2), we have $\mathfrak{p} = \varphi^{-1}(S\mathfrak{p})$. Let E be the multiplicative set $R \setminus \mathfrak{p}$. Then $\varphi(E)$ is a multiplicative set in S disjoint from $S\mathfrak{p}$. If $\mathfrak{p}'$ is an ideal containing $S\mathfrak{p}$ maximal with respect to being disjoint from $\varphi(E)$, then $\mathfrak{p}'$ is a prime ideal in S by a familiar argument in commutative algebra, and we clearly have $\varphi^{-1}(\mathfrak{p}') = \mathfrak{p}$, as desired. □

(4.77) Definition. If a ring homomorphism $\varphi : R \to S$ satisfies the conditions in (4.74), we shall say that φ is (right) faithfully flat, or that S is a *(right) faithfully flat extension* of R. (The word "extension" is justified here, since φ has to be injective by (4.74)(4).)

(4.78) Examples.

(1) Any polynomial extension $R \subseteq R[X]$ is a (right) faithfully flat extension.

(2) If R is a commutative noetherian ring, then, for any ideal $\mathfrak{A}$ in the Jacobson radical of R, the $\mathfrak{A}$-adic completion of R is a faithfully flat extension of R. For a proof of this, we refer the reader to Matsumura [86].

(3) Let R be any commutative coherent ring. Then the ring $S = \prod_{\mathfrak{m}} R_{\mathfrak{m}}$ ($\mathfrak{m}$ ranging over all maximal ideals of R) is a faithfully flat extension of R with respect to the natural map $\varphi : R \to S$. Here, the flatness of S as an R-module follows from (4.47). To see that S_R is *faithfully* flat, note that, for any maximal ideal $\mathfrak{m}' \subset R$:

$$\left(\prod_{\mathfrak{m}} R_{\mathfrak{m}}\right)\mathfrak{m}' \subseteq \prod_{\mathfrak{m}} (R_{\mathfrak{m}}\,\mathfrak{m}') \subsetneq \prod_{\mathfrak{m}} R_{\mathfrak{m}},$$

and apply (4.71).

Some key properties of faithfully flat extensions are given below in (4.79) and (4.80). In applications, these and other similar properties are used to carry out "descent" arguments (deducing information over R from information over S).

(4.79) Lemma. *Let $\varphi : R \to S$ be (right) faithfully flat. Then a left R-module M is f.g. (resp. f.p.) iff the left S-module $S \otimes_R M$ is f.g. (resp. f.p.).*

Proof. ("If" part.) First assume $S \otimes_R M$ is f.g. We can pick a finite number of S-module generators to be $\{1 \otimes m_i : 1 \le i \le n\}$. Let $M_0 = \sum R m_i \subseteq M$. Then $S \otimes_R M_0 \to S \otimes_R M \to 0$ is exact. Since S_R is faithfully flat, $M_0 \to M \to 0$ is exact, i.e. $M_0 = M$. Next, assume that $S \otimes_R M$ is f.p. By the above, ${}_RM$ is

f.g. Fix an exact sequence $0 \to K \to R^n \to M \to 0$ in ${}_R\mathfrak{M}$. By tensoring with S, we get an exact sequence

$$0 \to S \otimes_R K \to S \otimes_R R^n (\cong S^n) \to S \otimes_R M \to 0.$$

Since $S \otimes_R M$ is f.p., (4.26)(b) implies that $S \otimes_R K$ is f.g., and hence (by the first part again) ${}_RK$ is f.g. This shows that ${}_RM$ is f.p. □

(4.80) Proposition. *Let $\varphi : R \to S$ be (right) faithfully flat, and assume that $\varphi(R) \subseteq Z(S)$ (the center of S).*

(1) *Let $f : N \to M$ be a homomorphism in ${}_R\mathfrak{M}$, where M is f.p. Then f is a split surjection iff $1 \otimes f : S \otimes_R N \to S \otimes_R M$ is.*
(2) *A left R-module M is f.g. projective iff the left S-module $S \otimes_R M$ is f.g. projective.*
(3) *A left R-module M is flat (resp. faithfully flat) iff the left S-module $S \otimes_R M$ is flat (resp. faithfully flat).*

Proof. Since φ is injective, the hypothesis $\varphi(R) \subseteq Z(S)$ implies that R is commutative (and that S is an R-algebra). In particular, for R-modules M, N, the abelian groups $\mathrm{Hom}_R(M, N)$, $M \otimes_R N$ have natural R-module structures. This observation will be used freely below.

(1) Using Exercise 12 (for the third "$\Longleftrightarrow$"), we have:

$$\begin{aligned} f \text{ is a split surjection} &\Longleftrightarrow \mathrm{Hom}_R(M, N) \xrightarrow{f_*} \mathrm{Hom}_R(M, M) \text{ is onto} \\ &\Longleftrightarrow S \otimes_R \mathrm{Hom}_R(M, N) \to S \otimes_R \mathrm{Hom}_R(M, M) \text{ is onto} \\ &\Longleftrightarrow \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R M) \text{ is onto} \\ &\Longleftrightarrow S \otimes_R f \text{ is a split surjection.} \end{aligned}$$

(2) ("If" part.) Assume that $S \otimes_R M$ is f.g. projective. By Exercise 1(a), it is f.p., so by (4.79), ${}_RM$ itself is f.p. By (1), it follows that any epimorphism $N \to M$ in ${}_R\mathfrak{M}$ splits, so ${}_RM$ is f.g. projective.

(3) The "only if" parts follow from (the left module analogues of) (4.1) and (4.72)(7). For the converses, assume first $S \otimes_R M$ is flat. Let $A \to B$ be a monomorphism in $\mathfrak{M}_R$. Then $A \otimes_R S \to B \otimes_R S$ is a monomorphism in $\mathfrak{M}_S$, since ${}_RS$ is flat over R.[38] Tensoring with the flat left S-module $S \otimes_R M$, we get another monomorphism α in:

$$(4.81) \qquad \begin{array}{ccc} (A \otimes_R S) \otimes_S (S \otimes_R M) & \xrightarrow{\alpha} & (B \otimes_R S) \otimes_S (S \otimes_R M) \\ \cong\uparrow & & \cong\uparrow \\ S \otimes_R (A \otimes_R M) & \xrightarrow{\beta} & S \otimes_R (B \otimes_R M) \end{array}$$

[38] It does not matter much whether we view S as a left or a right R-module, since $\varphi(R) \subseteq Z(S)$.

Using the isomorphisms shown above, we see that β is also a monomorphism. Since S_R is *faithfully* flat, it follows that $A\otimes_R M \to B\otimes_R M$ is a monomorphism, so we have shown that ${}_RM$ is flat. Finally, assume that $S\otimes_R M$ is faithfully flat. For any $A_R \neq (0)$, we have $A\otimes_R S \neq (0)$ and hence $(A\otimes_R S)\otimes_S(S\otimes_R M) \neq (0)$. By the isomorphism on the left side of the above diagram, we have $S\otimes_R(A\otimes_R M) \neq (0)$, so in particular $A \otimes_R M \neq (0)$. Since we already know ${}_RM$ is flat, this shows that it is *faithfully* flat. □

(4.82) Remark. The example $\mathbb{Z} \subset \mathbb{Q}$ shows that the "if" parts of (1), (2), (3) Proposition (4.80) need not hold if $R \subseteq S$ is only assumed to be a flat extension (instead of a *faithfully flat* extension).

§4J. Pure Exact Sequences

The following notion of a pure short exact sequence is closely related to the notion of flatness and will be developed in some detail in this subsection.[39]

(4.83) Definition. A (short) exact sequence $\mathcal{E} : 0 \to A \xrightarrow{\varphi} B \to C \to 0$ in $\mathfrak{M}_R$ is said to be *pure (exact)* if $\mathcal{E} \otimes_R C'$ is an exact sequence (of abelian groups) for any left R-module C'. (Of course, only the *injectivity* of $A \otimes_R C' \to B \otimes_R C'$ is at stake.) If this is the case, we say that $\varphi(A)$ is a *pure submodule* of B (or that B is a *pure extension* of $\varphi(A)$).

(4.84) Examples.

(a) Any split short exact sequence is pure.

(b) Any direct sum of pure exact sequences is pure.

(c) More generally, the direct limit of any direct system of pure short exact sequences is pure exact. (The proof results from the fact that tensor product commutes with direct limits.) In particular, the direct limit of any direct system of split short exact sequences is pure exact. We shall show later in this subsection that *any* pure short exact sequence arises in this way.

(d) For any family of right R-modules $\{B_i\}$ $(i \in I)$, $\bigoplus_{i\in I} B_i$ is always a pure submodule of $\prod_{i\in I} B_i$. Indeed, for any left R-module C', we have a commutative diagram

$$\begin{array}{ccc} (\prod_i B_i) \otimes_R C' & \xrightarrow{\varepsilon} & \prod_i (B_i \otimes_R C') \\ {\scriptstyle\alpha}\uparrow & & \uparrow \\ (\bigoplus_i B_i) \otimes_R C' & \xrightarrow{\cong} & \bigoplus_i (B_i \otimes_R C') \end{array}$$

where ε is defined as in (4.41). This clearly implies that α is an injection.

[39] My thanks go to I. Emmanouil whose notes on pure exact sequences were most helpful to me in the writing of this subsection.

(e) Let $A \subseteq B \subseteq D$ be right R-modules. If A is pure in D, then A is also pure in B. Conversely, if A is pure in B and B is pure in D, then A is pure in D. The proofs are straightforward by using Def. (4.83). (For more information, see Exercises (30) and (31).)

(f) Let $f : R \to S$ be a ring homomorphism, whereby S is viewed as a left R-module. If $\mathcal{E}$ is pure exact in $\mathfrak{M}_R$, then $\mathcal{E} \otimes_R S$ is pure exact in $\mathfrak{M}_S$. Again, the proof is routine by using the definition of purity.

(g) If R is a von Neumann regular ring, then any short exact sequence in $\mathfrak{M}_R$ is pure. The converse is true also. See Exercise 29.

The basic relationship between flat modules and pure exact sequences is given in the following theorem.

(4.85) Theorem. *A right R-module C is flat iff any short exact sequence*

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathfrak{M}_R$ is pure.

Proof. First assume C is flat. Let C' be any left R-module, and fix an exact sequence $\mathcal{E}' : 0 \to A' \to B' \to C' \to 0$ in ${}_R\mathfrak{M}$, where B' is free. Since C is flat, $C \otimes_R \mathcal{E}'$ is exact. By *FC*–(24.22)(1) it follows that $\mathcal{E} \otimes_R C'$ is exact, so we have proved that $\mathcal{E}$ is pure. Conversely, assume that C_R is such that any exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{M}_R$ is pure. Fix such a sequence $\mathcal{E}$ with B free, and let

$$\mathcal{E}' : 0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

be any exact sequence in ${}_R\mathfrak{M}$. By assumption, $\mathcal{E} \otimes_R C'$ is exact, so by *FC*–(24.22)(2), $C \otimes_R \mathcal{E}'$ is exact. This shows that C is flat. □

The Theorem we just proved deserves to be in §4A. However, we have promised to make the first few subsections of §4 independent of *FC*–§24. Since the proof above depends on *FC*–(24.22) (actually the proof of *FC*–(24.22) is only a straightforward diagram chase), we have reserved (4.85) for this subsection.

(4.86) Corollary. *Let $\mathcal{E} : 0 \to A \to B \to C \to 0$ be exact in $\mathfrak{M}_R$.*

(1) *Assume B is flat. Then $\mathcal{E}$ is pure iff C is flat.*
(2) *Assume C is flat. Then B is flat iff A is flat.*

Proof. (1) The "if" part follows from the Theorem (and does not require B to be flat). For the "only if" part, assume $\mathcal{E}$ is pure. Repeating the second half of the proof of (4.85), we see that $C \otimes_R \mathcal{E}'$ is exact for any exact sequence $\mathcal{E}'$ in ${}_R\mathfrak{M}$, so C is flat.

(2) Let $M \to N$ be any monomorphism in ${}_R\mathfrak{M}$. We have a commutative diagram:

$$
(4.87)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes_R M & \longrightarrow & B \otimes_R M & \longrightarrow & C \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A \otimes_R N & \longrightarrow & B \otimes_R N & \longrightarrow & C \otimes_R N & \longrightarrow & 0
\end{array}
$$

Here, both rows are exact since the flatness of C implies that $\mathcal{E}$ is pure (by (4.85)). But also, the flatness of C implies that γ is injective. A simple diagram chase in (4.87) shows that β is injective iff α is injective. (2) follows immediately from this observation. (A somewhat different proof for the "if" part of (2) was given earlier in (4.13)). □

(4.88) Example. There do exist pure short exact sequences $0 \to A \to B \to C \to 0$ that are not split, and in which A, B, C are all non-flat. For instance, over $R = \mathbb{Z}$, take $B_0 = \mathbb{Z} \times \mathbb{Z} \times \cdots$, $A_0 = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, and $C_0 = B_0/A_0$. Here, A_0, B_0, C_0 are torsion-free, and hence $\mathbb{Z}$-flat (by (4.20)). In particular, $\mathcal{E}_0 : 0 \to A_0 \to B_0 \to C_0 \to 0$ is pure.[40] However, $\mathcal{E}_0$ is non-split. (In fact, any homomorphism $g : C_0 \to B_0$ must kill $x = (2, 2^2, 2^3, \ldots) + A_0$ since $x \in 2^i C_0$ for any $i \geq 1$.) If $\mathcal{E}_1$ denotes the direct sum exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

then $\mathcal{E}_0 \oplus \mathcal{E}_1$ remains pure, non-split, and the three modules involved in this new sequence are now non-flat.

We now come to the main characterization theorem for pure exact sequences, due to Cohn, Fieldhouse, Warfield, and others. One of the most appealing conditions characterizing purity of $A \subseteq B$ is (3) below in terms of solving inhomogeneous systems of linear equations. Another interesting characterization, (6), is in terms of direct limits of *split* exact sequences.

(4.89) Theorem. *For any short exact sequence $\mathcal{E} : 0 \to A \hookrightarrow B \to C \to 0$ in $\mathfrak{M}_R$, the following are equivalent:*

(1) *$\mathcal{E}$ is pure exact.*
(2) *$\mathcal{E} \otimes_R C'$ is exact for any f.p. left R-module C'.*
(3) *If $a_j \in A$ $(1 \leq j \leq n)$, $b_i \in B$ $(1 \leq i \leq m)$ and $s_{ij} \in R$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ are given such that $a_j = \sum_i b_i s_{ij}$ for all j, then there exist $a_i' \in A$ $(1 \leq i \leq m)$ such that $a_j = \sum_i a_i' s_{ij}$ for all j.*

[40] Of course, we could have seen this more directly by using (4.84)(d).

(4) *Given any commutative diagram (in $\mathfrak{M}_R$):*

$$(4.90)\qquad \begin{array}{ccc} R^n & \xrightarrow{\ \sigma\ } & R^m \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ A & \hookrightarrow & B \end{array} \qquad (m, n < \infty)$$

there exists $\theta \in \mathrm{Hom}_R(R^m, A)$ *such that* $\theta\sigma = \alpha$.

(5) $\mathrm{Hom}_R(M, \mathcal{E})$ *is exact for any f.p. right* R*-module* M *(i.e., for any such* M*, any homomorphism* γ *from* M *to* C *can be lifted to a homomorphism* λ *from* M *to* B*).*

(6) $\mathcal{E}$ *is the direct limit of a direct system of split exact sequences*

$$0 \longrightarrow A \longrightarrow B_i \longrightarrow C_i \longrightarrow 0 \quad (i \in I),$$

where the C_i*'s are f.p. right* R*-modules.*

Before we proceed to the proof of the theorem, a word of caution is in order. In several textbooks, some of the characterizing conditions for purity in this theorem have been incorrectly stated. In Rotman [79: p. 95], the condition (4) was stated in terms of the existence of a $\theta \in \mathrm{Hom}_R(R^m, A)$ "making the diagram (4.90) commutative." This is not the case, as we only require the *upper triangle* to be commutative. Similarly, the formulation of the linear equations condition (3) in the form "$a_j \in \sum_i Bs_{ij} \Longrightarrow a_j \in \sum_i As_{ij}$" in Rowen [89: p. 338] is not correct. (Why?) Finally, we should note that, in contrast to (2), (5) is *not* equivalent to $\mathrm{Hom}_R(N, \mathcal{E})$ being exact for every N_R. In fact, the latter condition in the special case $N = C$ would have already implied that $\mathcal{E}$ splits!

Proof of (4.89). (1)$\Longrightarrow$(2) is a tautology.

(2)$\Longrightarrow$(3). Given the equations $a_j = \sum_i b_i s_{ij}$ as in (3), we define K to be the submodule of ${}_RR^n = \bigoplus_{j=1}^n Re_j$ generated by $\{\sum_j s_{ij}e_j : 1 \le i \le m\}$. By (2), the map

$$\rho : A \otimes_R (R^n/K) \longrightarrow B \otimes_R (R^n/K)$$

is injective. We shall identify $B \otimes_R (R^n/K)$ as a quotient of $B \otimes_R R^n$ by $\mathrm{im}(B \otimes_R K)$ (and likewise for $A \otimes_R (R^n/K)$). Then

$$\rho\left(\overline{\sum_j a_j \otimes e_j}\right) = \sum_j \sum_i \overline{b_i s_{ij} \otimes e_j} = \sum_i \overline{b_i \otimes \sum_j s_{ij} e_j} = 0,$$

so $\sum_j a_j \otimes e_j \in \mathrm{im}(A \otimes_R K)$. This means that, for some $a_1', \ldots, a_m' \in A$:

$$\sum_j a_j \otimes e_j = \sum_i a_i' \otimes \sum_j s_{ij} e_j = \sum_j \left(\sum_i a_i' s_{ij}\right) \otimes e_j .$$

Since $A \otimes_R R^n = \bigoplus_j (A \otimes e_j)$, it follows that $a_j = \sum_i a_i' s_{ij}$ for all j, as desired.

(3)$\Longrightarrow$(4) is straightforward.

(4)$\Longrightarrow$(5). Fix a finite presentation $R^n \xrightarrow{\sigma} R^m \xrightarrow{\tau} M \to 0$ in $\mathfrak{M}_R$. We can construct a commutative diagram:

$$\begin{array}{ccccccccc} & & R^n & \xrightarrow{\sigma} & R^m & \xrightarrow{\tau} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

(First construct β by using the freeness of R^m; then note that

$$\psi\beta\sigma = \gamma\tau\sigma = 0 \Longrightarrow \beta\sigma(R^n) \subseteq A,$$

and take $\alpha = \beta\sigma$.) By (4), there exists $\theta : R^m \to A \subseteq B$ such that $\theta\sigma = \alpha$. For $\beta' = \beta - \theta : R^m \to B$, we have

$$\beta'\sigma = \beta\sigma - \theta\sigma = \beta\sigma - \alpha = 0,$$

so there exists $\lambda : M \to B$ such that $\beta' = \lambda\tau$. But then

$$\psi\lambda\tau = \psi\beta' = \psi(\beta - \theta) = \psi\beta = \gamma\tau,$$

and the fact that τ is an epimorphism implies $\psi\lambda = \gamma$, as desired.

(5)$\Longrightarrow$(6). Represent C as a direct limit $\varinjlim C_i$ where the C_i 's are f.p. right R-modules (see the footnote to the proof of (4.34)). Consider the commutative diagram with exact rows:

$$\begin{array}{cccccccccc} \mathcal{E}_i: & 0 & \longrightarrow & A & \longrightarrow & B_i & \xrightarrow{\psi_i} & C_i & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\beta_i} & & \downarrow{\scriptstyle\gamma_i} & & \\ \mathcal{E}: & 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \end{array}$$

where $B_i = \{(x, y) \in B \oplus C_i : \psi(x) = \gamma_i(y)\}$ is the pullback of ψ and γ_i. By (5), γ_i can be lifted to a homomorphism $\lambda_i : C_i \to B$. The rule $\rho_i(y) = (\lambda_i(y), y)$ for $y \in C_i$ now defines a splitting $\rho_i : C_i \to B_i$ for ψ_i. Thus, each $\mathcal{E}_i$ is a *split* exact sequence. It is easy to show that $\mathcal{E} \cong \varinjlim \mathcal{E}_i$, so we have proved (6).

(6)$\Longrightarrow$(1) follows from (4.84)(c). □

(4.91) Corollary. *If C is a f.p. right R-module, then an exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathfrak{M}_R$ is pure iff it is split. In particular, if R is a right noetherian ring and B_R is f.g., then the pure submodules of B are just its direct summands.

(4.92) Corollary. *If A is a pure submodule of B_R, then $A \cap B\mathfrak{A} = A\mathfrak{A}$ for any left ideal $\mathfrak{A} \subseteq R$.*

Proof. This follows by applying (4.89)(3) for $n = 1$. (Or more directly, note that $A \cap B\mathfrak{A} = A\mathfrak{A}$ amounts to the injectivity of $A \otimes_R (R/\mathfrak{A}) \to B \otimes_R (R/\mathfrak{A})$.) □

For certain classes of rings, we also have a converse to (4.92).

(4.93) Corollary. *Let R be a commutative PID, and $A \subseteq B$ be right R-modules. Then A is pure in B iff $A \cap Br = Ar$ for any $r \in R$.*

Proof. ("If" part.) Assume that $A \cap Br = Ar$ for all $r \in R$. Then $A \to B$ remains injective upon tensoring by any cyclic R-module. Since any f.g. R-module is a direct sum of cyclic modules, it follows that $A \to B$ remains injective upon tensoring by any f.g. R-module. Therefore, A is pure in B by (1)$\Longleftrightarrow$(2) in (4.89). (A slight modification of the argument also shows that A is pure in B iff $A \cap Bp^n = Ap^n$ for any $n \geq 1$ and any nonzero prime element p of R.) □

In Rowen's "Ring Theory, I" [88: p. 338], there is an exercise asking the reader to prove the converse of (4.92) in general. Unfortunately, the converse of (4.92) is false, even over a commutative noetherian ring. We shall conclude this subsection by recording a counterexample to this effect, shown to me by I. Emmanouil.

For any nonzero commutative ring R, let $\mathcal{F}$ be the family of nonzero ideals of R. To begin with, we consider any $R \neq (0)$ with the property that $\bigcap_{\mathfrak{B} \in \mathcal{F}} \mathfrak{B} = 0$.[41] Let

(4.94) $$A = R \quad \text{and} \quad B = \prod_{\mathfrak{B} \in \mathcal{F}} R/\mathfrak{B} \ \text{in } \mathfrak{M}_R,$$

and let $i : A \to B$ be the natural map (given by the various projections from R to $R/\mathfrak{B}$). The assumption we made on $\mathcal{F}$ means that i is injective. We claim that:

(4.95) *The injectivity of $i : A \to B$ is preserved by tensoring with $R/\mathfrak{A}$, where $\mathfrak{A} \subseteq R$ is any ideal.*

Indeed, assuming $\mathfrak{A} \neq 0$ (as we may) and taking the projection map from B to $R/\mathfrak{A}$, we get a map

$$B \otimes_R \frac{R}{\mathfrak{A}} \longrightarrow \frac{R}{\mathfrak{A}} \otimes_R \frac{R}{\mathfrak{A}} \cong \frac{R}{\mathfrak{A}} \cong A \otimes_R \frac{R}{\mathfrak{A}}$$

in $\mathfrak{M}_R$. Therefore, $i \otimes_R (R/\mathfrak{A})$ is in fact a split monomorphism. Note that, if we view i as an inclusion of A in B, (4.95) means exactly that $A \cap B\mathfrak{A} = A\mathfrak{A}$ for any ideal $\mathfrak{A} \subseteq R$.

Next we want to show that A *need not be pure in* B. This is to be accomplished via the following observation.

[41] In the setting of *FC*–§12, these are the "subdirectly reducible rings". However, it is not necessary to bring in this terminology here.

(4.96) *Suppose N is a f.p. left R-module such that $N_0 := \bigcap_{\mathfrak{B}\in\mathcal{F}} \mathfrak{B}N \neq 0$. Then the injectivity of $i : A \to B$ is lost upon tensoring with N.*

In fact, by (4.44), we can identify $B \otimes_R N$ with $\prod ((R/\mathfrak{B}) \otimes N) \cong \prod N/\mathfrak{B}N$. If we also identify $A \otimes_R N$ with N, then $i \otimes N$ has clearly kernel $N_0 \neq 0$.

To complete our construction, we need only come up with a module N as in (4.96), over a suitable ring R. Take, for instance, the 3-dimensional commutative algebra

(4.97) $$R = k[x, y]/(x^2, xy, y^2) = k \oplus kx \oplus ky$$

over any field k. The condition $\bigcap_{\mathfrak{B}\in\mathcal{F}} \mathfrak{B} = 0$ is clear since kx and ky are ideals with $kx \cap ky = 0$. Now take $N = k \oplus k \oplus k$ with R-action well-defined by:

(4.98) $$x(a, b, c) = (0, 0, a) \quad \text{and} \quad y(a, b, c) = (0, 0, b).$$

Of course ${}_RN$ is f.p. To check the rest of (4.96), it suffices to show that $(0, 0, 1) \in \mathfrak{B}N$ for any nonzero ideal $\mathfrak{B} \subseteq R$. Fix an element $\beta = a + bx + cy \in \mathfrak{B}\backslash\{0\}$. If $a \neq 0$, we have $x = a^{-1}x\beta \in \mathfrak{B}$, so $(0, 0, 1) = x(1, 0, 0) \in \mathfrak{B}N$. Now assume $a = 0$. If $c = 0$, then $x = b^{-1}\beta \in \mathfrak{B}$ and we are done as before. If $c \neq 0$, then $\beta(0, 1, 0) = (0, 0, c)$, so again $(0, 0, 1) = (c^{-1}\beta)(0, 1, 0) \in \mathfrak{B}N$. (In fact, this computation shows that the N_0 in (4.96) is precisely $0 \oplus 0 \oplus k \subset N$.)

The choice of B as $\prod_{\mathfrak{B}\in\mathcal{F}} R/\mathfrak{B}$ above should not give the impression that we get a counterexample to the converse of (4.92) only by taking B to be a "large" R-module. In fact, if we take the algebra R in (4.97) over a finite field k, then $\mathcal{F}$ is a finite family, and A, B are both R-modules with finite cardinalities.

Although the converse of (4.92) is false in general, it turns out to be true if B is a *projective* module. This is a result of Fieldhouse; see Exercise 41.

Exercises for §4

1. For any ring R, show that

   (a) every f.g. projective right R-module is f.p., and that
   (b) every f.p. right R-module is projective iff R is von Neumann regular.

2. Prove the following slight generalization of (4.5): If every f.g. submodule P_0 of a module P_R is contained in a flat submodule P_1 of P, then P itself is flat.

3. In a ring theory text, the following statement appeared: "A module is flat iff every f.g. submodule is flat." Give a counterexample to the "only if" part of this statement. (The "if" part is true by (4.4).)

4. In a ring theory text, the following statement appeared: "If $0 \to C \to Q \to P \to 0$ is exact with C and Q f.g., then P is f.p." Give a counterexample.

5. In a ring theory text, the following statement appeared: "For right R-modules $N \subseteq M$, if $N \cap Mr = Nr$ for every $r \in R$, then $N \cap M\mathfrak{A} = N\mathfrak{A}$ for every left ideal $\mathfrak{A} \subseteq R$." Give a counterexample.

6. (a) Let M, N be submodules of a module E such that $M + N$ is flat. Show that $M \cap N$ is flat iff M and N are both flat.
(b) Give an example of a flat module with two submodules M, N such that M, N, $M \cap N$ are all flat, but $M + N$ is not flat.

7. Show that $\mathbb{Q}$ is isomorphic to a direct summand of $G = \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \times \cdots$.

8. Prove (1), (2), and (3) in (4.54).

9. Let $0 \to M' \to M \to M'' \to 0$ be exact in ${}_R\mathfrak{M}$. If M, M'' are f.p., and M' is f.g., is M' necessarily f.p.?

10. Let F_1, F_2 be left exact contravariant additive functors from ${}_R\mathfrak{M}$ to abelian groups, and let $\theta : F_1 \to F_2$ be a natural transformation. If $\theta(R) : F_1(R) \to F_2(R)$ is a monomorphism (resp. isomorphism), show that $\theta(M) : F_1(M) \to F_2(M)$ is also a monomorphism (resp. isomorphism) for every f.g. (resp. f.p.) module ${}_RM$. State and prove the analogue of this for right exact covariant additive functors.

11. Recall that, for arbitrary right R-module P and M, there exists a natural map $\sigma_{M,P} : P \otimes_R M^* \to \mathrm{Hom}_R(M, P)$, where $M^* := \mathrm{Hom}_R(M, R)$ is viewed, as usual, as a left R-module (see Exercise (2.20)).

(1) Assume that P is flat and M is f.p. Show that $\sigma_{M,P}$ is an isomorphism. Using this, give another proof for the "only if" part of (4.32). (**Hint.** Viewing P as fixed, define $F_1(M) = P \otimes_R M^*$, $F_2(M) = \mathrm{Hom}_R(M, P)$, and apply Ex. 10.)
(2) Show that $\sigma_{M,P}$ is also an isomorphism if we assume, instead, that M is projective and P is f.p.

12. Let $\varphi : R \to R'$ be a ring homomorphism. Assume that R is commutative, $\varphi(R)$ is in the center of R', and that R' is a flat R-module via φ. Let M be a f.g. (resp. f.p.) left R-module. Show that, for any left R-module N, the natural map

$$\theta_{M,N} : R' \otimes_R \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_{R'}(R' \otimes_R M, R' \otimes_R N)$$

is a monomorphism (resp. isomorphism).

13. Let $\mathcal{E} : 0 \to A \to B \to C \to 0$ be an exact sequence in $\mathfrak{M}_R$. Assume that R is commutative and C is f.p. Show that $\mathcal{E}$ is split iff the localization of $\mathcal{E}$ at every maximal ideal is split. Does this remain true if C is not f.p.?

14. Over a commutative ring R, show that a module P is flat iff, for any maximal ideal $\mathfrak{m} \subset R$, the localization $P_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$.

15. Show that the following are equivalent for any f.g. module P over a commutative ring R: (1) P is flat; (2) for any maximal ideal $\mathfrak{m} \subset R$, $P_{\mathfrak{m}}$

is free over $R_{\mathfrak{m}}$. If P is f.p., show that (1) and (2) are further equivalent to P being projective.

16. (Vasconcelos) Let P be a f.g. flat module over a *commutative* ring R. Define the nth *invariant factor* of P to be $I_n(P) = \text{ann}\big(\bigwedge^n(P)\big)$, where $\bigwedge^n(P)$ denotes the nth exterior power of P. Let rk P : Spec $R \to \mathbb{Z}$ be the *rank function* of P, as defined in Exercise (2.21). Show that:

    (1) For any $\mathfrak{p} \in \text{Spec } R$, $(\text{rk } P)(\mathfrak{p}) \geq n$ iff $I_n(P) \subseteq \mathfrak{p}$.
    (2) $\{\mathfrak{p} \in \text{Spec } R : (\text{rk } P)(\mathfrak{p}) = n\} = V(I_n(P)) \backslash V(I_{n+1}(P))$, where $V(\mathfrak{A})$ denotes the Zariski closed set $\{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{A}\}$ in Spec R.
    (3) For any n and any prime $\mathfrak{p}$, $I_n(P)_{\mathfrak{p}}$ is either (0) or $R_{\mathfrak{p}}$. Using this, show that $I_n(P)^2 = I_n(P)$.
    (4) Show that P is projective iff $I_n(P)$ is f.g. for all n.
    (5) Show that, if R has no nontrivial idempotent ideals, any f.g. flat module P_R is projective.
    (6) Deduce from (4) that a f.g. ideal $P \subseteq R$ is projective iff P is flat and ann(P) is f.g.

17. (Vasconcelos) Construct a principal ideal $P = aR$ in a commutative ring R such that P is flat but not projective, as follows. Let $R_0 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \cdots$, viewed as a (commutative) ring without 1, with addition and multiplication defined componentwise. Let $R = \mathbb{Z} \oplus R_0$ be the ring obtained by adjoining an identity $1 \in \mathbb{Z}$ to R_0. For $a = (2, 0) \in \mathbb{Z} \oplus R_0 = R$, show that

    (1) the principal ideal $P = aR$ is not f.p. (so R is not coherent), and
    (2) P is flat but not projective.

18. Show that, over a commutative ring, the tensor product of any two flat (resp. faithfully flat) modules is flat (resp. faithfully flat).

19. Let P_R be a flat right module and ${}_RM$ be a left module with submodules M_1, M_2. Show that $P \otimes_R (M_1 \cap M_2) = (P \otimes_R M_1) \cap (P \otimes_R M_2)$ in $P \otimes_R M$.

20. Let P_R be a projective module, and K be a submodule of rad P (the intersection of maximal submodules of P; see *FC*–(24.3)). If P/K is flat, show that $K = 0$.

21. (This problem, due to H. Bass, assumes familiarity with the class of semiperfect rings introduced in *FC*–§23.) Let R be a semiperfect ring. Use Exercise 20 to show that any f.g. flat module M_R is projective.

22. (This problem, due to S. Chase, assumes familiarity with the class of right perfect rings introduced in *FC*–§23.) Let R be a commutative ring. Show that R is coherent and perfect iff R is artinian. (**Sketch** (for "only if"). Assume R is coherent and perfect. By *FC*–(23.11), one may assume R is local, say, with maximal ideal $\mathfrak{m}$. Since R satisfies DCC on principal ideals by *FC*–(23.20), it has a minimal ideal $\mathfrak{A}$. Thus $R/\mathfrak{m} \cong \mathfrak{A}$ is f.p.,

whence $\mathfrak{m}$ is f.g. Since $\mathfrak{m}$ is nil, it must then be nilpotent. Each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, being f.g. and semisimple, has finite length, so R_R has finite length.)

23. Let J be a f.g. left ideal in a left coherent ring R. For any finite set $A \subseteq R$, show that $K = \{r \in R : rA \subseteq J\}$ is a f.g. left ideal. From this, conclude that, for any two f.g. ideals I, J in a commutative coherent ring, $(J : I) = \{r \in R : rI \subseteq J\}$ is also a f.g. ideal.

24. In an algebra text, the following statement appeared: "A direct sum $\bigoplus_{i \in I} M_i$ of R-modules is faithfully flat iff each M_i is flat and at least one of the M_i 's is faithfully flat." Give a counterexample to the "only if" part of this statement. (**Hint.** Use (4.72)(1).)

25. For any ring extension $R \subseteq S$, show that the following are equivalent:
(1) $R \subseteq S$ is a (right) faithfully flat extension.
(2) S is a pure, flat extension of R in $\mathfrak{M}_R$.
(3) For any system of linear equations

$$\sum_{i=1}^{m} x_i b_{ij} = a_j \qquad (a_j,\, b_{ij} \in R,\ \ 1 \leq j \leq n),$$

any solution $(s_1, \ldots, s_m) \in S^m$ can be expressed in the form $s_i = r_i + \sum_k t_k c_{ki}$, where $(r_1, \ldots, r_m)$ is a solution of the system in R^m, and for each k, $t_k \in S$ and $(c_{k1}, \ldots, c_{km})$ is a solution of the associated homogeneous system $\sum_{i=1}^{m} x_i b_{ij} = 0$ in R^m.

26. Let $\mathcal{E} : 0 \to A \to B \to C \to 0$ be an exact sequence in $\mathfrak{M}_R$, where A, C are flat and one of them is faithfully flat. Show that B must be faithfully flat.

27. Let $R \subseteq S$ be a (right) faithfully flat extension. If S is left noetherian (resp. artinian), show that R is also left noetherian (resp. artinian).

28. Deduce the characterization of flat modules in (4.33) from the characterization of pure exact sequences in (4.89)(5). (**Hint.** Use (4.85).)

29. Show that a ring R is von Neumann regular iff all short exact sequences in $\mathfrak{M}_R$ are pure, iff all right ideals are pure in R_R. (**Hint.** Use (4.21), (4.85), and (4.86).)

30. Let $K \subseteq A \subseteq B$ be right R-modules. Show that (1) if A is pure in B, then A/K is pure in B/K, and (2) if we assume K is pure in B, the converse of (1) also holds.

31. Let A, A' be submodules of a module B_R.
(1) If $A + A'$ and $A \cap A'$ are pure in B, show that A and A' are pure in B.
(2) If A, A' are pure in B and $A + A'$ is flat, show that $A \cap A'$ is pure in B.
(3) Construct an example of $A, A' \subseteq B$ such that $A, A', A + A'$ are all pure in B, but $A \cap A'$ is not.

(4) Construct an example of A, $A' \subseteq B$ such that A, A', $A \cap A'$ are all pure in B, but $A + A'$ is not.

32. Let A be a submodule of a f.p. module B_R. Show that A is a direct summand of B iff A is f.g. and pure in B.

33. Show that, over any domain R, a right ideal $\mathfrak{B}$ is pure in R_R iff $\mathfrak{B}$ is (0) or R.

34. Over a commutative ring R, show that $A \subseteq B$ is pure (in $\mathfrak{M}_R$) iff $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ is pure (in $\mathfrak{M}_{R_{\mathfrak{m}}}$) for every maximal ideal $\mathfrak{m} \subset R$.

35. (Prüfer) Show that a subgroup A of an abelian group B is pure iff any coset β with respect to the subgroup A contains an element b whose order ($\leq \infty$) equals the order of β in B/A.

36. For a module B_R over a commutative domain R, let B_t denote its torsion submodule $\{b \in B : br = 0 \text{ for some } r \in R\backslash\{0\}\}$. Is B_t always a pure submodule of B?

37. Let B be an additive abelian group, viewed as a $\mathbb{Z}$-module. *True or False*: the (pure) torsion subgroup B_t is always a direct summand of B?

38. A monomorphism $\varphi : A \to B$ in $\mathfrak{M}_R$ is said to be *locally split* if, for any $a \in A$, there exists $\sigma \in \mathrm{Hom}_R(B, A)$ such that $\sigma(\varphi(a)) = a$. In this case, an argument used in the last part of the proof of (4.23) shows that, for any $a_1, \ldots, a_n \in A$, there exists $\sigma \in \mathrm{Hom}_R(B, A)$ such that $\sigma(\varphi(a_i)) = a_i$ for all i. Using this, show that if φ is locally split, then $0 \to A \xrightarrow{\varphi} B \to B/A \to 0$ is pure. (**Hint.** Check condition (3) in (4.89).)

39. (Azumaya) An epimorphism $\psi : B \to C$ in $\mathfrak{M}_R$ is said to be *locally split* if, for any $c \in C$, there exists $\tau \in \mathrm{Hom}_R(C, B)$ such that $\psi\tau(c) = c$. In this case, prove the following statements.

(1) For any $c_1, \ldots, c_n \in C$, there exists $\tau_n \in \mathrm{Hom}_R(C, B)$ such that $\psi\tau_n(c_i) = c_i$ for $1 \leq i \leq n$.
(2) For any countably generated submodules $D \subseteq C$, the epimorphism $\psi^{-1}(D) \to D$ induced by ψ is split. (In particular, if C itself is countably generated, then ψ is already split.)

(3) The short exact sequence $\mathcal{E} : 0 \to \ker\psi \to B \xrightarrow{\psi} C \to 0$ is pure.

(**Hint.** For (1), construct τ_n 's by induction on n. For (3), check that condition (5) in (4.89) holds here for any finitely (or even countably) generated module M.)

40. Let $\mathcal{E} : 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be exact in $\mathfrak{M}_R$.

(1) If φ is locally split, does it follow that ψ is locally split?
(2) If $\mathcal{E}$ is pure, does it follow that one of φ, ψ is locally split?

41. (Fieldhouse) For any submodule A of a projective module B_R, show that the following are equivalent:

(1) The inclusion map $A \hookrightarrow B$ is locally split.
(2) A is pure in B.
(3) $A \cap B\mathfrak{A} = A\mathfrak{A}$ for any left ideal $\mathfrak{A} \subseteq R$.

(**Hint** (for (3)$\Longrightarrow$(1)). Take B' such that $F := B \oplus B'$ is free, and consider the projection $\pi : F \to B$.)

42. Let $\mathcal{E} : 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be a short exact sequence in $\mathfrak{M}_R$ where B is projective.

    (1) If A is f.g. and pure in B, show that A is a direct summand of B (and hence also a projective R-module).
    (2) If ψ is locally split, show that φ is also locally split.

43. (Zimmermann-Huisgen) Let R be a left noetherian ring, and C be an arbitrary direct product R^I, viewed as a *right* R-module. For any f.g. submodule $D \subseteq C$, show that there exists $\rho \in \text{Aut}(C_R)$ such that $\rho(D) \subseteq R^J$ for some *finite* subset $J \subseteq I$, where, by R^J, we mean the direct summand of R^I consisting of $(x_i)_{i \in I}$ with $x_i = 0$ for all $i \notin J$.

44. (Zimmermann-Huisgen) Let $\mathcal{E} : 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be a short exact sequence of right R-modules over a left noetherian ring R. If C is a direct product $(R^I)_R$ where I is any set, show that ψ is locally split.

45. Use Exercises (42) and (44) to construct a short exact sequence

$$\mathcal{E} : 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

over some ring R in which φ and ψ are both locally split, but $\mathcal{E}$ itself is not split.

46. Construct a short exact sequence $\mathcal{E} : 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ where ψ is locally split, but φ is not.

47. Let C be a right R-module and M, N be left R-modules. Let

$$\mathcal{F} : 0 \longrightarrow C' \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in ${}_R\mathfrak{M}$, where $C' = \text{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$ is the character module of C. Show that if $\mathcal{F}$ is pure, then it splits.

48. For any exact sequence $\mathcal{E} : 0 \to A \to B \to C \to 0$ in $\mathfrak{M}_R$, show that the following are equivalent: (1) $\mathcal{E}$ is pure; (2) $\mathcal{E}'$ is pure; (3) $\mathcal{E}'$ is split.

49. Show that any right R-module A is naturally embedded in A'' as a pure submodule.

50. Let R be a (commutative) UFD, and let x, y be two nonzero elements of R with $\gcd(x, y) = 1$. If the ideal $\mathfrak{A} = xR + yR$ is flat, show that $\mathfrak{A} = R$.

51. Let R be a (commutative) UFD. Show that R is a PID iff all ideals of R are flat, iff all torsion-free R-modules are flat.

§5. Homological Dimensions

§5A. Schanuel's Lemma and Projective Dimensions

After discussing the notions of projective, injective and flat modules, it is natural to include a treatment of homological dimensions of modules in general. Therefore, we devote the present section §5 to an introduction to the theory of *projective dimension*, *injective dimension*, and *flat dimension* of modules. This theory leads us to the definition of some new numerical invariants of rings, called their *global dimensions*. There are several kinds of global dimensions, and right global dimensions need not always be the same as left global dimensions. These homological invariants offer new tools with which to study (both commutative and noncommutative) rings, and by now there is a rather extensive literature on this theory.

Due to limitation of space, our coverage of homological dimensions of modules will be primarily limited to a discussion of the main definitions, some basic examples, and a selection of principal results. In particular, in order to avoid a long digression on the theory of derived functors, we shall develop the definition of homological dimensions without using "Ext" or "Tor", but rather using the projective, injective, and flat "shift operators". In doing so, we follow closely the exposition in Kaplansky [72]. However, to minimize the overlap, we shall offer a few different proofs.

(5.1) Schanuel's Lemma. *Let M be a right module over any ring R, and let*

$$0 \longrightarrow K \longrightarrow P \stackrel{\alpha}{\longrightarrow} M \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow Q \stackrel{\beta}{\longrightarrow} M \longrightarrow 0$$

be short exact sequences in $\mathfrak{M}_R$, where P_R is projective. Then there exists a short exact sequence

$$(*) \qquad 0 \longrightarrow K \longrightarrow L \oplus P \longrightarrow Q \longrightarrow 0.$$

In particular, if Q_R is also projective, then we have $K \oplus Q \cong L \oplus P$ in $\mathfrak{M}_R$.

Proof. We shall prove the existence of $(*)$ in two ways. First, let

$$X = \{(p, q) \in P \oplus Q : \alpha(p) = \beta(q)\},$$

which is a submodule of $P \oplus Q$. The map $X \stackrel{\pi_1}{\longrightarrow} P$ obtained by first coordinate projection is surjective. In fact, for any $p \in P$, there exists $q \in Q$ such that $\beta(q) = \alpha(p)$ since β is surjective. This gives $(p, q) \in X$ with $\pi_1(p, q) = p$. Next,

$$\begin{aligned} \ker \pi_1 &= \{(0, q) : (0, q) \in X\} \\ &= \{(0, q) : \beta(q) = 0\} \\ &\cong \ker \beta \cong L. \end{aligned}$$

Therefore, we have an exact sequence

$$0 \longrightarrow L \longrightarrow X \xrightarrow{\pi_1} P \longrightarrow 0, \tag{5.2}$$

and similarly, we have

$$0 \longrightarrow K \longrightarrow X \xrightarrow{\pi_2} Q \longrightarrow 0. \tag{5.3}$$

Since P is projective, (5.2) splits and we have $X \cong L \oplus P$. Replacing X by $L \oplus P$ in (5.3) yields $(*)$.

The second proof is essentially a reformulation of the first, but it yields a more explicit construction of $(*)$. To simplify the notation, it is convenient to think of K (resp. L) as a submodule of P (resp. Q). By the projectivity of P, we can find $\gamma : P \to Q$ with $\alpha = \beta \circ \gamma$. Then we construct

$$0 \longrightarrow K \xrightarrow{\psi} L \oplus P \xrightarrow{\varphi} Q \longrightarrow 0 \tag{5.4}$$

by $\varphi(\ell,\, p) = \gamma(p) - \ell$, and $\psi(k) = (\gamma(k),\, k)$. A routine check shows that (5.4) is a short exact sequence. □

Schanuel's Lemma was conceived by S. Schanuel when I. Kaplansky taught a course on homological ring theory at the University of Chicago in the Fall of 1958. The circumstances of discovery are vividly described by the following words of Kaplansky [72: p. 165]: "Early in the course I formed a one-step projective resolution of a module, and remarked that if the kernel was projective in one resolution it was projective in all. I added that, although the statement was so simple and straightforward, it would be a while before we proved it. Steve Schanuel spoke up and told me and the class that it was quite easy, and thereupon sketched what has come to be known as "Schanuel's Lemma". It took a couple of days and a half-dozen conversations before the proof was fully in hand."

The following long exact sequence version of Schanuel's Lemma is just an easy self-strengthening of (5.1).

(5.5) Corollary. *Suppose we have long exact sequences*

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\alpha} M \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \xrightarrow{\beta} M \longrightarrow 0$$

in $\mathfrak{M}_R$, where the P_i's and Q_i's are projective modules. Then

$$K \oplus Q_{n-1} \oplus P_{n-2} \oplus Q_{n-3} \oplus \cdots \cong L \oplus P_{n-1} \oplus Q_{n-2} \oplus P_{n-3} \oplus \cdots . \tag{5.6}$$

Proof. Let $K' = \ker \alpha$ and $L' = \ker \beta$. By (5.1), $K' \oplus Q_0 \cong L' \oplus P_0$. We can form the following new (shorter) exact sequences:

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \oplus Q_0 \longrightarrow K' \oplus Q_0 \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow Q_1 \oplus P_0 \longrightarrow L' \oplus P_0 \longrightarrow 0,$$

where the "middle" modules are all projective. Invoking an inductive hypothesis at this point, we get the desired conclusion (5.6). □

Schanuel's Lemma suggests naturally the following definition. For $A, B \in \mathfrak{M}_R$, we say that A, B are *projectively equivalent* (written $A \sim B$) if there exist projective modules $P, Q \in \mathfrak{M}_R$ such that $A \oplus P \cong B \oplus Q$. We verify quickly that " $\sim$ " is an equivalence relation. Let us denote the equivalence class of A by $[A]$, and write G for the set of all such equivalence classes.

The binary operation on G given by $[A] + [B] = [A \oplus B]$ is easily checked to be well-defined, and it has the following properties.

(5.7) Proposition. *$(G, +)$ is an abelian semigroup with identity (henceforth written 0) given by the class of the zero module. A class $[P]$ has an inverse in G iff $[P] = 0$, iff P is a projective module.*

The proof is completely routine and is therefore left to the reader.

For any $M \in \mathfrak{M}_R$, we can define a class $\mathcal{P}(M) \in G$ as follows. Take any short exact sequence

$$(*) \qquad 0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is projective,[42] and take $\mathcal{P}(M) = [K] \in G$. Schanuel's Lemma says exactly that this is well-defined.

(5.8) Proposition. *Let $M, N \in \mathfrak{M}_R$. Then*

(1) *$\mathcal{P}(M \oplus N) = \mathcal{P}(M) + \mathcal{P}(N) \in G$.*
(2) *$\mathcal{P}(M)$ depends only on $[M]$ (so it is legitimate to write $\mathcal{P}(M)$ as $\mathcal{P}[M]$).*
(3) *$\mathcal{P} : G \to G$ is an endomorphism of the semigroup G. (We call $\mathcal{P}$ the projective shift on right R-modules.)*

Proof. (1) follows by taking the direct sum of two short exact sequences of the type $(*)$, one for M and one for N. For (2), assume that $[M] = [M']$, so we have $M \oplus P \cong M' \oplus Q$ for some projective modules P, Q. By (1) we have $\mathcal{P}(M) + \mathcal{P}(P) = \mathcal{P}(M') + \mathcal{P}(Q) \in G$, and hence $\mathcal{P}(M) = \mathcal{P}(M')$, since $\mathcal{P}(P) = \mathcal{P}(Q) = 0$. Finally (3) also follows from (1). □

The iterates $\mathcal{P}^n[M]$ $(n \geq 0)$ can be found in one step as follows. Let

$$(5.9) \qquad \cdots \longrightarrow P_n \xrightarrow{\alpha_n} P_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

be a long exact sequence where the P_n's are projective modules. (We say that (5.9) is a *projective resolution* for M.) Let $K_n = \operatorname{im}(\alpha_n)$ $(n \geq 0)$; by induction,

[42] Of course such an exact sequence exists. For instance, we can take P to be a big free module that maps onto M.

it follows immediately that $\mathcal{P}^n[M] = [K_n]$ for all n. We say that K_n (or, more precisely, the class $[K_n]$) is the nth *syzygy* of the module M. The fact that the class $[K_n]$ is uniquely determined by M and n can, of course, also be seen directly from (5.5).

(5.10) Definition. The *projective dimension* of $M \in \mathfrak{M}_R$ (or of the class $[M]$) is defined to be:

$$\operatorname{pd}(M) = \operatorname{pd}_R(M) = \min\{n : \mathcal{P}^n[M] = 0\}.$$

If no such n exists, we define $\operatorname{pd}_R(M)$ to be ∞. (Clearly, $\operatorname{pd}(M) = 0$ iff M is a projective module.)

(5.11) Proposition. *For $M \in \mathfrak{M}_R$ and $n \geq 0$, the following statements are equivalent:*

(1) $\operatorname{pd}_R(M) \leq n$.

(2) *For any "partial" projective resolution*

$$P_{n-1} \xrightarrow{\alpha_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

the kernel of α_{n-1} is projective.[43]

(3) *There exists a finite projective resolution*

$$0 \longrightarrow P_n \xrightarrow{\alpha_n} P_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Moreover, for $n \geq 1$, we have $\operatorname{pd}_R(M) = n$ iff there exists a finite projective resolution as in (3) *where α_n is nonsplit.*

Proof. (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(1) all follow from the remarks made before Definition (5.10). To prove the last statement suppose $\operatorname{pd}_R(M) = n \geq 1$, and take a finite projective resolution as in (3). Then, for $K_{n-1} := \operatorname{im}(\alpha_{n-1})$,

$$(*) \qquad 0 \longrightarrow P_n \xrightarrow{\alpha_n} P_{n-1} \xrightarrow{\alpha_{n-1}} K_{n-1} \longrightarrow 0$$

is nonsplit, for otherwise K_{n-1} is projective and we would have $\mathcal{P}^{n-1}[M] = [K_{n-1}] = 0$. Conversely, if there is a finite projective resolution as in (3) where α_n is nonsplit, then, in $(*)$, K_{n-1} cannot be projective and we'll have $\mathcal{P}^{n-1}[M] = [K_{n-1}] \neq 0$. This implies that $\operatorname{pd}_R(M) = n$. ☐

For later reference, it is useful to note that, for any $M \in \mathfrak{M}_R$ which is not projective, we have always

$$(5.12) \qquad \operatorname{pd}_R(M) = 1 + \operatorname{pd}_R(\mathcal{P}(M)).$$

(5.13) Definition. The *right global dimension* of a ring R is defined to be

$$\text{r.gl.dim}\, R = \sup\{\operatorname{pd}_R(M) : M \in \mathfrak{M}_R\} \leq \infty.$$

[43] In the case $n = 0$, (2) should be interpreted to mean simply "M is projective".

In the case when this supremum is finite, it is precisely the *index of nilpotency* of the shift operator $\mathcal{P}$ on G. The *left global dimension* of R, denoted by l.gl.dim R, is defined similarly. If R is commutative, we shall write gl.dim R for the common value of r.gl.dim R and l.gl.dim R.

The next proposition clarifies the meaning of rings with small right global dimensions. With this hindsight, it is now clear why we took an interest in (right) hereditary rings in §2.

(5.14) Proposition. (1) r. gl. dim $R = 0$ *iff R is semisimple.* (2) r. gl. dim $R \leq 1$ *iff R is right hereditary.*

Proof. (1) We have seen in *FC*–(2.8) (right module analogue) that R is semisimple iff all $M \in \mathfrak{M}_R$ are projective. This immediately gives (1). For (2), *first assume* r.gl.dim $R \leq 1$. For any right ideal $\mathfrak{A} \subseteq R$, consider the short exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow R \longrightarrow R/\mathfrak{A} \longrightarrow 0.$$

Since $\text{pd}_R(R/\mathfrak{A}) \leq 1$, we have $[\mathfrak{A}] = \mathcal{P}[R/\mathfrak{A}] = 0$, so $\mathfrak{A}_R$ is projective. *Conversely, assume that R is a right hereditary* ring. For any $M \in \mathfrak{M}_R$, take a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $F \in \mathfrak{M}_R$ free. By (2.24), K_R is projective, so by (5.11), $\text{pd}_R(M) \leq 1$. This implies that r.gl.dim $R \leq 1$. □

(5.15) Remark. If r.gl.dim $R = 0$, then R is semisimple, and so l.gl.dim $R = 0$ as well. In (2.33), however, we have seen that there exists a right hereditary ring R that is not left hereditary. For such a ring R, we have therefore r.gl.dim $R = 1$ but l.gl.dim $R \geq 2$. Kaplansky has constructed such a ring with gl.dim $R = 2$, and Small has constructed one with gl.dim $R = 3$. For more information on the relation between r.gl.dim R and l.gl.dim R (or the lack of it), see (5.59)–(5.61), and (5.70)–(5.71).

Following an idea of Kaplansky, we can make up easily some examples of modules of infinite projective dimensions. As in *First Course*, we write $\text{ann}_r(S)$ for the right annihilator of a set S (in a given ring).

(5.16) Lemma. *Let $a, b \in R$ be such that* $\text{ann}_r(a) = bR$ *and* $\text{ann}_r(b) = aR$. *Then we have* $\text{pd}(aR) = \text{pd}(bR) = \infty$ *unless $aR \oplus bR \cong R$ (in which case, of course,* $\text{pd}(aR) = \text{pd}(bR) = 0$*).*

Proof. The surjection $R \to aR$ defined by $x \mapsto ax$ has kernel $\text{ann}_r(a) = bR$, so we have

(5.16A) $$0 \longrightarrow bR \longrightarrow R \longrightarrow aR \longrightarrow 0.$$

This gives $\mathcal{P}[aR] = [bR]$, and we have similarly $\mathcal{P}[bR] = [aR]$. If aR is not projective, then $\mathcal{P}^n[aR]$ and $\mathcal{P}^n[bR]$ are never 0, so $\text{pd}(aR) = \text{pd}(bR) = \infty$. On the other hand, if aR is projective, then (5.16A) splits, and we have $aR \oplus bR \cong R$. □

It is worth noting that, in the above situation, we have the following infinite "free resolution"

$$(5.17) \qquad \cdots \longrightarrow R \xrightarrow{\alpha_2} R \xrightarrow{\alpha_1} R \xrightarrow{\alpha_0} aR \longrightarrow 0,$$

where $\alpha_n(x) = ax$ when n is even, and $\alpha_n(x) = bx$ when n is odd.

We list below several examples of a ring R with a pair of elements a, b satisfying the hypothesis of (5.16). The verifications for $\text{ann}_r(a) = bR$ and $\text{ann}_r(b) = aR$ are easy in all cases, and will be left to the reader. Also, in each case, $aR \oplus bR$ will *not* be isomorphic to R, so we shall always end up with $\text{pd}(aR) = \text{pd}(bR) = \infty$.

(5.18) Examples.

(1) $R = k[t]$ with a relation $t^n = 0$, where $n \geq 2$ is fixed, and k is a nonzero ring. We take $a = t$ and $b = t^{n-1}$. Here we cannot have an R-isomorphism $aR \oplus bR \cong R$, since $(aR \oplus bR)t^{n-1} = 0$ but $Rt^{n-1} \neq 0$. Therefore, we have $\text{pd}(aR) = \text{pd}(bR) = \infty$, and hence r.gl.dim $R = \infty$. (Note that bR is the R-module k_R on which t acts trivially.)

(2) $R = k[x, y]$ with the relation $xy = 0$, where k is any nonzero ring. We take $a = x$ and $b = y$. *Then* (5.16A) *is a nonsplit* sequence. For, if (5.16A) splits, then bR is a direct summand of R_R, and must therefore contain a nonzero idempotent. But

$$bR = yk[x, y] = yk[y] \subseteq k[y]$$

shows that bR cannot contain a nonzero idempotent. Therefore, we have again $\text{pd}(aR) = \text{pd}(bR) = \infty$, and r.gl.dim $R = \infty$.

(3) Let R be the integral group ring $\mathbb{Z}G$ where G is the cyclic group $\langle\sigma\rangle$ of order $n \geq 2$. Take

$$a = 1 + \sigma + \cdots + \sigma^{n-1} \quad \text{and} \quad b = \sigma - 1.$$

Here, bR is exactly the "augmentation ideal" of $R = \mathbb{Z}G$ (the kernel of the augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ defined by $\varepsilon(g) = 1$ for all $g \in G$). The other module aR is just $a \cdot \mathbb{Z}$, with trivial G-action. The sequence (5.16A) is again *nonsplit* in $\mathfrak{M}_R$, for, if $R = bR \oplus M$ for some ideal M, then $M \cong aR$ implies that $M = c \cdot \mathbb{Z}$ for some $c \in R$ with $\varepsilon(c) = \pm 1$ and $c \cdot g = c$ for all $g \in G$. But the latter implies that $c \in \text{ann}_r(b) = aR$, and hence $\varepsilon(c) \in \varepsilon(a)\mathbb{Z} = n\mathbb{Z}$, a contradiction. Therefore, we have $\text{pd}(aR) = \text{pd}(bR) = \infty$, and gl.dim $R = \infty$. In this example, (5.17) yields the infinite free resolution

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{\alpha_2} \mathbb{Z}G \xrightarrow{\alpha_1} \mathbb{Z}G \xrightarrow{\alpha_0} \mathbb{Z} \longrightarrow 0,$$

where α_n is multiplication by b when n is odd, and multiplication by a when $n \neq 0$ is even, and $\alpha_0 = \varepsilon$. (For convenience, we have replaced aR by $\mathbb{Z}$, the so-called trivial $\mathbb{Z}G$-module.) The resolution above is well-known in homological group theory, where it is used to compute the group cohomology of the cyclic group $\langle\sigma\rangle$ with coefficients in any $\mathbb{Z}G$-module.

(4) $R = \mathbb{Z}/ab\,\mathbb{Z}$ where $a,\ b > 1$ are not relatively prime. For any common prime divisor p of a and b,

$$aR \oplus bR \cong \mathbb{Z}/b\,\mathbb{Z} \oplus \mathbb{Z}/a\,\mathbb{Z}$$

contains a copy of $\mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}$, so it cannot be isomorphic to the cyclic group R. Therefore, again, $\text{pd}(aR) = \text{pd}(bR) = \infty$, and gl.dim $R = \infty$. This observation leads immediately to the following computation of the global dimension of any quotient ring of $\mathbb{Z}$.

(5.19) Corollary. *The global dimension of the ring $\mathbb{Z}/n\mathbb{Z}$ $(n > 0)$ is 0 when n is square-free, and is ∞ otherwise.*

Next, let us study the relationship among the projective dimensions of the three modules $A,\ B,\ C$ in a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{M}_R$. The following result holds over any ring R. As suggested by Kaplansky [72: p. 169], we should view this result as an attempt to compute $\text{pd}(C)$ via $\text{pd}(A)$ and $\text{pd}(B)$: the attempt is successful except in the "ambiguous" case $\text{pd}(A) = \text{pd}(B)$, where we have only an upper bound estimate on $\text{pd}(C)$. (In this result, the usual conventions about the symbol ∞ apply.)

(5.20) Theorem. *Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathfrak{M}_R$. If two of* $\text{pd}(A)$, $\text{pd}(B)$, $\text{pd}(C)$ *are finite, so is the third. In any case,*

(1) *If* $\text{pd}(A) < \text{pd}(B)$, *then* $\text{pd}(C) = \text{pd}(B)$.
(2) *If* $\text{pd}(A) > \text{pd}(B)$, *then* $\text{pd}(C) = \text{pd}(A) + 1$.
(3) *If* $\text{pd}(A) = \text{pd}(B)$, *then* $\text{pd}(C) \leq \text{pd}(A) + 1$.

Proof. First assume C is projective. Then, the given sequence splits, so $B \cong A \oplus C$ and $\text{pd}(A) = \text{pd}(B)$. Here, only Case (3) can occur, and the desired conclusion is a tautology. Next assume B is projective; that is, $\text{pd}(B) = 0$. Here, $\text{pd}(A) < \text{pd}(B)$ is impossible. If $\text{pd}(A) > \text{pd}(B)$, then A and hence C are not projective, and we have $\text{pd}(C) = \text{pd}(A) + 1$ by (5.12). If $\text{pd}(A) = \text{pd}(B)$, then A is projective, and $\text{pd}(C) \leq 1 = \text{pd}(A) + 1$.

In the following, we may therefore assume that neither B nor C is projective. We express B as a quotient module P/K where P is projective, and take A as Q/K, so $C \cong P/Q$. Then

(5.21) $$\text{pd}(B) = \text{pd}(K) + 1 \quad\text{and}\quad \text{pd}(C) = \text{pd}(Q) + 1,$$

and we have a new exact sequence

(5.22) $$0 \longrightarrow K \longrightarrow Q \longrightarrow A \longrightarrow 0.$$

The first conclusion of the theorem now follows by induction on the sum of the two finite projective dimensions (the inductive hypothesis being applied to (5.22)).

If at least two of $\text{pd}(A)$, $\text{pd}(B)$, $\text{pd}(C)$ are ∞, the conclusions in Cases (1), (2), and (3) are easy to check. Thus, for the remainder of the proof, we may assume that $\text{pd}(A)$, $\text{pd}(B)$, and $\text{pd}(C)$ are finite; we then induct on their sum. Applying the inductive hypothesis to (5.22), we have the following.

(1)′ If $\text{pd}(K) < \text{pd}(Q)$ (i.e., $\text{pd}(B) < \text{pd}(C)$), then

$$\text{pd}(A) = \text{pd}(Q) = \text{pd}(C) - 1 \geq \text{pd}(B).$$

(2)′ If $\text{pd}(K) > \text{pd}(Q)$ (i.e., $\text{pd}(B) > \text{pd}(C)$), then

$$\text{pd}(A) = \text{pd}(K) + 1 = \text{pd}(B).$$

(3)′ If $\text{pd}(K) = \text{pd}(Q)$ (i.e., $\text{pd}(B) = \text{pd}(C)$), then

$$\text{pd}(A) \leq \text{pd}(K) + 1 = \text{pd}(B).$$

Now suppose $\text{pd}(A) < \text{pd}(B)$. We can only be in Case (3)′, so $\text{pd}(C) = \text{pd}(B)$, as desired. Next, suppose $\text{pd}(A) > \text{pd}(B)$. Here, we can only be in Case (1)′, so $\text{pd}(C) = \text{pd}(A) + 1$, as desired. Finally, suppose $\text{pd}(A) = \text{pd}(B)$. If we are in Cases (2)′ or (3)′, then

$$\text{pd}(C) \leq \text{pd}(B) < \text{pd}(A) + 1;$$

and if we are in Case (1)′, then $\text{pd}(C) = \text{pd}(A) + 1$. □

(5.23) Corollary. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence in* $\mathfrak{M}_R$. *Then*

$$\text{pd}(B) \leq \max\{\text{pd}(A),\ \text{pd}(C)\},$$

with equality unless $\text{pd}(C) = \text{pd}(A) + 1$.

Proof. (1) Suppose $\text{pd}(A) < \text{pd}(B)$. By (5.20), we have $\text{pd}(C) = \text{pd}(B) > \text{pd}(A)$, so

$$\max\{\text{pd}(A),\ \text{pd}(C)\} = \text{pd}(C) = \text{pd}(B).$$

(2) Suppose $\text{pd}(A) > \text{pd}(B)$. By (5.20), $\text{pd}(C) = \text{pd}(A) + 1$; here,

$$\max\{\text{pd}(A),\ \text{pd}(C)\} = \text{pd}(A) + 1 \geq \text{pd}(B) + 2.$$

(3) Suppose $\text{pd}(A) = \text{pd}(B)$. By (5.20), $\text{pd}(C) \leq \text{pd}(A) + 1$. If $\text{pd}(C) < \text{pd}(A) + 1$, then $\max\{\text{pd}(A),\ \text{pd}(C)\} = \text{pd}(A) = \text{pd}(B)$. Otherwise,

$$\max\{\text{pd}(A),\ \text{pd}(C)\} = \text{pd}(A) + 1 = \text{pd}(B) + 1.$$

□

It can be seen (cf. Exercise 0) that (5.23) is in fact an equivalent formulation of (5.20). The following is an easy consequence.

(5.24) Corollary. *Let* $0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n = B$ *be a finite filtration of a module* B_R. *Then* $\text{pd}(B) \leq \max\{\text{pd}(B_{i+1}/B_i)\}$.

We also take this opportunity to record the following related result.

(5.25) Proposition. *Let* $M = \bigoplus_i M_i$. *Then* $\text{pd}(M) = \sup\{\text{pd}(M_i)\}$.

Proof. If K_i is a module representing $\mathcal{P}^n[M_i]$, then $\mathcal{P}^n[M]$ is represented by $\bigoplus_i K_i$. In particular, $\mathcal{P}^n[M] = 0$ iff $\mathcal{P}^n[M_i] = 0$ for all i, iff $n \geq \sup\{\text{pd}(M_i)\}$. □

§5B. Change of Rings

Our next goal is to construct a module of a given finite projective dimension over a suitable ring, using the idea of "regular sequences". We start by considering a general quotient map $\varphi : R \to R/xR$, where x is a central element of R that is not a 0-divisor. In the following, we shall write $\overline{R}$ for the quotient ring R/xR. To avoid the trivial case $\overline{R} = 0$, we shall also assume that x is a nonunit in R. Via the map φ, we can view any (say, right) $\overline{R}$-module M as a (right) R-module. The relationship between $\text{pd}_{\overline{R}}(M)$ and $\text{pd}_R(M)$ is given by the following basic result of Kaplansky.

(5.26) Change of Rings Lemma. *Assume that M is a nonzero right $\overline{R}$-module with* $n := \text{pd}_{\overline{R}}(M) < \infty$. *Then* $\text{pd}_R(M) = n + 1$.

(Note that the conclusion is not true in general if $\text{pd}_{\overline{R}}(M) = \infty$. For instance, for $R = \mathbb{Q}[t]$ and $x = t^2$, the $\overline{R}$-module $M = \mathbb{Q}$ with trivial $\bar{t}$-action has $\text{pd}_{\overline{R}}(M) = \infty$ by (5.18)(1). But $\text{pd}_R(M) = 1$, since M is a nonprojective module over the principal ideal domain R.)

Proof of (5.26). We induct on n. First assume $n = 0$, that is, M is $\overline{R}$-projective. Then M is a direct summand of a free $\overline{R}$-module, say, $\overline{F}$. Consider the exact sequence

$$0 \longrightarrow xR \longrightarrow R \longrightarrow \overline{R} \longrightarrow 0 \quad \text{in } \mathfrak{M}_R.$$

Since $xR \cong R$, we have $\text{pd}_R(\overline{R}) \leq 1$. This implies that $\text{pd}_R(M) \leq \text{pd}_R(\overline{F}) \leq 1$, by (5.25). On the other hand, since $Mx = 0$ and x is not a 0-divisor in R, $M_R \neq 0$ cannot be embedded in a free R-module, and in particular is non-projective. This shows that $\text{pd}_R(M) = 1$, as desired.

To treat the case $n \geq 1$, fix an exact sequence

$$0 \longrightarrow K \longrightarrow \overline{F} \longrightarrow M \longrightarrow 0 \quad \text{in } \mathfrak{M}_{\overline{R}},$$

with $\overline{F}$ free. Then $\text{pd}_{\overline{R}}(K) = n-1$, and the inductive hypothesis gives $\text{pd}_R(K) = n$. If $n \geq 2$, then $\text{pd}_R(K) > 1 \geq \text{pd}_R(\overline{F})$ and (5.20)(2) yields

$$\text{pd}_R(M) = \text{pd}_R(K) + 1 = n + 1.$$

We are thus left with the case $n = 1$, for which (5.20) yields $\text{pd}_R(M) \le 2$. Take an exact sequence

$$(5.27) \qquad 0 \longrightarrow T \hookrightarrow F \longrightarrow M \longrightarrow 0 \quad \text{in } \mathfrak{M}_R\,,$$

with F free. Since $Mx = 0$, we have $Fx \subseteq T$, so we have an exact sequence

$$(5.28) \qquad 0 \longrightarrow T/Fx \longrightarrow F/Fx \longrightarrow M \longrightarrow 0 \quad \text{in } \mathfrak{M}_{\overline{R}}.$$

Therefore, $\text{pd}_{\overline{R}}(T/Fx) = \text{pd}_{\overline{R}}(M) - 1 = 0$, and so the exact sequence

$$(5.29) \qquad 0 \longrightarrow Fx/Tx \longrightarrow T/Tx \longrightarrow T/Fx \longrightarrow 0 \quad \text{in } \mathfrak{M}_{\overline{R}}$$

splits. Identifying $M \cong F/T$ with Fx/Tx (by multiplication by x), we see that M is a direct summand of T/Tx. If T is R-projective, T/Tx and hence M would be $\overline{R}$-projective, which is not the case. Therefore, we must have $\text{pd}_R(M) = 1 + \text{pd}_R(T) \ge 2$ from (5.27), whence $\text{pd}_R(M) = 2$. □

(5.30) Corollary. *In the above notations, if* $R \neq 0$ *and* $n := \text{r.gl.dim}\,\overline{R} < \infty$, *then* $\text{r.gl.dim}\,R \ge n + 1$.

(Again, the example $R = \mathbb{Q}[t]$ with $x = t^2$ shows that, in this corollary, the hypothesis r.gl.dim $\overline{R} < \infty$ is essential!)

To construct a module with a given projective dimension, we shall use the notion of a regular sequence defined below.

(5.31) Definition. An ordered sequence of central elements $x_1, \ldots, x_n$ in a ring R is called a *regular sequence* if $\sum x_i R \neq R$ and, for any $i \ge 1$, the image of x_i is not a zero-divisor in the ring $R/(x_1 R + \cdots + x_{i-1} R)$. (Note that, by this definition, the empty sequence is regular if $R \neq 0$.)

The next Proposition gives the basic connection between regular sequences and projective dimensions.

(5.32) Proposition. *Let* $x_1, \ldots, x_n$ *be a regular sequence in* R, *and let* $I = \sum_{i=1}^n x_i R$. *Then* $\text{pd}((R/I)_R) = n$.

Proof. We induct on n, the case $n = 0$ being trivial. For $n > 0$, the images $\overline{x}_2, \ldots, \overline{x}_n$ clearly form a regular sequence in $\overline{R} := R/x_1 R$. By the inductive hypothesis, we have then

$$\text{pd}\big((\overline{R}/(\overline{x}_2\overline{R} + \cdots + \overline{x}_n\overline{R}))_{\overline{R}}\big) = n - 1.$$

Identifying $\overline{R}/(\overline{x}_2\overline{R} + \cdots + \overline{x}_n\overline{R})$ with $R/(x_1 R + \cdots + x_n R) = R/I$, (5.26) yields

$$\text{pd}\big((R/I)_R\big) = (n-1) + 1 = n.$$

□

In the case when R is a commutative ring, there is, in fact, a "canonical" resolution

$$(5.33) \qquad 0 \longrightarrow F_n \xrightarrow{\alpha_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} R/I \longrightarrow 0$$

with F_i free in $\mathfrak{M}_R$. This is called the *Koszul resolution* for R/I, and is defined as follows:

$$F_0 = R, \qquad F_1 = \bigoplus_{i=1}^{n} e_i R, \qquad F_r = \bigwedge^r (F_1) \quad (1 \le r \le n),$$

where $\bigwedge^r(F_1)$ denotes the rth exterior power of F_1. The map α_0 is the projection map from R to R/I, and α_1 sends each $e_i \in F_1$ to $x_i \in I$. For $r \ge 2$, α_r is defined by

$$(5.34) \qquad \alpha_r\left(e_{i_1} \wedge \cdots \wedge e_{i_r}\right) = \sum_{j=1}^{r} (-1)^{j-1}\left(e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_r}\right) x_{i_j},$$

where the hat means "omission". For the proof of the exactness of (5.33), see Matsumura [80]. Note that the *injection α_n is non-split.* In fact, if α_n is split by some $\varphi : F_{n-1} \to F_n$, then, writing

$$\varphi(e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n) = (e_1 \wedge \cdots \wedge e_n)\, b_i\,,$$

we have

$$e_1 \wedge \cdots \wedge e_n = \varphi\alpha_n(e_1 \wedge \cdots \wedge e_n) = (e_1 \wedge \cdots \wedge e_n) \sum_{j=1}^{n} (-1)^{j-1} b_j x_j.$$

This implies that $\sum_{j=1}^{n}(-1)^{j-1} b_j x_j = 1$, a contradiction. Therefore, once we know that the Koszul resolution (5.33) is exact, the last part of (5.11) will also give $\text{pd}(R/I) = n$, yielding a more explicit proof of (5.32) in the commutative case.

Since (5.33) provides one of the few known constructions of a finite free resolution, it is worthwhile to work out some of its special cases. The case $n = 1$ is essentially trivial, and the case $n = 2$ is very easy. So let us work out the case $n = 3$. Here (5.33) takes the form

$$(5.35) \qquad 0 \longrightarrow R \xrightarrow{\alpha} R^3 \xrightarrow{\beta} R^3 \xrightarrow{\gamma} R \longrightarrow R/(x, y, z) \longrightarrow 0,$$

where we have rewritten x_1, x_2, x_3 as x, y, z. We shall express elements in R^r as column r-vectors, and express a homomorphism $R^r \to R^s$ by an $s \times r$ matrix. Using the basis

$$f_1 = e_2 \wedge e_3, \quad f_2 = -e_1 \wedge e_3, \quad f_3 = e_1 \wedge e_2$$

on $\bigwedge^2(R^3) \cong R^3$, and the natural basis e_1, e_2, e_3 on $\bigwedge^1(R^3) = R^3$, (5.34) yields

$$\beta(f_1) = e_3 y - e_2 z, \qquad \beta(f_2) = -e_3 x + e_1 z, \qquad \beta(f_3) = e_2 x - e_1 y,$$

so β has the matrix $\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$. On the other hand, γ has matrix $(x,\ y,\ z)$, and α has matrix $(x,\ y,\ z)^t$. Thus, (5.35) is a rather "symmetrical" free revolution, with $\alpha = \gamma^t$ and β given by an alternating matrix. By matrix multiplication, we see directly that $\beta\alpha = 0$ and $\gamma\beta = 0$. Using the fact that $x,\ y,\ z$ form a regular sequence, we can further check that $\ker\beta \subseteq \operatorname{im}\alpha$, and $\ker\gamma \subseteq \operatorname{im}\beta$ (cf. Exercise 6). Therefore, (5.35) gives indeed a free resolution for $R/(x,\ y,\ z)R$.

For an explicit example of a regular sequence, we can take $x_1, \ldots, x_n$ in a polynomial ring $R = A[x_1, \ldots, x_n]$ over any nonzero ring A. Here

$$R/(x_1R + \cdots + x_{i-1}R) \cong A[x_i, \ldots, x_n],$$

in which x_i is clearly not a 0-divisor. The right R-module $R/(x_1R + \cdots + x_nR)$ is just A with all x_i acting trivially; Prop. (5.32) shows that $\mathrm{pd}(A) = n$. In the case when A is a semisimple ring, our forthcoming result (5.36) will show that r.gl.dim $R = n$, so A is a right R-module of the largest possible projective dimension.

Our next goal is to prove the following important result on the right global dimension of a polynomial extension of a ring.

(5.36) Theorem. *For any nonzero ring A and $R = A[x]$, we have*

$$\mathrm{r.\,gl.\,dim}\ R = 1 + \mathrm{r.\,gl.\,dim}\ A.$$

(By induction, it follows that $\mathrm{r.\,gl.\,dim}\ A[x_1, \ldots, x_n] = n + \mathrm{r.\,gl.\,dim}\ A$.*)*

Proof. For any right A-module M, let us write $M[x]$ for the R-module $M \otimes_A R$. We can think of elements of $M[x]$ as "polynomials " of the form $\sum m_i x^i$, on which x acts by right multiplication. It is easy to see that M is projective over A iff $M[x]$ is projective over R. Since the functor "$- \otimes_A R$" is exact, it follows readily that

$$\mathrm{pd}_A(M) = \mathrm{pd}_R(M[x]). \tag{5.37}$$

To prove the theorem, first consider the case r.gl.dim $A = \infty$. In this case, there exists a right A-module M with $\mathrm{pd}_A(M) = \infty$. (The proof of this is left as Exercise 3 in this section.) From (5.37), we see immediately that r.gl.dim $R = \infty$. Thus, we may now assume $d = \mathrm{r.\,gl.\,dim}\ A < \infty$. It suffices to show that

$$\mathrm{r.\,gl.\,dim}\ R \leq d + 1, \tag{5.38}$$

for, once this is proved, we can apply (5.30) to the quotient map $R \to R/xR \cong A$ to get r.gl.dim $R \geq d + 1$. Consider any right R-module M: we think of M as a right A-module given with an A-endomorphism f. As in the beginning of the proof, we can form the R-module $M[x]$ with elements $\sum m_i x^i$. (Here $m_i x^i$ is just an abbreviation for $m_i \otimes x^i$, not to be confused with the action of x^i on $m_i \in M_R$.) We shall construct a short exact sequence

$$0 \longrightarrow M[x] \xrightarrow{\ \psi\ } M[x] \xrightarrow{\ \varphi\ } M \longrightarrow 0 \tag{5.39}$$

in $\mathfrak{M}_R$. First, we define φ by $\varphi(\sum m_i x^i) = \sum f^i(m_i)$. Clearly, φ is a surjective R-homomorphism from $M[x]$ onto M. Next, observe that $\tilde{f} := f \otimes_A \mathrm{Id}_R$ is an R-endomorphism of $M[x]$, sending $\sum m_i x^i$ to $\sum f(m_i)x^i$. We also have the R-endomorphism $\tilde{x}$ of $M[x]$ sending $\sum m_i x^i$ to $\sum m_i x^{i+1}$. Define ψ in (5.39) to be the difference $\tilde{x} - \tilde{f}$. Then

$$\psi\left(\sum_{i=0}^{r} m_i x^i\right) = -f(m_0) + (m_0 - f(m_1))x + \cdots + (m_{r-1} - f(m_r))x^r + m_r x^{r+1},$$

so ψ is clearly injective. A routine calculation shows that $\varphi\psi = 0$, so the exactness of (5.39) boils down to $\ker \varphi \subseteq \operatorname{im} \psi$. Suppose $\varphi(\sum m_i x^i) = \sum f^i(m_i) = 0$. Then

$$\sum_{i\geq 0} m_i x^i = \sum_{i\geq 0} (m_i x^i - f^i(m_i)) = \sum_{i\geq 1} (\tilde{x}^i - \tilde{f}^i)(m_i),$$

which is clearly in $\operatorname{im}(\tilde{x} - \tilde{f}) = \operatorname{im}(\psi)$. Since (5.39) is now exact, (5.20)(3) applies to give

$$\mathrm{pd}_R M \leq 1 + \mathrm{pd}_R M[x] = 1 + \mathrm{pd}_A M \leq 1 + d,$$

in view of (5.37). This establishes (5.38). □

In the classical case *when A is a field*, (5.36) shows that $S = A[x_1, \ldots, x_n]$ has global dimension n. This result essentially goes back to Hilbert. In fact, Hilbert proved the following interesting statement: *for any ideal $\mathfrak{A} \subseteq S$, there exists a resolution*

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathfrak{A} \longrightarrow 0,$$

where the F_i's are f.g. free S-modules. This is known in the literature as Hilbert's Syzygy Theorem.[44] By a slight abuse of terminology, Theorem 5.36 is sometimes called Hilbert's Syzygy Theorem as well.

Hilbert's classical result suggests that we can somehow get by with the use of f.g. free modules in the place of f.g. projective modules over the polynomial ring $S = A[x_1, \ldots, x_n]$. In 1955, Serre raised the question whether all f.g. projective S-modules are free. The affirmative answer to this question soon became known as "Serre's Conjecture". After much work in the 60s and 70s, this famous conjecture was proved independently by D. Quillen and A. Suslin in 1976. For an exposition of their proofs and a historical survey, see Lam [78].

§5C. Injective Dimensions

We shall now turn our attention to the *injective dimensions* of modules. Here the situation is dual to that of projective dimensions, so we can state various facts without repeating their proofs. To begin with, we have the following.

[44]In view of Auslander's Theorem, to be proved in (5.51) below, this result of Hilbert does give the upper bound gl.dim $S \leq n$.

(5.40) (Injective) Schanuel's Lemma. *Let N be a right module over any ring R, and let*

(5.41) $0 \longrightarrow N \longrightarrow I \longrightarrow S \longrightarrow 0$ *and* $0 \longrightarrow N \longrightarrow J \longrightarrow T \longrightarrow 0$

be short exact sequences in $\mathfrak{M}_R$, where I_R is injective. Then there exists a short exact sequence

$$0 \longrightarrow J \longrightarrow T \oplus I \longrightarrow S \longrightarrow 0.$$

In particular, if J_R is also injective, then we have $S \oplus J \cong T \oplus I$ in $\mathfrak{M}_R$.

By analogy with the projective case, we define two (right) modules S, T to be *injectively equivalent* if $S \oplus J \cong T \oplus I$ for suitable injective modules I and J. Whenever no confusion is liable, we shall write $[S]$ for the "injective equivalence class" of S. As before, we get an additive semigroup G' of such classes, whose identity 0 is given by the class of all injective modules. Again Schanuel's Lemma (5.40) enables us to define an additive shift operator $\mathcal{I} : G' \to G'$. Here, $\mathcal{I}(N) = [S]$ whenever we have a short exact sequence $0 \to N \to I \to S \to 0$ with I_R injective. We shall call $\mathcal{I}$ the *injective shift* on right R-modules. As before, the $\mathcal{I}(N)$'s are given by the kernels (or syzygies) of an *injective resolution* (with all I_n 's injective):

$$0 \longrightarrow N \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots .$$

Note that such a resolution exists since (by (3.20)) any module can be embedded in an injective module.

(5.42) Definition. The *injective dimension* of $N \in \mathfrak{M}_R$ (or of the class $[N] \in G'$) is defined to be

(5.43) $$\mathrm{id}(N) = \mathrm{id}_R(N) = \min\{n : \mathcal{I}^n[N] = 0\}.$$

If no such n exists, we define $\mathrm{id}_R(N)$ to be ∞. The *right injective global dimension* of R is defined to be

(5.44) $$\text{r. inj. gl. dim } R = \sup\{\mathrm{id}_R(N) : N \in \mathfrak{M}_R\} \leq \infty.$$

The *left injective global dimension* (l.inj.gl.dim R) of R is defined similarly, by using left R-modules.

If r.gl.dim $R = 0$, then R is semisimple and hence all right R-modules are injective, which shows r.inj.gl.dim $R = 0$. If r.gl.dim $R = 1$, then R is not semisimple but right hereditary. Here, not every right R-module is injective by *FC*–(2.9), but any quotient of an injective right R-module is injective by (3.22). Therefore, from (5.44), we have r.inj.gl.dim $R = 1$. As it turns out, these facts are not accidents; they are the first two cases of the following beautiful result in the theory of global dimensions of rings.

(5.45) Theorem. *For any ring R,* r. gl. dim R = r. inj. gl. dim R *(and similarly for left dimensions).*

In most standard textbooks in homological algebra, this theorem is proved by the use of the "Ext" functors. However, defining these "Ext" functors properly would take us too far afield, and in any case we do not need these functors in the sequel. Therefore, we shall try to give a proof for (5.45) without introducing the full machinery of the "Ext" functors. This approach to (5.45), which makes use of only the vanishing of Ext^1, comes from Kaplansky's authoritative treatment [Kaplansky: 72]. The outline of our proof is the same as Kaplansky's. However, the details of our proof are different, in that we make full use of the functorial nature of the "Hom" functors. In fact, the steps of the proof are arranged in such a way that hardly any arguments are necessary for each step! Moreover, this approach can be carried over *mutatis mutandis* for the treatment of left and right weak global dimensions a little later in this section. The heart of the matter lies in the following observation.

(5.46) Lemma. *Let* $\mathcal{E} : 0 \to A \to B \to C \to 0$ *and* $\mathcal{E}' : 0 \to C' \to B' \to A' \to 0$ *be an exact sequence in* $\mathfrak{M}_R$.

(1) *Assume* B *is projective. Then*

(a) $\mathrm{Hom}_R(\mathcal{E}, A')$ *exact* $\Longrightarrow$ $\mathrm{Hom}_R(A, \mathcal{E}')$ *exact.*
(b) $\mathrm{Hom}_R(\mathcal{E}, C')$ *exact* $\Longrightarrow$ $\mathrm{Hom}_R(C, \mathcal{E}')$ *exact.*

(2) *Assume* B' *is injective. Then*

(a) $\mathrm{Hom}_R(A, \mathcal{E}')$ *exact* $\Longrightarrow$ $\mathrm{Hom}_R(\mathcal{E}, A')$ *exact.*
(b) $\mathrm{Hom}_R(C, \mathcal{E}')$ *exact* $\Longrightarrow$ $\mathrm{Hom}_R(\mathcal{E}, C')$ *exact.*

Proof. It suffices to prove (1), since the proof for (2) is completely dual. Consider the following exact commutative diagram (where "Hom" means "Hom_R"):

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(C, C') & \longrightarrow & \mathrm{Hom}(B, C') & \xrightarrow{\varphi} & \mathrm{Hom}(A, C') \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(C, B') & \longrightarrow & \mathrm{Hom}(B, B') & \longrightarrow & \mathrm{Hom}(A, B') \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\
0 & \longrightarrow & \mathrm{Hom}(C, A') & \longrightarrow & \mathrm{Hom}(B, A') & \xrightarrow{\beta} & \mathrm{Hom}(A, A')
\end{array}
$$

Here α is surjective, since we assume B is projective. If $\mathrm{Hom}(\mathcal{E}, A')$ is exact, then β is surjective, so γ is surjective, which means that $\mathrm{Hom}(A, \mathcal{E}')$ is exact. This proves (1)(a). To prove (1)(b), we need to show that if φ is surjective, then ψ is surjective. This is done by an easy diagram chase, using again the surjectivity of α. □

(5.47) Theorem. *For any two given right R-modules C, C', the following are equivalent:*

(1) $\mathrm{Hom}_R(C, \mathcal{E}')$ *is exact for some* $\mathcal{E}' : 0 \to C' \to B' \to A' \to 0$ *where* B' *is injective.*
(1)′ $\mathrm{Hom}_R(C, \mathcal{E}')$ *is exact for all* $\mathcal{E}' : 0 \to C' \to B' \to A' \to 0$.
(2) $\mathrm{Hom}_R(\mathcal{E}, C')$ *is exact for some* $\mathcal{E} : 0 \to A \to B \to C \to 0$ *where* B *is projective.*
(2)′ $\mathrm{Hom}_R(\mathcal{E}, C')$ *is exact for all* $\mathcal{E} : 0 \to A \to B \to C \to 0$.
(3) *The only extension of* C' *by* C *is the split extension (i.e. any short exact sequence* $0 \to C' \to X \to C \to 0$ *in* $\mathfrak{M}_R$ *splits).*

Proof. (1)$\Longrightarrow$ (2)′ follows from (5.46)(2b), and (2)$\Longrightarrow$ (1)′ follows from (5.46)(1b). Since (1)′$\Longrightarrow$(1) and (2)′$\Longrightarrow$(2) are tautologies, (1), (1)′, (2), (2)′ are equivalent.

(1)′$\Longrightarrow$(3). Applying (1)′ to $\mathcal{E}' : 0 \to C' \to X \to C \to 0$, we see that

$$\mathrm{Hom}_R(C, X) \longrightarrow \mathrm{Hom}_R(C, C)$$

is onto. This means that $\mathcal{E}'$ splits.

(3)$\Longrightarrow$ (1)′. Consider *any* $\mathcal{E}' : 0 \to C' \to B' \xrightarrow{f} A' \to 0$. To show that $\mathrm{Hom}_R(C, \mathcal{E}')$ is exact, we must try to "lift" any $g : C \to A'$ to a homomorphism $C \to B'$. The argument here is standard. We form the "pullback"

$$X = \{(b', c) \in B' \oplus C : \ f(b') = g(c)\}$$

and get an exact sequence $0 \to C' \to X \to C \to 0$. By (3), the surjection $X \to C$ splits by some $h : C \to X$. Then the composition of h with the projection of X to B' gives the desired lifting of g to a homomorphism $C \to B'$. □

(5.48) Definition. If the right R-modules C, C' satisfy the (equivalent) conditions in (5.47), we say that $\mathrm{Ext}(C, C') = 0$.

In the standard notation of homological algebra, $\mathrm{Ext}^1_R(C, C')$ is the group of isomorphism classes of extensions of C' by C. Here, we have avoided a formal introduction of the group $\mathrm{Ext}^1_R(C, C')$, but shall try to get by with the notion of $\mathrm{Ext}^1_R(C, C') = 0$. In fact, for the purposes of proving (5.45), we need only know the equivalence of (1), (1)′, (2), (2)′ in (5.47). In other words, we could have defined $\mathrm{Ext}(C, C') = 0$ via these conditions without ever using the interpretation in (5.47)(3)! Thus, even the proof for (3)$\Longrightarrow$(1)′ above could have been omitted. The reader should check that (5.47)(3) is indeed never used below.

(5.49) Lemma. *For any right R-module C', the following are equivalent:*

(1) C' *is injective.*
(2) $\mathrm{Ext}(C, C') = 0$ *for all* C_R.
(3) $\mathrm{Ext}(R/\mathfrak{A}, C') = 0$ *for all right ideals* $\mathfrak{A} \subseteq R$.

Similarly, a right module C is projective iff $\mathrm{Ext}(C, C') = 0$ *for all* C'_R*; however, there is no analogue for* (3).

Proof. (1)$\Longleftrightarrow$(2) follows by using the condition (5.47)(2′), so it only remains to prove (3)$\Longrightarrow$(1). But (3) implies that $\mathrm{Hom}_R(\mathcal{E}, C')$ is exact for

$$\mathcal{E}:\ 0 \longrightarrow \mathfrak{A} \longrightarrow R \longrightarrow R/\mathfrak{A} \longrightarrow 0,$$

where $\mathfrak{A}$ is any right ideal. By Baer's Criterion (3.7), the *surjectivity* of

$$\mathrm{Hom}_R(R, C') \to \mathrm{Hom}_R(\mathfrak{A}, C')$$

for all right ideals $\mathfrak{A} \subseteq R$ implies that C' is an injective R-module. □

From Def. (5.48), we see quite generally that

$$\mathrm{Ext}(C_1 \oplus C_2, C') = 0 \iff \mathrm{Ext}(C_i, C') = 0 \ \text{ for } i = 1, 2;$$

and similarly for $\mathrm{Ext}(C, C'_1 \oplus C'_2) = 0$. Using these in conjunction with (5.49), it follows that $\mathrm{Ext}(C, C') = 0$ is a property depending only on the projective equivalence class of C and the injective equivalence class of C'. Keeping this observation in mind, we now prove the following crucial property.

(5.50) Theorem. *For any right R-module C, C', we have* $\mathrm{Ext}(\mathcal{P}(C), C') = 0$ *iff* $\mathrm{Ext}(C, \mathcal{I}(C')) = 0$.

Proof. Fix exact sequences

$$\mathcal{E}: 0 \to A \to B \to C \to 0, \quad \text{and} \quad \mathcal{E}': 0 \to C' \to B' \to A' \to 0,$$

where B is projective and B' is injective. By (5.46)(1(a) and 2(a)), $\mathrm{Hom}_R(\mathcal{E}, A')$ is exact iff $\mathrm{Hom}_R(A, \mathcal{E}')$ is exact. Using Def. (5.48), this means that $\mathrm{Ext}(C, A') = 0$ iff $\mathrm{Ext}(A, C') = 0$. This is exactly (5.50), since $A' = \mathcal{I}(C')$, and $A = \mathcal{P}(C)$. □

Equipped with the above relationship between the shift operators $\mathcal{P}$ and $\mathcal{I}$, we can now return to the main theorem (5.45).

Proof of (5.45). We have r.gl.dim $R \le n$ iff $\mathcal{P}^n(C)$ is projective for all $C \in \mathfrak{M}_R$, iff $\mathrm{Ext}(\mathcal{P}^n C, C') = 0$ for all $C, C' \in \mathfrak{M}_R$. Similarly, r.inj.gl.dim $R \le n$ iff $\mathrm{Ext}(C, \mathcal{I}^n C') = 0$ for all $C, C' \in \mathfrak{M}_R$. But by repeated use of (5.50), $\mathrm{Ext}(\mathcal{P}^n C, C') = 0$ iff $\mathrm{Ext}(C, \mathcal{I}^n C') = 0$. Therefore, r.gl.dim $R \le n$ iff r.inj.gl.dim $R \le n$, which proves (5.45).

(5.51) Corollary (Auslander). r. gl. dim $R = \sup\{\mathrm{pd}(C)\}$, *where C ranges over all cyclic right R-modules.*

Proof. We may assume that the supremum above is a finite number. Then, for any right ideal $\mathfrak{A} \subseteq R$, we have

$$\mathrm{Ext}(\mathcal{P}^n(R/\mathfrak{A}),\ C') = 0 \quad \text{and so} \quad \mathrm{Ext}(R/\mathfrak{A},\ \mathcal{I}^n(C')) = 0,$$

for any right R-module C'. By (5.49), this implies that $\mathcal{I}^n$ is the zero operator, so $n \geq$ r.inj.dim $R =$ r.gl.dim R, by (5.45). □

We may think of (5.51) as a way of computing r.gl.dim R via the projective dimensions of the right ideals of R. In fact, for any non-semisimple ring R, (5.51) gives the formula

(5.51′) $$\mathrm{r.gl.dim}\ R = 1 + \sup\{\mathrm{pd}(\mathfrak{A})\},$$

where $\mathfrak{A}$ ranges over all right ideals of R.

Auslander's result (5.51) leads one to wonder if r.gl.dim R can also be computed via the *injective* dimensions of the cyclic right R-modules. Unfortunately, this is not the case in general. However, if R is a *right noetherian* ring, then the "injective analogue" of (5.51) does hold:

$$\mathrm{r.gl.dim}\ R = \sup\{\mathrm{id}(C)\},$$

where C ranges over all *cyclic* right R-modules. This result is due to B. Osofsky; for a sketch of the proof, see Exercise 27.

(5.52) Remark. If we define $\mathrm{Ext}^1_R(M,\ N)$ to be the group of isomorphism classes of extensions of N by M, it can be shown quite generally that $\mathrm{Ext}^1_R(M,\ N)$ depends only on the projective (resp. injective) equivalence class of M (resp. N), and that

$$\mathrm{Ext}^1_R(\mathcal{P}(C),\ C') \cong \mathrm{Ext}^1_R(C,\ \mathcal{I}(C'))$$

as groups. The higher Ext-groups $\mathrm{Ext}^{n+1}_R(C,\ C')$ can then be taken as

$$\mathrm{Ext}^1_R(\mathcal{P}^n(C),\ C') \cong \mathrm{Ext}^1_R(C,\ \mathcal{I}^n(C')).$$

Using these definitions, we have then r.gl.dim $R \leq n$ iff $\mathrm{Ext}^{n+1}_R \equiv 0$ (the zero bifunctor). (Since r.inj.gl.dim R is always equal to r.gl.dim R, we shall henceforth discard the former notation.)

§5D. *Weak Dimensions of Rings*

We now go on to study the *flat dimensions* of right R-modules. To define this notion, we shall make use of the formation of *character modules*. At the beginning of §4, we have defined the character module of a right R-module M to be the left R-module $M' = \mathrm{Hom}_{\mathbb{Z}}(M,\ \mathbb{Q}/\mathbb{Z})$. In this section, it will be convenient to use a different notation, M^0, for the character module of M, so that we can free up the "prime" notation for other uses. Indeed, we have been using "primes" freely in this section anyway in the absence of character modules. In the following, therefore,

M^0 shall denote the character module of M, while M' shall just denote some other module.

Since we do not have a "flat Schanuel's Lemma" (cf. Exercise 3), we adopt the following definition of a "flat equivalence."

(5.53) Definition. Two right R modules K_1, K_2 are said to be *flat equivalent* if there exist flat right modules F_1, F_2 such that $(K_1 \oplus F_1)^0 \cong (K_2 \oplus F_2)^0$ as left R-modules. (Since "^{0}" is additive over direct sums and the F_i^0's are injective, this implies, in particular, that K_1^0 and K_2^0 are injectively equivalent in ${}_R\mathfrak{M}$.)

The above being obviously an equivalence relation, we have now the notion of *flat equivalence classes*. In general, the partition of isomorphism classes of right modules into flat classes is a *coarsening* of the partition into projective classes.

The definition in (5.53) is chosen such that we can define a "flat shift" operator on the semigroup, say, H, of flat equivalence classes. For any module M_R, take any exact sequence $0 \to K \to F \to M \to 0$ in $\mathfrak{M}_R$ where F is flat; we then take $\mathcal{F}(M) = [K] \in H$. Note that if $0 \to K' \to F' \to M \to 0$ is also exact with F' flat, then

$$0 \to M^0 \to F^0 \to K^0 \to 0, \qquad 0 \to M^0 \to F'^0 \to K'^0 \to 0$$

are exact (by (4.8)) with F^0, F'^0 injective (by (4.9)). By the Injective Schanuel's Lemma (5.40), we have

$$(K' \oplus F)^0 \cong K'^0 \oplus F^0 \cong K^0 \oplus F'^0 \cong (K \oplus F')^0,$$

so K, K' are flat equivalent by Def. (5.53). This checks that $\mathcal{F}(M) = [K] \in H$ is well-defined, and, as we can see easily, $\mathcal{F}(M)$ depends only on the flat class of M. Therefore, $\mathcal{F}$ is an additive shift operator on H. Clearly, this operator is "induced" by the projective shift operator $\mathcal{P}$ defined on the semigroup G of the projective equivalence classes. Finally, note that

(5.54) $$[M] = 0 \text{ in } H \text{ iff } M_R \text{ is flat.}$$

In fact, if $[M] = 0 \in H$, then $(M \oplus F)^0 \cong F'^0$ for suitable flat modules F, F'. But then $M^0 \oplus F^0 \cong F'^0$ implies that M^0 is injective, and hence M is flat, by (4.9).

(5.55) Definition. The *flat dimension* of M_R is defined to be

$$\text{fd}(M) = \text{fd}_R(M) = \min\{n : \mathcal{F}^n(M) = 0 \in H\}.$$

In particular, $\text{fd}(M) \leq \text{pd}(M)$. The *right weak dimension* of R is defined to be

(5.56) $$\text{r. wd}(R) = \sup\{\text{fd}(M_R)\} \leq \text{r. gl. dim } R.$$

The *left weak dimension of* R (l.wd(R)) is defined similarly, by using left R-modules.

While $\mathcal{F}$ is related to the projective shift $\mathcal{P}$ by "coarsening", it is also related to the injective shift $\mathcal{I}$ (on *left* modules) through the character module formation.

This relation can be expressed in the form of a commutative diagram, where H is as above, and H' is the semigroup of injective equivalence classes of *left* R-modules:

$$\begin{array}{ccc} H & \xrightarrow{\mathcal{F}} & H \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ H' & \xrightarrow{\mathcal{I}} & H' \end{array}$$

In a formula, we have $\mathcal{I}(M^0) = \mathcal{F}(M)^0$ for any M_R. This together with (5.54) (or (4.9)) immediately imply that

$$\text{(5.57)} \qquad \mathrm{fd}(M) = \mathrm{id}(M^0) \quad \text{for any } M_R.$$

Of course, $\mathrm{id}(M^0)$ here denotes the injective dimension of the *left* R-module M^0. Using (5.57) in conjunction with $\mathrm{fd}(M) \leq \mathrm{pd}(M)$ and (5.45), we obtain the following.

(5.58) Theorem. $\mathrm{r.\,wd}(R) \leq \min\{\mathrm{r.\,gl.\,dim}\ R,\ \mathrm{l.\,gl.\,dim}\ R\}$.

This inequality is perhaps the main reason for the choice of the term "weak dimension". We also have the following two important results due to M. Auslander.

(5.59) Theorem. *Let R be any right noetherian ring. Then* $\mathrm{fd}(M) = \mathrm{pd}(M)$ *for any f.g. right R-module M, and* $\mathrm{r.\,gl.\,dim}\ R = \mathrm{r.\,wd}(R) \leq \mathrm{l.\,gl.\,dim}\ R$. *(For instance, if R is left hereditary, it is right hereditary.)*

Proof. Since R is right noetherian, there exists a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{\alpha_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

with f.g. projective right modules $\{P_n\}$. Then $K_n := \mathrm{im}(\alpha_n)$ is f.g., and represents the flat class $\mathcal{F}^n(M)$ as well as the projective class $\mathcal{P}^n(M)$. But if K_n is flat, then it is also projective, by Case (1) of (4.38). This shows that $\mathrm{pd}(M) \leq \mathrm{fd}(M)$, and hence the equality. For the last part of the Theorem, it suffices to show that r.gl.dim $R \leq$ r.wd(R). This follows from the above, since, by (5.51), r.gl.dim R can be computed as $\sup\{\mathrm{pd}(M)\}$ where M ranges over all f.g. right R-modules. □

(5.60) Corollary. *For any noetherian ring R, we have*

$$\mathrm{r.\,wd}(R) = \mathrm{l.\,wd}(R) = \mathrm{r.\,gl.\,dim}\ R = \mathrm{l.\,gl.\,dim}\ R.$$

(5.61) Remark. For right noetherian rings that are not left noetherian, we have mentioned before (cf. (5.15)) that r.gl.dim $R <$ l.gl.dim R is possible. C. Jensen showed, in 1966, that if all ideals of R are countably generated, then the two dimensions differ at most by 1. On the other hand, in 1969, A. V. Jategaonkar produced examples of right noetherian rings R with r.gl.dim $R = m$ *and* l.gl.dim $R = n$ where m, n are arbitrary, with $0 < m \leq n \leq \infty$.

We now turn our attention to the relationship between r.wd(R) and l.wd(R) for general rings. First, some examples are in order.

(5.62a) Example. A ring R has r.wd$(R) = 0$ iff all right R-modules are flat, iff R is von Neumann regular (by (4.21)). It follows that r.wd$(R) = 0$ iff l.wd$(R) = 0$. (Using (5.59), we obtain a homological proof for the fact (see *FC*–(4.25)) that a right noetherian von Neumann regular ring must be semisimple.)

(5.62b) Example. Let R be a ring that is not von Neumann regular. Then r.wd$(R) = 1$ (resp. l.wd$(R) = 1$) iff submodules of flat right (resp. left) modules are flat. From (4.66), it follows that r.wd$(R) = 1$ iff l.wd$(R) = 1$, iff right (resp. left) ideals are flat. For commutative domains R, this condition means precisely that R is a Prüfer domain (and not a field); see (4.69).

For a noncommutative example, consider Small's triangular ring

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Since R is right noetherian and right hereditary (cf. (2.33)), (5.60) applies and we have

$$1 = \text{r. gl. dim } R = \text{r. wd}(R) = \text{l. wd}(R).$$

But by (2.33) R is *not* left hereditary, so l.gl.dim $R \geq 2$. As it turns out, l.gl.dim $R = 2$. (For more details, see Exercise 23.)

The equality of r.wd(R) and l.wd(R) in the above examples is, again, a special case of the following general theorem in the theory of homological dimensions.

(5.63) Theorem. *For any ring R,* $\text{r. wd}(R) = \text{l. wd}(R)$.

This theorem is possibly a bit surprising, since the same result does not hold for (projective or injective) global dimensions. We shall show, however, that (5.63) can be proved by essentially the same argument used earlier for proving r.gl.dim R = r.inj.gl.dim R. First, we have the following analogue of (5.46).

(5.64) Lemma. *Let $\mathcal{E} : 0 \to K \to F \to M \to 0$ and $\mathcal{E}' : 0 \to K' \to F' \to M' \to 0$ be exact sequences in, respectively, $\mathfrak{M}_R$ and ${}_R\mathfrak{M}$.*

(1) *Assume F is flat. Then*

(a) $\mathcal{E} \otimes_R K'$ *exact* $\Longrightarrow$ $K \otimes_R \mathcal{E}'$ *exact.*
(b) $\mathcal{E} \otimes_R M'$ *exact* $\Longrightarrow$ $M \otimes_R \mathcal{E}'$ *exact.*

(2) *Assume F' is flat. Then*

(a) $K \otimes_R \mathcal{E}'$ *exact* $\Longrightarrow$ $\mathcal{E} \otimes_R K'$ *exact.*
(b) $M \otimes_R \mathcal{E}'$ *exact* $\Longrightarrow$ $\mathcal{E} \otimes_R M'$ *exact.*

Proof. As in (5.46), the proof is an easy diagram chase in:

$$\begin{array}{ccccccc}
K \otimes K' & \longrightarrow & F \otimes K' & \longrightarrow & M \otimes K' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K \otimes F' & \longrightarrow & F \otimes F' & \longrightarrow & M \otimes F' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K \otimes M' & \longrightarrow & F \otimes M' & \longrightarrow & M \otimes M' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

(where $\otimes = \otimes_R$). In fact, we have already proved (1)(b) and (2)(b) in FC–(24.22) by chasing the same diagram. □

As before, we can prove by using (5.64)(1(b) and 2(b)):

(5.65) Theorem. *For a right R-module M and a left R-module M', the following are equivalent:*

(1) *$M \otimes_R \mathcal{E}'$ is exact for some $\mathcal{E}' : 0 \to K' \to F' \to M' \to 0$ where F' is flat.*
(1′) *$M \otimes_R \mathcal{E}'$ is exact for all $\mathcal{E}' : 0 \to K' \to F' \to M' \to 0$.*
(2) *$\mathcal{E} \otimes_R M'$ is exact for some $\mathcal{E} : 0 \to K \to F \to M \to 0$ where F is flat.*
(2′) *$\mathcal{E} \otimes_R M'$ is exact for all $\mathcal{E} : 0 \to K \to F \to M \to 0$.*

(5.66) Definition. If M, M' satisfy the (equivalent) conditions in (5.65), we shall say that $\operatorname{Tor}(M, M') = 0$. (Unfortunately, there is no description for this relation analogous to the simple description (5.47)(3) for $\operatorname{Ext}(C, C') = 0$. This is part of the reason why we have tried to "avoid" using the description (5.47)(3) before.)

Again, we shall try to get by with the notion of $\operatorname{Tor}(M, M') = 0$, thus avoiding a full introduction of the higher Tor-functors. As before, we can deduce the following characterization of flatness from (5.65)((1′) and (2′)).

(5.67) Lemma. *For any right R-module M, the following are equivalent:*

(1) *M is flat.*
(2) $\operatorname{Tor}(M, M') = 0$ *for all* $_RM'$.
(3) $\operatorname{Tor}(M, R/\mathfrak{A}) = 0$ *for all left ideals* $\mathfrak{A} \subseteq R$.

A similar result holds for the characterization of left flat modules.

Here the proof of (3)$\Longrightarrow$(1) uses the Modified Flatness Test (4.12).

As before, $\mathrm{Tor}(M, M') = 0$ is a property depending only on the flat equivalence classes of M_R and ${}_RM'$. Keeping this observation in mind, we deduce from (5.64)(1(a) and 2(a)):

(5.68) Theorem. *For any right R-module M and left R-module M', we have* $\mathrm{Tor}(\mathcal{F}(M), M') = 0$ *iff* $\mathrm{Tor}(M, \mathcal{F}(M')) = 0$. *(Of course, the second $\mathcal{F}$ here refers to the flat shift on left R-modules.)*

Using this and (5.67), we deduce immediately Theorem (5.63), as well as the following flat analogue of Auslander's result (5.51).

(5.69) Theorem. $\mathrm{r.wd}(R) = \sup\{\mathrm{fd}(M)\}$, *where M ranges over all cyclic right R-modules. Equivalently, if R is not a von Neumann regular ring, then*

$$\mathrm{r.wd}(R) = 1 + \sup\{\mathrm{fd}(\mathfrak{A})\},$$

where $\mathfrak{A}$ ranges over all right ideals of R.

Note that, by (5.63), $\mathrm{r.wd}(R) = \mathrm{l.wd}(R) \leq \mathrm{l.gl.dim}\, R$. This provides a proof for (5.58) (and hence (5.59), (5.60)) without the use of character modules. Of course, in view of (5.63), we should now write $\mathrm{wd}(R)$ for the common value of $\mathrm{r.wd}(R)$ and $\mathrm{l.wd}(R)$, and call it simply the *weak dimension* of R.

In homological algebra, $\mathrm{Tor}_n^R(M, M')$ $(n \geq 1)$ are a sequence of abelian groups associated with $M \in \mathfrak{M}_R$ and $M' \in {}_R\mathfrak{M}$. They depend only on the flat classes of M and M', and are "covariant" in these variables. Moreover, we have

$$\mathrm{Tor}_1^R(\mathcal{F}(M), M') \cong \mathrm{Tor}_1^R(M, \mathcal{F}(M')),$$

which, of course, subsumes (5.68). The higher Tor-groups $\mathrm{Tor}_{n+1}^R(M, M')$ may be taken as $\mathrm{Tor}_1^R(\mathcal{F}^n(M), M') \cong \mathrm{Tor}_1^R(M, \mathcal{F}^n(M'))$. Finally, $\mathrm{wd}(R) \leq n$ iff $\mathrm{Tor}_{n+1}^R \equiv 0$ (the zero bifunctor). These facts about the Tor_n-functors together with those about the Ext^n-functors certainly give a broader and more complete view of the theory of homological dimensions of modules and rings. However, it is remarkable that one can also prove most of the main results about homological dimensions without a full introduction of these higher functors.

§5E. *Global Dimensions of Semiprimary Rings*

Let us now give some applications of the theorems obtained above by studying certain special classes of rings. The next group of results, from (5.70) to (5.75), assumes some familiarity with the material on right perfect rings developed in *FC*–§§23–24. Readers not familiar with the notion of right perfect rings may skip to (5.76) without loss of continuity.

(5.70) Theorem (Bass). *Let R be a right perfect ring. Then for any right R-module M,* $\mathrm{fd}(M) = \mathrm{pd}(M)$. *In particular,* $\mathrm{r.gl.dim}\, R = \mathrm{wd}(R) \leq \mathrm{l.gl.dim}\, R$. *If R is also left perfect, equality holds.*

Proof. Over R, every right flat module is projective, by *FC*–(24.25). Applying the argument in the proof of (5.59) to M gives the desired conclusion. □

To note a useful special case of (5.70), let us recall the class of semiprimary rings mentioned earlier in §3I. A ring R is called *semiprimary* if its Jacobson radical $J = \text{rad } R$ is nilpotent, and the quotient ring R/J is semisimple. Since any semiprimary ring is (left and right) perfect by *FC*–(23.19), (5.70) yields the following.

(5.71) Corollary. *For any semiprimary ring* R, *we have*

$$\text{wd}(R) = \text{r. gl. dim } R = \text{l. gl. dim } R.$$

For any semiprimary ring R, we may write gl.dim R for the three equal numbers above. This number may be computed as follows.

(5.72) Theorem (Auslander). *Let R be any semiprimary ring with Jacobson radical* $J = \text{rad } R$. *Let $\{C_i\}$ be a complete set of simple right R-modules (up to isomorphism). Then*

$$\text{gl. dim } R = \max\{\text{pd}(C_i)\} = \text{pd}((R/J)_R). \tag{5.73}$$

In particular, the following are equivalent:

(1) *R is right semihereditary.*
(2) *R is right hereditary.*
(3) *J_R is a projective R-module.*
(4) *J_R is a flat R-module.*
(1)′-(4)′ *The left analogues of* (1)-(4).

Proof. The C_i 's are just the simple right modules over the semisimple ring $\overline{R} := R/J$ (up to isomorphism). In particular, they are finite in number. This justifies the notation "max" in (5.73). Let $m = \max\{\text{pd}(C_i)\} \leq \infty$. Let C be any right R-module with $CJ = 0$. Then C is semisimple and is isomorphic to a direct sum of the C_i 's (with multiplicities). By (5.25), $\text{pd}(C_R) \leq m$. Now consider *any* right R-module M. If, say, $J^n = 0$, we have a finite filtration

$$0 = MJ^n \subseteq \cdots \subseteq MJ^2 \subseteq MJ \subseteq M.$$

Since each filtration factor is killed by J, (5.24) gives

$$\text{pd}(M) \leq \max\{\text{pd}(MJ^i/MJ^{i+1})\} \leq m,$$

and so r.gl.dim $R = m$. The equality $m = \text{pd}((R/J)_R)$ follows from (5.25) as well since $(R/J)_R$ is a (finite) direct sum of the C_i 's, each occurring at least once.

(1)$\Longrightarrow$(4) follows from (4.6).

(4)$\Longrightarrow$(3). Since R is right perfect, any flat right R-module is projective (by *FC*–(24.24)).

(3)$\Longrightarrow$(2). The sequence $0 \to J \to R \to R/J \to 0$ in $\mathfrak{M}_R$ shows that

$$1 \geq \text{pd}((R/J)_R) = \text{gl. dim } R \quad (\text{by } (5.73)).$$

(2)$\Longrightarrow$(1) is a tautology.

(2) $\Longleftrightarrow$ (2)′ follows from l.gl.dim R = r. gl. dim R, and the equivalence of (1)′-(4)′ follows from left-right symmetry. □

To obtain a nice consequence of (5.72), we shall use informally the following notion: a ring R is called *right Kasch* if every simple right R-module is isomorphic to a (minimal) right ideal of R. This notion will be studied more systematically in §8C; right now, we just have the following application in mind.

(5.74) Corollary. *Let* R *be a semiprimary ring that is right (or left) Kasch. Then* gl. dim R *is* 0 *or* ∞.

Proof. Assume, instead, that gl.dim R is a positive integer m. By (5.73), m = pd(C) for some simple right R-module C. Since R is right Kasch, C_R is isomorphic to a suitable right ideal $\mathfrak{A}$. The module $(R/\mathfrak{A})_R$ is not projective, for otherwise

$$(*) \qquad 0 \longrightarrow \mathfrak{A} \longrightarrow R \longrightarrow R/\mathfrak{A} \longrightarrow 0$$

would split and we would have $m = \text{pd}(\mathfrak{A}) = 0$. Therefore, (∗) gives

$$\text{pd}(R/\mathfrak{A}) = \text{pd}(\mathfrak{A}) + 1 = m + 1,$$

a contradiction! □

(5.75) Corollary. *Let* R *be a semiprimary ring, with* J = rad R. *Assume either* R/J *is a simple ring or* R *is commutative. Then,* R *is right (resp. left) Kasch, and in particular* gl. dim R *is* 0 *or* ∞.

Proof. *First assume* R/J *is an (artinian) simple ring.* Then R has, up to isomorphism, a *unique* (say, right) simple module C. Let n be the least integer such that $J^n = 0$. We may assume that $n > 1$ (for otherwise gl.dim $R = 0$ already). Then $J^{n-1} \neq 0$, and (being killed by J) it is a semisimple right R-module. In particular, it must contain a minimal right ideal, necessarily isomorphic to C. Therefore, R is right (and similarly, left) Kasch, and (5.74) applies. *Next, assume R is commutative.* Since R is semiperfect, it decomposes into a finite direct product of local rings $(R_i, \mathfrak{m}_i)$ (by *FC*–(23.11)). Here the $\mathfrak{m}_i$'s remain nilpotent, and each $R_i/\mathfrak{m}_i$ is a field. Applying the first case of the Corollary, we see that each R_i is Kasch, and gl.dim $R_i \in \{0, \infty\}$. From this, we see easily that R is also Kasch, and that gl.dim $R \in \{0, \infty\}$. □

Since any right or left artinian ring is semiprimary, the results (5.71)-(5.75) apply well to any such ring R. For instance, if R is any right or left artinian local ring, then $R/\text{rad}\ R$ is a division ring, and (5.75) implies that gl.dim R is 0 or ∞. For another example, let R be any right or left artinian ring that is right self-injective (for instance any group algebra kG over a finite group G). This is what we shall later call a "quasi-Frobenius ring": such a ring will be shown to be always Kasch (see the proof of (15.1)). Therefore, again, (5.75) implies that gl.dim R is 0 or ∞. On the other hand, the following example shows that there do exist finite-dimensional (necessarily noncommutative) algebras over any field k with *arbitrary* finite global dimension $n \geq 1$.

(5.76) Example. For any field k and any natural number n, let R be the $(2n+1)$-dimensional k-algebra on the basis

$$\{e_0, e_1, \ldots, e_n, v_1, \ldots, v_n\}$$

with multiplication defined as follows:

$$(5.77) \qquad \begin{aligned} e_i e_j &= \delta_{ij} e_i\,, & v_i v_j &= 0\,,\\ v_i e_j &= \delta_{ij} v_i\,, & e_j v_i &= \delta_{j+1,i} v_i\,. \end{aligned}$$

Since many multiplications simply give 0, it is easy to check the associative law for the basis elements. (The most significant case we need to check is

$$e_{i-1}(v_i e_i) = (e_{i-1} v_i) e_i \quad (i \geq 1).$$

Here, both sides are equal to v_i.) Note that $e_0, e_1, \ldots, e_n$ are mutually orthogonal idempotents, and that $1 := e_0 + e_1 + \cdots + e_n$ is the identity for R. Actually, if we let S be the ring $ke_0 \times \cdots \times ke_n$, and equip $J := kv_1 \oplus \cdots \oplus kv_n$ with the (S, S)-bimodule structure given by the bottom formulas of (5.77), then R is precisely the "trivial extension" of J by S defined in (2.22)(A). A couple of more concrete realizations of R will be given later, which will give us a broader view of this ring. Right now, we shall just work formally with the multiplication formulas given in (5.77); these will enable us to make quick computations.

Since $J := \bigoplus_{i=1}^{n} kv_i$ is an ideal with square zero and

$$R/J \cong S \cong k \times \cdots \times k \quad (n+1 \text{ copies}),$$

we see that $J = \text{rad}\ R$, and that there are $n+1$ distinct simple left R-modules $C_0, \ldots, C_n$, where $\dim_k C_j = 1$, and e_i acts as $\delta_{ij} I$ on C_j. (It is a bit more convenient to work with left R-modules here.) An easy computation shows that

$$P_j := Re_j = ke_j \oplus kv_j, \qquad JP_j = Je_j = kv_j \quad (j \geq 0),$$

where, for convenience, we have set $v_0 = 0$. In view of this computation, we may identify C_j with P_j/JP_j. Since this module is simple, P_j must be indecomposable. Therefore, $\{P_0, \ldots, P_n\}$ gives a full set of principal indecomposable (projective) left modules over R. Also, noting that $JP_j = kv_j \cong C_{j-1}$, we have a short exact sequence

$$0 \longrightarrow C_{j-1} \longrightarrow P_j \longrightarrow C_j \longrightarrow 0$$

for each $j \geq 1$. This enables us to determine $\text{pd}(C_j)$. To begin with, $\text{pd}(C_0) = \text{pd}(P_0) = 0$. It is easy to check that the above sequence does not split for $j = 1$, so we have $\text{pd}(C_1) = 1$. Since $\mathcal{P}(C_j) = [C_{j-1}]$, we conclude by induction that $\text{pd}(C_j) = j$ for all j. *It now follows from* (5.72) *that* gl.dim $R = n$. Also, by "connecting" the short exact sequences obtained above, we can write down for each $j \geq 0$, an explicit projective resolution for C_j:

$$0 \longrightarrow P_0 \longrightarrow \cdots \longrightarrow P_i \xrightarrow{\varphi_i} P_{i+1} \longrightarrow \cdots \longrightarrow P_j \longrightarrow C_j \longrightarrow 0.$$

Here, φ_i is simply right multiplication by v_{i+1}. (Note that, by (5.77), $P_i v_{i+1} = Re_i v_{i+1} = Rv_{i+1} = kv_{i+1} \subseteq P_{i+1}$.)

The $(2n+1)$-dimensional k-algebra R can be identified more explicitly as follows. Let k' be the "ring of dual numbers" over k; that is, $k' = k[\varepsilon]$ with $\varepsilon^2 = 0$. It is easy to see that R is isomorphic to the following k-subalgebra of the matrix algebra $\mathbb{M}_{n+1}(k')$:

$$A = \begin{pmatrix} k & k\varepsilon & & 0 \\ & k & \ddots & \\ & & \ddots & k\varepsilon \\ 0 & & & k \end{pmatrix}.$$

An explicit k-algebra isomorphism from R to A is given by:

$$\varphi(e_j) = E_{j+1,j+1} \quad (j \geq 0) \quad \text{and} \quad \varphi(v_j) = \varepsilon E_{j,j+1} \quad (j \geq 1),$$

where the E_{ij}'s (i, j ranging from 1 to $n+1$) are the matrix units of $\mathbb{M}_{n+1}(k')$. It is straightforward to check that φ respects all the relations in (5.77). Thus, in retrospect, we could have taken R to be the matrix algebra $A \subset \mathbb{M}_{n+1}(k')$. Note that if we identify R with A, the jth principal indecomposable module P_j $(0 \leq j \leq n)$ is simply the left ideal of A consisting of matrices in A with nonzero entries only on the $(j+1)^{\text{th}}$ column. We have shown that gl.dim $A = n$. In particular, when $n = 1$, A is a hereditary k-algebra. This algebra is, however, nothing new: it is just isomorphic to $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ by the isomorphism that sends $\begin{pmatrix} a & b\varepsilon \\ 0 & c \end{pmatrix}$ to $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ (for $a, b, c \in k$). In fact, for any $n \geq 1$, it is not difficult to see (by constructing a similar isomorphism) that A is isomorphic to the ring $T/(\text{rad } T)^2$, where T is the ring of $(n+1) \times (n+1)$ upper triangular matrices over k, and $(\text{rad } T)^2$ is the square of the Jacobson radical of T.

After the foregoing analysis on the algebra R, the following remark on the Kasch property is in order. We know that, among the $n+1$ simple left R-modules C_j $(0 \leq j \leq n)$, $C_0, \ldots, C_{n-1}$ do embed into ${}_RR$, namely, as the minimal left ideals $kv_1, \ldots, kv_n$. Of course, C_0 also embeds as ke_0, but this is relatively insignificant. The truly significant thing here is that gl. dim $R = n$. From this, for instance, we could have predicted from (5.75) that R is not left Kasch, and this necessarily means that the remaining simple module C_n *cannot* be embedded in ${}_RR$. We invite the reader to verify this directly. Similarly, if $C_0', \ldots, C_n'$ are

the distinct simple right R-modules (labeled in accordance with the idempotents $e_0, \ldots, e_n$), then $C'_1, \ldots, C'_n$ embed in R_R as minimal right ideals $kv_1, \ldots, kv_n$, but R is also not right Kasch so there is no such embedding for C'_0.

To restore the Kasch property, we can make the J bigger by adding a v_0; in other words, take $J = \bigoplus_{i=0}^{n} kv_i$. We can keep the multiplication formulas in (5.77), with the sole modification that $e_n v_0 = v_0$. The new ring R' obtained this way is now left and right Kasch, so we have $\text{gl. dim } R' = \infty$ according to (5.75). *But what about our cute little calculation of the global dimension using the projective shifts?* We'll leave it to the reader to find out exactly what goes wrong with that argument for the new algebra R'.

To conclude the present subsection §5E, we shall mention a relevant open problem. Given a ring R, it is of interest to find out when r.gl.dim R can be computed via knowledge of the numbers $\{\text{pd}(C_i)\}$, where C_i ranges over the simple right R-modules. Auslander's Theorem (5.72) gives the simple equation

$$\text{r. gl. dim } R = \sup\{\text{pd}(C_i)\}$$

for any *semiprimary* ring R. It is known that this equation is not true in general for 1-sided noetherian rings, but it has remained an open question *whether the equation holds for (2-sided) noetherian rings.* Affirmative answers are known in several cases, of which we shall mention two. The first case is when R is commutative: this will be handled below in (5.92). The second case is when r.gl.dim $R < \infty$ (work of Bhatwadekar and Goodearl). The unknown case is when (R is noetherian and) r.gl.dim $R = \infty$: here, one must decide if $\sup\{\text{pd}(C_i)\} = \infty$ also.

A few other interesting open problems concerning homological dimensions of modules and rings (with detailed commentary) can be found in the Appendix of [Goodearl-Warfield: 89].

§5F. Global Dimensions of Local Rings

We turn our attention now to the class of local rings R. Throughout the treatment of local rings, we let $\mathfrak{m} = \text{rad } R$, and k be the residue division ring $R/\mathfrak{m}$. Via the projection map $R \to k$, we can view k as a right (resp. left) R-module. It turns out that this unique simple right (resp. left) R-module controls much of the homological behavior of the modules in $\mathfrak{M}_R$ (resp. ${}_R\mathfrak{M}$). To develop this theme, we start with the following elementary observation.

(5.78) Lemma. *Let* $(R, \mathfrak{m})$ *be a local ring as above, and* C *be a f.g. right* R*-module. If* $\text{Hom}_R(C, k) = 0$, *then* $C = 0$.

Proof. Assume $C \neq 0$. By Nakayama's Lemma *FC*–(4.22), $C \neq C\mathfrak{m}$. Fix a k-vector space epimorphism $f : C/C\mathfrak{m} \to k$. Composing this with $C \to C/C\mathfrak{m}$, we see that $\text{Hom}_R(C, k) \neq 0$. □

(5.79) Proposition. *Let* $(R, \mathfrak{m})$ *be a right noetherian local ring, and* C *be a f.g. right* R*-module. Then the following are equivalent:*

(1) *C is projective.*
(1′) *C is free.*
(2) $\text{Ext}(C, k_R) = 0$.
(2′) $\text{Hom}_R(\mathcal{E}, k)$ *is exact for any exact sequence* $\mathcal{E} : 0 \to A \to B \to C \to 0$ *in* $\mathfrak{M}_R$.
(3) $\text{Tor}(C, {}_Rk) = 0$.
(3′) $\mathcal{E} \otimes_R k$ *is exact for any* $\mathcal{E}$ *as in* (2′).

Proof. (1)$\Longleftrightarrow$(1′) is well known; see *FC*–(19.29).

(2)$\Longleftrightarrow$(2′) and (3)$\Longleftrightarrow$(3′) are "definitions", from our perspective.

(1)$\Longrightarrow$(2′) is clear, since $\mathcal{E}$ must split under (1).

(2′)$\Longrightarrow$(1). Fix an exact sequence $0 \to K \xrightarrow{\varphi} F \to C \to 0$ as in the proof of (4.38)(2). In particular, $K \subseteq F\mathfrak{m}$. By (2′), we have the following exact sequence:

$$(5.80) \qquad 0 \longrightarrow \text{Hom}_R(C, k) \longrightarrow \text{Hom}_R(F, k) \xrightarrow{\varphi^*} \text{Hom}_R(K, k) \longrightarrow 0.$$

Any homomorphism $F \to k$ is zero on $F\mathfrak{m}$, and hence on K. This means that $\varphi^* = 0$, so the exactness of (5.80) implies that $\text{Hom}_R(K, k) = 0$. But K_R is f.g. since R is right noetherian. Therefore, $K = 0$ by (5.78) and $C \cong F$ is free.

(1)$\Longrightarrow$ (3′) is clear as before.

(3′) $\Longrightarrow$(1). Proceeding as in the proof of (2′) $\Longrightarrow$(1), we get an exact sequence

$$0 \longrightarrow K \otimes_R k \longrightarrow F \otimes_R k \longrightarrow C \otimes_R k \longrightarrow 0, \quad \text{or}$$

$$0 \longrightarrow K/K\mathfrak{m} \xrightarrow{\psi^*} F/F\mathfrak{m} \longrightarrow C/C\mathfrak{m} \longrightarrow 0$$

under the usual identifications. Now $K \subseteq F\mathfrak{m}$ implies that $\psi^* = 0$, so $K/K\mathfrak{m} = 0$ and hence $K = 0$ by Nakayama's Lemma. We conclude again that $C \cong F$ is free. □

At this point, the reader should recall our earlier remark about the higher Ext and Tor functors, made in (5.52) and in the second paragraph after (5.69). In keeping with the general notations used in the literature, let us define $\text{Ext}_R^{n+1}(A_R, B_R) = 0$ to mean $\text{Ext}(\mathcal{P}^n(A), B) = 0$ (equivalently, $\text{Ext}(A, \mathcal{I}^n(B)) = 0$), and $\text{Tor}_{n+1}^R(A_R, {}_RB) = 0$ to mean $\text{Tor}(\mathcal{F}^n(A), B) = 0$ (equivalently, $\text{Tor}(A, \mathcal{F}^n(B)) = 0$). Using this suggestive notation, we now deduce the following.

(5.81) Corollary. *Let M be any f.g. right R-module, where R is as above. Given any $n \geq 0$, the following are equivalent:*

(1) $\text{pd}(M) \leq n$.
(2) $\text{Ext}_R^{n+1}(M, k) = 0$.
(3) $\text{Tor}_{n+1}^R(M, k) = 0$.

Proof. We need only prove (2)$\Longrightarrow$(1) and (3)$\Longrightarrow$(1). Let C be a f.g. right R-module representing $\mathcal{P}^n(M)$. If (2) or (3) holds, then $\text{Ext}(C, k) = 0$ or $\text{Tor}(C, k) = 0$, and (5.79) implies that C is projective. This means that $\mathcal{P}^n(M) = 0$, and so (1) holds. □

(5.82) Theorem. *For any right noetherian local ring R:*

$$\text{r. gl. dim } R = \text{id}(k_R) \leq \text{pd}({}_R k).$$

Proof. (1) Suppose $\text{pd}({}_R k) \leq n$. Then for any f.g. M_R, $\text{Tor}(M, \mathcal{F}^n({}_R k)) = 0$, and so $\text{Tor}_{n+1}^R(M, k) = 0$. By (5.81), we have $\text{pd}(M) \leq n$. In view of (5.51), this proves that r.gl.dim $R \leq \text{pd}({}_R k)$.

(2) Now suppose $\text{id}(k_R) \leq n$. Then for any f.g. M_R, $\text{Ext}(M, \mathcal{I}^n(k_R)) = 0$, and so $\text{Ext}_R^{n+1}(M, k) = 0$. By (5.81), we have $\text{pd}(M) \leq n$, thus proving r.gl.dim $R \leq \text{id}(k_R)$. On the other hand, we have always

$$\text{id}(k_R) \leq \text{r. inj. gl. dim } R = \text{r. gl. dim } R\,,$$

so equality holds. □

In the case of a *commutative* R, the result above can be further refined, as follows.

(5.83) Corollary. *Let R be a commutative noetherian local ring. Then:*

(1) $\text{gl. dim } R = \text{id}(k) = \text{pd}(k)$.
(2) $\text{gl. dim } R \leq n$ *iff* $\text{Tor}_{n+1}^R(k, k) = 0$, *iff* $\text{Ext}_R^{n+1}(k, k) = 0$.

Proof. (1) follows from (5.82) since, in this *commutative* case,

$$\text{pd}({}_R k) = \text{pd}(k_R) \leq \text{r. gl. dim } R.$$

For (2), it suffices to prove the "if" parts. Assume either $\text{Tor}_{n+1}^R(k, k) = 0$ or $\text{Ext}_R^{n+1}(k, k) = 0$. Applying (5.81) to $M = k_R$, we see that $\text{pd}(k_R) \leq n$, and therefore, by (1), gl.dim $R \leq n$. □

For commutative noetherian local rings, there is an extensive theory of homological and cohomological dimensions of modules. The first major result is the following, which identifies the class of commutative noetherian local rings of finite global dimension.

(5.84) Theorem (Serre, Auslander-Buchsbaum). *Let $(R, \mathfrak{m})$ be a commutative noetherian local ring. Then* $\text{gl. dim } R < \infty$ *iff R is a regular local ring. In this case,* $\text{gl. dim } R = \dim R$ *(the Krull dimension of R).*

Recall that the Krull dimension of a commutative ring R is the supremum of the lengths of all chains of prime ideals in R. Unless stated otherwise, $(R, \mathfrak{m})$ *shall*

denote a commutative noetherian local ring below, with residue field $k = R/\mathfrak{m}$. In this case, dim R is just the height of $\mathfrak{m}$, and by a theorem of Krull, this height is bounded by the minimum number $V(R)$ of generators of the ideal $\mathfrak{m}$. By Nakayama's Lemma, $V(R) = \dim_k \mathfrak{m}/\mathfrak{m}^2$. If the inequality $\dim R \leq V(R)$ happens to be an equality, $(R, \mathfrak{m})$ is said to be a *regular local ring*. It is known that a regular local ring is always an integral domain;[45] we shall assume this fact without proof.

Let us first prove the "if" part of (5.84), and its last statement. Suppose R is regular, with $\dim R = d$. Then $\mathfrak{m} = \sum_{i=1}^{d} x_i R$ for suitable $x_i \in R$. Since R is a domain, x_1 is not a 0-divisor in R. It is easy to see that $R/x_1 R$ is a regular local ring of dimension $d-1$, with maximal ideal generated by the images of $x_2, \ldots, x_d$. Therefore, by induction on d, we see that $x_1, \ldots, x_d$ form a regular sequence in R. By (5.83)(1) and (5.32), we have then

$$\text{gl. dim } R = \text{pd}(k) = \text{pd}(R/\mathfrak{m}) = d = \dim R.$$

The "only if" part of (5.84) is quite a bit harder. Its proof depends on the following lemma on commutative noetherian local rings $(R, \mathfrak{m})$.

(5.85) Lemma. *Suppose* $\mathfrak{m} \setminus \mathfrak{m}^2$ *consists of* 0*-divisors of* R. *Then* $a\,\mathfrak{m} = 0$ *for some nonzero element* $a \in R$.

Proof. Let $\text{Ass}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, so that $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r$ is the set of 0-divisors of R (see Exercise (3.40E)). By the given hypothesis, we have thus

$$\mathfrak{m} \setminus \mathfrak{m}^2 \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r. \tag{5.86}$$

We may assume that $\mathfrak{m} \neq 0$ (for otherwise we can take $a = 1$). By Nakayama's Lemma, there exists an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. For any $y \in \mathfrak{m}^2$, we have $x + y^p \in \mathfrak{m} \setminus \mathfrak{m}^2$ for $p \geq 1$, so by the Pigeon-Hole Principle, $x + y^p$, $x + y^q \in \mathfrak{p}_i$ for suitable p, q, i with $p < q$. By subtraction, we have $y^p(1 - y^{q-p}) \in \mathfrak{p}_i$, so $y \in \mathfrak{p}_i$. This (together with (5.86)) yields

$$\mathfrak{m} \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r\,. \tag{5.87}$$

By the Lemma of Prime Avoidance,[46] this implies that $\mathfrak{m} \subseteq \mathfrak{p}_j$ for some j. Since $\mathfrak{p}_j$ has the form $\text{ann}(a)$ for some $a \neq 0$, we have $a\,\mathfrak{m} = 0$ as desired. □

We can now prove the "only if" part of (5.84), by induction on $n := \text{gl. dim } R < \infty$. If $n = 0$, R is semisimple. Then R is a field, and we are done. Now assume

[45]This can be proved, for instance, by working with the associated graded ring $R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \ldots$ of R. For more details, see Zariski-Samuel [58: vol. 2, p. 302].

[46]This is a well-known classical lemma in commutative algebra due to Neal McCoy. The nomenclature of "prime avoidance" used here follows Eisenbud [95: pp. 90-91]. Eisenbud explained this lemma in the following words: "If an ideal I (in a commutative ring) is not contained in any of a finite number of prime ideals $\mathfrak{p}_j$, then there is an element of I that "avoids" being contained in any of the $\mathfrak{p}_j$'s." We shall use this basic fact freely in the following.

$n > 0$. *We claim that* $\mathfrak{m} \setminus \mathfrak{m}^2$ *contains a non* 0*-divisor.* For, if otherwise, the lemma above yields an element $a \neq 0$ with $a\mathfrak{m} = 0$. Fix a f.g. R-module P with $\text{pd}(P) = 1$, and take an exact sequence $0 \to K \to F \to P \to 0$ as in the proof of (4.38)(2). Then K is f.g. projective, and hence free. But $Ka \subseteq F\mathfrak{m}a = 0$, so $K = 0$. This implies that $P \cong F$ is free, a contradiction. Now fix a non 0-divisor $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, and let $\overline{R} = R/(x)$. This is a noetherian local ring with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/(x)$. We make the following two crucial claims:

(A) *As an* R*-module,* $\overline{\mathfrak{m}}$ *is isomorphic to a direct summand of* $\mathfrak{m}/x\mathfrak{m}$.
(B) $\text{pd}_{\overline{R}}\, \mathfrak{m}/x\mathfrak{m} < \infty$.

Assuming these claims, we can complete the proof as follows. From (A) and (B), we have $\text{pd}_{\overline{R}}\overline{\mathfrak{m}} < \infty$ and hence $\text{pd}_{\overline{R}}\, k < \infty$. By (5.26) and (5.83), it follows that

$$\text{gl. dim}\, \overline{R} = \text{pd}_{\overline{R}}\, k = \text{pd}_R\, k - 1 = n - 1.$$

By the inductive hypothesis, $\overline{R}$ is a regular local ring, and by what we have already shown, $\dim \overline{R} = n - 1$. (This can also be gotten, if we wish, as part of the inductive hypothesis.) Therefore, there exists a prime chain

$$\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \quad \text{in } R \text{ with } (x) \subseteq \mathfrak{p}_1.$$

Since x is a non 0-divisor, an easy localization argument shows that $\mathfrak{p}_1$ is not a minimal prime, so $\dim R \geq n$. On the other hand, from $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, we deduce readily that

$$V(R) = 1 + V(\overline{R}) = 1 + \dim \overline{R} = n.$$

Since $V(R) \geq \dim R$ in general, it follows that $V(R) = \dim R = n$, so R is a regular local ring, of dimension n.

It remains to prove the two outstanding claims.

Proof of (A). Pick $\{y_i\}$ such that $\{x, y_1, \ldots, y_t\}$ is a minimal set of generators for $\mathfrak{m}$, and let $S = x\mathfrak{m} + \sum y_i R$. We claim that $x\mathfrak{m} \subseteq S \cap (x)$ is an equality. In fact, if $z \in S \cap (x)$, then

$$z = xa = y_1 b_1 + \cdots + y_t b_t + xm$$

for some $a \in R$, $b_i \in R$, and $m \in \mathfrak{m}$. Therefore,

$$xa - (y_1 b_1 + \cdots + y_t b_t) = xm \in \mathfrak{m}^2.$$

Since the images of $x, y_1, \ldots, y_t$ in $\mathfrak{m}/\mathfrak{m}^2$ are k-independent, we must have $a \in \mathfrak{m}$, and hence $z = xa \in x\mathfrak{m}$. From $S \cap (x) = x\mathfrak{m}$ and $S + (x) = \mathfrak{m}$, it follows that

$$\mathfrak{m}/x\mathfrak{m} \cong S/x\mathfrak{m} \oplus (x)/x\mathfrak{m}.$$

This implies (A), since, by Noether's Isomorphism Theorem, $\overline{\mathfrak{m}} := \mathfrak{m}/(x) \cong S/x\mathfrak{m}$.

Proof of (B). More generally, let us show that, for any R-module M on which x acts as a non 0-divisor,

$$\mathrm{pd}_R M = r < \infty \implies \mathrm{pd}_{\overline{R}}\, M/xM \le r.$$

The proof is by induction on r. If $r = 0$, M is projective over R. Then M/xM is projective over $\overline{R}$, so $\mathrm{pd}_{\overline{R}} M/xM = 0$. If $r > 0$, fix an exact sequence

$$(*) \qquad 0 \longrightarrow K \hookrightarrow F \xrightarrow{\varphi} M \longrightarrow 0 \quad \text{in } {}_R\mathfrak{M},$$

where F is free. We claim that

$$(**) \qquad 0 \longrightarrow K/xK \longrightarrow F/xF \longrightarrow M/xM \longrightarrow 0$$

is exact in $\mathfrak{M}_{\overline{R}}$. For this, it suffices to check that $K \cap xF \subseteq xK$. But if $a \in K$ has the form xb where $b \in F$, then

$$0 = \varphi(a) = \varphi(xb) = x\varphi(b) \implies \varphi(b) = 0.$$

Hence, $b \in K$ and $a = xb \in xK$. From $(*)$, $\mathrm{pd}_R K = r - 1$. Since x is a non 0-divisor on F and hence on K, our inductive hypothesis gives $\mathrm{pd}_{\overline{R}} K/xK \le r - 1$, so from $(**)$, we conclude that

$$\mathrm{pd}_{\overline{R}}\, M/xM \le 1 + \mathrm{pd}_{\overline{R}}\, K/xK \le r.$$

This completes the proof of Serre's Theorem (5.84). Before we move on, we should stress once more that the *noetherian* assumption on the local ring R in (5.84) is truly essential. In fact, B. Osofsky has constructed commutative local rings R with gl.dim $R < \infty$ that are not integral domains (and therefore necessarily non-noetherian). She has also shown that there exist commutative non-noetherian valuation domains with any prescribed global dimension. Therefore, the theory of global dimensions for *non-noetherian* local rings is an entirely different ball game (which we shall not further pursue).

Returning to the theorem of Serre and Auslander-Buchsbaum, let us now record what is perhaps its most remarkable consequence:

(5.88) Corollary. *Let R be any regular local ring. Then, for any prime ideal $\mathfrak{p}$ in R, the localization $R_{\mathfrak{p}}$ is also a regular local ring.*

Proof. Since any $R_{\mathfrak{p}}$-module is the localization of some R-module, we have gl.dim $R_{\mathfrak{p}} \le$ gl. dim R. Now apply (5.84). □

The result proved above was a conjecture of Krull made in the 1930s. It had remained open for years, and was solved only with the advent of the homological methods. To the best of my knowledge, there is still no known proof of (5.88) using only classical commutative algebra techniques. This is why we have chosen to include a full treatment of (5.84) here.

The homological theory of modules over commutative noetherian local rings was developed in the late 1950s by Auslander, Buchsbaum, and Serre. Without going into any details, let us just mention two of the most beautiful results in the

theory (beyond (5.84)). The first is the theorem that *any regular local ring is a unique factorization domain* — another long-standing conjecture. The second is a formula due to Auslander and Buchsbaum, valid for any f.g. module $M \neq 0$ of finite projective dimension over any (commutative) noetherian local ring $(R, \mathfrak{m})$:

$$\text{(5.89)} \qquad \mathrm{pd}(M) + \mathrm{depth}(M) = \mathrm{depth}(R) \leq V(R).$$

Here, the "depth" of a f.g. R-module $M \neq 0$ is defined to be the length of a maximal "M-regular sequence" in $\mathfrak{m}$. It is also the smallest integer d such that $\mathrm{Ext}^d_R(k, M) \neq 0$. (The latter description of $\mathrm{depth}(M)$ is to be compared with the description of $\mathrm{pd}(M)$ in (5.81).) In the case when R is a regular local ring, (5.89) holds for *all* f.g. R-modules $M \neq 0$, and $\mathrm{depth}(R) = V(R) = \dim R$.

§5G. Global Dimensions of Commutative Noetherian Rings

While we do not go into the finer theory of local homological algebra here, we would like to conclude this section by showing how to "globalize" some of our earlier results. This will lead us to the notion of a regular ring, in generalization of the notion of a regular local ring. We begin with the following basic observation.

(5.90) Lemma. *Let R be a commutative noetherian ring, and let M be any f.g. R-module. Then there exists a maximal ideal $\mathfrak{m}_1 \subset R$ such that*

$$\text{(5.91)} \qquad \mathrm{pd}(M) = \mathrm{pd}_{R_{\mathfrak{m}_1}}(M_{\mathfrak{m}_1}) \geq \mathrm{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$$

for all $\mathfrak{m} \in \mathrm{Max}(R)$ (the spectrum of maximal ideals of R).

Proof. That $\mathrm{pd}(M) \geq \mathrm{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ follows from the exactness of localization (and requires neither the ACC on R nor the finite generation of M). The existence of $\mathfrak{m}_1$ for (5.91) depends on a "compactness" argument. We may clearly assume that

$$d(\mathfrak{m}) := \mathrm{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) < \infty \quad \text{for all } \mathfrak{m}.$$

For each integer n, let M_n be a f.g. R-module representing the projective equivalence class of $\mathcal{P}^n(M)$. For any $\mathfrak{m} \in \mathrm{Max}(R)$, $(M_n)_{\mathfrak{m}}$ represents the projective equivalence class of $\mathcal{P}^n(M_{\mathfrak{m}})$, so $(M_{d(\mathfrak{m})})_{\mathfrak{m}}$ is free, say, of rank $r(\mathfrak{m})$. Thus, there exists an R-homomorphism $\varphi(\mathfrak{m}) : R^{r(\mathfrak{m})} \to M_{d(\mathfrak{m})}$ which, when localized at $\mathfrak{m}$, is an $R_{\mathfrak{m}}$-isomorphism. This means that, for a suitable element $s_{\mathfrak{m}} \notin \mathfrak{m}$, we have

$$s_{\mathfrak{m}} \cdot \ker(\varphi(\mathfrak{m})) = s_{\mathfrak{m}} \cdot \mathrm{coker}(\varphi(\mathfrak{m})) = 0.$$

The elements $\{s_{\mathfrak{m}}\}$ $(\mathfrak{m} \in \mathrm{Max}(R))$ generate the unit ideal R, so there exist $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \in \mathrm{Max}(R)$ such that $\{s_{\mathfrak{m}_1}, \ldots, s_{\mathfrak{m}_k}\}$ already generate R. Let

$$d = \max\{d(\mathfrak{m}_1), \ \ldots, \ d(\mathfrak{m}_k)\} = d(\mathfrak{m}_1) \ \text{(say)}.$$

We claim that M_d is projective. If so, then we'll have

$$\mathrm{pd}_R(M) \leq d(\mathfrak{m}_1) = \mathrm{pd}_{R_{\mathfrak{m}_1}}(M_{\mathfrak{m}_1}),$$

proving (5.91). Now, by (4.29), M_d is a finitely presented R-module. To show that M_d is projective, it suffices to show, according to Exercise (4.15), that $(M_d)_{\mathfrak{m}}$ is free for any $\mathfrak{m} \in \operatorname{Max}(R)$. Fix any index i $(1 \le i \le k)$ such that $s_{\mathfrak{m}_i} \notin \mathfrak{m}$. Since $s_{\mathfrak{m}_i}$ kills both $\ker(\varphi(\mathfrak{m}_i))$ and $\operatorname{coker}(\varphi(\mathfrak{m}_i))$,

$$\varphi(\mathfrak{m}_i):\ R^{r(\mathfrak{m}_i)} \longrightarrow M_{d(\mathfrak{m}_i)}$$

localizes to an isomorphism at $\mathfrak{m}$; that is, $(M_{d(\mathfrak{m}_i)})_{\mathfrak{m}}$ is free over $R_{\mathfrak{m}}$. On the other hand, $d \ge d(\mathfrak{m}_i)$, so $(M_d)_{\mathfrak{m}}$ can be obtained by applying a power of the projective shift operator (over $R_{\mathfrak{m}}$) to $(M_{d(\mathfrak{m}_i)})_{\mathfrak{m}}$. This shows that $(M_d)_{\mathfrak{m}}$ is free, as desired. □

(5.92) Theorem. *Let R be a commutative noetherian ring. Then, for any $\mathfrak{m} \in \operatorname{Max}(R)$, we have $\operatorname{gl.dim} R_{\mathfrak{m}} = \operatorname{pd}_R(R/\mathfrak{m})$, and*

(5.93) $$\operatorname{gl.dim} R = \sup\{\operatorname{gl.dim} R_{\mathfrak{m}}\} = \sup\{\operatorname{pd}_R(S)\},$$

where $\mathfrak{m}$ ranges over all maximal ideals of R, and S ranges over a complete set of simple R-modules.

Proof. For a fixed $\mathfrak{m} \in \operatorname{Max}(R)$, apply (5.90) to $M = R/\mathfrak{m}$. Since $M_{\mathfrak{m}} \cong R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ $(\cong R/\mathfrak{m})$ and $M_{\mathfrak{m}'} = 0$ for other maximal ideals $\mathfrak{m}'$, (5.90) and (5.83)(1) give

$$\operatorname{pd}_R(R/\mathfrak{m}) = \operatorname{pd}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}) = \operatorname{gl.dim} R_{\mathfrak{m}}.$$

In view of this, it is sufficient to prove the first equality in (5.93). Here, the equality "$\ge$" is true for any commutative R, since any $R_{\mathfrak{m}}$-module is the localization of an R-module. The inequality "$\le$" for noetherian R follows from (5.91). □

We are now in a position to derive the following global version of the result of Serre–Auslander-Buchsbaum (5.84).

(5.94) Theorem. *For any commutative noetherian ring R, the following are equivalent:*

(1) *$R_{\mathfrak{p}}$ is a regular local ring for any $\mathfrak{p} \in \operatorname{Spec} R$.*
(2) *$R_{\mathfrak{m}}$ is a regular local ring for any $\mathfrak{m} \in \operatorname{Max}(R)$.*
(3) *$\operatorname{pd}_R(\mathfrak{m}) < \infty$ for any $\mathfrak{m} \in \operatorname{Max}(R)$.*
(4) *$\operatorname{pd}_R(\mathfrak{p}) < \infty$ for any $\mathfrak{p} \in \operatorname{Spec} R$.*
(5) *$\operatorname{pd}_R(M) < \infty$ for any f.g. R-module M.*

If any of these conditions holds (and R is noetherian), we say that R is a regular ring.[47] *For such a ring R, we have* $\operatorname{gl.dim} R = \dim R$.

Proof. (5)$\Longrightarrow$(4) is clear since $\mathfrak{p}$ is f.g., and (4)$\Longrightarrow$(3) is a tautology.

[47]Not to be confused with a von Neumann regular ring!

(3)$\Longrightarrow$(2). In view of the exact sequence $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$, (3) implies that $\text{pd}(R/\mathfrak{m}) < \infty$, and hence gl.dim $R_\mathfrak{m} < \infty$ by the first part of (5.92). Therefore, by (5.84), $R_\mathfrak{m}$ is a regular local ring for any $\mathfrak{m} \in \text{Max}(R)$.

(2)$\Longrightarrow$(1) follows from (5.88) since any prime ideal $\mathfrak{p} \subset R$ is contained in a maximal ideal $\mathfrak{m}$, and $R_\mathfrak{p}$ is a localization of $R_\mathfrak{m}$.

(1)$\Longrightarrow$(5) follows from (5.91), and the "if" part of (5.84).

Now assume the above equivalent conditions hold. Then

$$\begin{aligned} \text{gl.dim}\, R &= \sup\{\text{gl.dim}\, R_\mathfrak{m} : \mathfrak{m} \in \text{Max}(R)\} && \text{(by (5.92))} \\ &= \sup\{\dim R_\mathfrak{m} : \mathfrak{m} \in \text{Max}(R)\} && \text{(by (5.84))} \\ &= \sup\{\text{height}(\mathfrak{m}) : \mathfrak{m} \in \text{Max}(R)\} \\ &= \dim R. \end{aligned}$$

□

(5.95) Corollary. *Let R be a commutative noetherian ring. If* gl.dim $R < \infty$, *then R is regular. The converse holds if R is a semilocal ring.*

Proof. The first part follows from the characterization (5.94)(5) of a regular ring. The second part follows from (5.93) (or from the equation gl.dim $R = \dim R$ for regular rings). □

In general, a commutative regular ring R need not have finite global (or Krull) dimension, even when R is an integral domain. Such an example can be gleaned from the Appendix of Nagata [62], as follows.

(5.96) Example (Nagata). Let $\bigcup_{i=1}^{\infty} I_i$ be a partition of the set of natural numbers $\mathbb{N} = \{1, 2, 3 \dots\}$ with the property that $|I_1| < |I_2| < |I_3| < \dots$, and let $A = K[x_1, x_2, \dots]$ where K is any field. Let $\mathfrak{p}_i$ be the prime ideal of A generated by $\{x_j : j \in I_i\}$, and $R = A_S$ be the localization of A at the multiplicative set $S = A \setminus \bigcup_{i=1}^{\infty} \mathfrak{p}_i$. Each prime $\mathfrak{p}_i$ can be seen to be of height $|I_i|$ in A, so it is clear that the localization of R of A has infinite Krull dimension. The fact that R is a *noetherian* domain is proved on p. 203 of Nagata [62]. It remains only to show that R is a regular ring, in the sense of (5.94). For this, let us check that, for any $\mathfrak{m} \in \text{Max}(R)$, $R_\mathfrak{m}$ is a regular local ring. It is easy to see that $\mathfrak{m}$ is of the form $\mathfrak{p}_i R$ for some i. (The proof of this is left as an exercise.) Therefore, the localization $R_\mathfrak{m}$ is isomorphic to $A_{\mathfrak{p}_i}$. To reach the latter localization, we can first localize A at the multiplicative set $K[\{x_j : j \notin I_i\}] \setminus \{0\}$. Letting K_i be the rational function field $K(\{x_j : j \notin I_i\})$, we have thus

$$R_\mathfrak{m} \cong K_i[x_{i_1}, \dots, x_{i_n}]_{(x_{i_1}, \dots, x_{i_n})},$$

where $I_i = \{i_1, \dots, i_n\}$. Since K_i is a field, this is a local ring of Krull dimension n, and it is a regular local ring since its maximal ideal is generated by the n

elements $x_{i_1}, \ldots, x_{i_n}$. This completes the proof that R is a regular ring, and by what we have said before,

$$\text{gl. dim } R = \dim R = \infty.$$

Admittedly, rings such as the R above are among what Nagata called "bad" noetherian rings. If we restrict our attention to (commutative) noetherian rings of finite Krull dimensions, then regular rings are exactly those of finite global dimensions. Many examples of regular rings can be obtained by using the remark (easily deducible from (5.94)) that *any localization of a regular ring remains regular.* Thus, for instance, if K is any field, any localization of $K[x_1, \ldots, x_n]$ is regular (recall (5.36)). Using the methods of classical algebraic geometry, it can also be shown that *the ring of regular functions of any nonsingular variety (say, over an algebraically closed field K) is regular.* This class of examples explains the origin of the name "regular rings".

In conclusion, we should also point out that Theorem (5.94) suggests a very good way to extend the definition of regular rings to the *noncommutative* setting. Namely, we can define a ring R to be *right (resp. left) regular* if R is right (resp. left) noetherian and every f.g. right (resp. left) R-module has finite projective dimension. Then, again, any right noetherian ring R with r.gl.dim $R < \infty$ is right regular, but a right regular ring R may have r.gl.dim $R = \infty$. And, not surprisingly, if R is right regular, then so is the localization of R at any central multiplicative set. Furthermore, Swan has proved the following result on the invariance of right regularity under a polynomial extension.

(5.97) Theorem. *If R is a right regular ring, then so is the polynomial ring $R[t]$.*

Of course, the fact that $R[t]$ is right noetherian follows from the Hilbert Basis Theorem. For the proof that any f.g. right $R[t]$-module has finite projective dimension, we refer the reader to Lam [78: p. 61]. In the special case when r.gl.dim $R < \infty$, the conclusion of (5.97) would have followed from (5.36). In the general case, the argument needed for proving (5.97) is somewhat similar.

The notion of right regularity defined above is particularly useful in algebraic K-theory. Due to limitation of space, however, we shall not pursue this matter here, and refer our reader to the excellent treatise [Bass: 68]. In closing, we should mention that, for ring-theoretic applications, there are other, perhaps even more appropriate notions of noncommutative regularity as well. These include "Auslander regularity" and "Artin-Schelter regularity", developed largely in the setting of graded rings. This is a topic of much current interest. For the details, we refer our reader to the recent papers of Artin-Schelter, Björk, Levasseur, and Stafford-Zhang.

Exercises for §5

0. Show that (5.23) is an equivalent formulation of (5.20).

1. Let R be a ring with r.gl.dim $R = n \geq 1$, and let B be a right R-module with $\text{pd}(B) = n-1$. Show that $\text{pd}(A) \leq n-1$ for any submodule $A \subseteq B$.

2. Show that r.gl.dim $R = \infty$ iff there exists a right R-module M such that $\text{pd}(M) = \infty$.

3. Let $0 \to K_i \to F_i \to M \to 0$ $(i = 1, 2)$ be exact in $\mathfrak{M}_R$, where F_1, F_2 are flat. Show that $K_1 \oplus F_2$ may not be isomorphic to $K_2 \oplus F_1$. (**Hint.** Let $M = \mathbb{Q}/\mathbb{Z}$, $K_1 = \mathbb{Z}$, $F_1 = \mathbb{Q}$, and let $0 \to K_2 \to F_2 \to M \to 0$ be any projective resolution of the $\mathbb{Z}$-module M.)

4. Recall that for any right R-module A, $A' = \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is a left R-module. Show that $A' \cong B'$ need not imply $A \cong B$. (**Hint.** For $R = \mathbb{Z}$, "dualize" the two exact sequences in Exercise 4.)

5. Let x, y, z be central elements in a ring R such that $xR \cap yR = zR$ and x, y are not 0-divisors. For ideal $I = xR + yR$, show that there exists a free resolution $0 \to R \to R^2 \to I_R \to 0$. (In particular, $\text{pd}_R(I) \leq 1$.)

6. Show that $\ker \beta \subseteq \text{im}\, \alpha$ and $\ker \gamma \subseteq \text{im}\, \beta$ in (5.35). (This amounts to the exactness of the Koszul resolution (5.33) for $n = 3$.)

7. Show that, if $I = \sum x_i R$, where $x_1, \ldots, x_n$ is a regular sequence in R (in the sense of (5.31)), then I/I^2 is a free right R/I-module with basis $x_1 + I^2, \ldots, x_n + I^2$. (**Hint.** Induct on $n \geq 0$, the case $n = 0$ being trivial.)

8. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring. Using Exercise 7, show that R is regular iff $\mathfrak{m}$ can be generated (as an ideal) by a regular sequence of R.

9. Let $(R, \mathfrak{m})$ be a right noetherian local ring with $\text{ann}_r(\mathfrak{m}) \neq 0$. Show that any f.g. right R-module P with $\text{pd}(P) < \infty$ is free. Deduce that r.gl.dim $R = \infty$ or else R is a division ring. (**Hint.** Generalize the argument in the beginning of the proof of the "only if" part of (5.84).)

   The next three exercises (with hints) are based on Kaplansky [72: pp. 176–181].

10. Let $\overline{R} = R/Rx$, where x is a central element in the ring R. Let M be a right R-module, and let $\overline{M} = M/Mx$. If x is not a 0-divisor on R_R and M_R, show that $\text{pd}_{\overline{R}}(\overline{M}) \leq \text{pd}_R(M)$. (**Hint.** For any exact sequence $0 \to K \to F \to M \to 0$ in $\mathfrak{M}_R$ with F free, show that the induced sequence $0 \to \overline{K} \to \overline{F} \to \overline{M} \to 0$ remains exact.)

11. Keep the hypotheses in Exercise 10, and assume, in addition, that R is right noetherian, $x \in \text{rad}\, R$, and M is f.g. Show that $\text{pd}_{\overline{R}}(\overline{M}) = \text{pd}_R(M)$.

[**Hint.** First show that $\overline{M}$ is free (resp. projective) over $\overline{R}$ iff M is free (resp. projective) over R.]

12. Keep the hypotheses in Exercise 11 and assume that $R \neq 0$ and $n = \text{r. gl. dim } \overline{R} < \infty$. Show that r.gl.dim $R = n + 1$. (**Hint.** For any f.g. M_R, fix a short exact sequence $0 \to K \to F \to M \to 0$ in $\mathfrak{M}_R$ with F f.g. free. Then apply Exercise 11 to K (not to M !), and invoke (5.30).)

13. (Auslander-Buchsbaum, Small, Strooker) For any right noetherian ring $A \neq (0)$, show that

$$\text{r. gl. dim } A[[x]] = 1 + \text{r. gl. dim } A,$$

where $A[[x]]$ denotes the power series ring in one variable over A.

14. A right R-module P is said to be *stably free* if, for some integer $n \geq 0$, $P \oplus R^n$ is free. If R is commutative, and P is a f.g. stably free R-module of rank 1, show that $P \cong R$. (**Hint.** Use a localization argument to show that $\bigwedge^k P = 0$ for $k \geq 2$.)

15. A right R-module M is said to have a *finite free resolution* (FFR) if, for some integer $n \geq 0$, there exists a long exact sequence

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \quad \text{in } \mathfrak{M}_R,$$

where the F_i 's are f.g. free right R-modules. If such a module M is projective, show by induction on n that M is stably free.

16. Let P be a f.g. projective module of rank 1 over a commutative ring R. If P has a FFR, use Exercises (14) and (15) to show that $P \cong R$.

17. Let R be a right coherent ring over which any f.g. projective right module is stably free. Show that any f.g. module M_R with $d = \text{pd}(M) < \infty$ has a FFR as in Exercise 15 with $n = 1 + d$.

The following three exercises assume some familiarity with UFDs *(Unique Factorization Domains) and the Auslander-Buchsbaum Theorem (that commutative regular local rings are* UFD's*).*

18. Let R be a commutative regular domain. Show that R is a UFD iff $\text{Pic}(R) = \{1\}$ (i.e., invertible ideals of R are all principal). [**Hint** (for sufficiency). The Auslander-Buchsbaum Theorem implies that any prime $\mathfrak{p} \subset R$ of height 1 is locally principal. Now apply (2.17).]

19. Let R be a commutative noetherian domain over which any f.g. module has a FFR. Show that R is a UFD. (**Hint.** Deduce that R is regular, and apply Exercises 16 and 18.)

20. Let R be a commutative regular domain over which all f.g. projectives are stably free. Show that R is a UFD.

21. Let R be a ring with IBN, and let M be a right R-module with a FFR as in Exercise 15. Using (5.5), show that the integer

$\chi(M) := \sum_{i=0}^{n}(-1)^i \operatorname{rank}(F_i)$ depends only on M; that is, it is independent of the particular FFR chosen. ($\chi(M)$ is called the *Euler characteristic* of M.)

(1) If R is right noetherian or commutative, show that $\chi(M) \geq 0$.
(2) Give an example to show that $\chi(M)$ may be negative in general.
(3) If R is a commutative domain with quotient field K, show that $\chi(M) = \dim_K(M \otimes_R K)$.

22. Let R be a ring with IBN, and let $0 \to M' \to M \to M'' \to 0$ be exact in $\mathfrak{M}_R$. If each of M', M'' has a FFR, show that M also has a FFR, and that $\chi(M) = \chi(M') + \chi(M'')$ in $\mathbb{Z}$.

23. (L. Small) Let R be a commutative noetherian domain with quotient field $K \neq R$, and let T be the triangular ring $\begin{pmatrix} R & K \\ 0 & K \end{pmatrix}$. Recall (from *FC*–(1.22)) that T is right noetherian but not left noetherian. Suppose gl.dim $R = n < \infty$.

(1) Show that r.gl.dim $T = n$.
(2) Show that l.gl.dim $T = n + 1$ if K contains an R-submodule M with $\text{pd}_R(M) = n$, and l.gl.dim $T = n$ otherwise.

(**Hint.** The one-sided ideals of T are classified in *FC*–(1.17). Compute their projective dimensions and use (5.51$'$). For (2) show that if A is any R-submodule of K, then $\mathfrak{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ is a left ideal of T, with $\text{pd}_T(\mathfrak{A}) = \text{pd}_R(A)$. Cf. (2.33) for the case $R = \mathbb{Z}$, $K = \mathbb{Q}$.)

24. Let $R = \left\{ \begin{pmatrix} x & u & v \\ 0 & x & w \\ 0 & 0 & y \end{pmatrix} \right\}$, where x, y, u, v, w are arbitrary elements in a division ring k. Show that the artinian ring R has exactly two simple right modules M_1, M_2, each 1-dimensional over k, with $\text{pd}_R(M_1) = \infty$ and $\text{pd}(M_2) = 0$. What are the projective shifts of M_1 and the Jacobson radical of R?

25. Let $\varphi : R \to S$ be a ring homomorphism, and let $n = \text{pd}(S_R)$ where S is viewed as a right R-module via φ. Show that, for any right S-module M, $\text{pd}(M_R) \leq n + \text{pd}(M_S)$.

26. (Bass) Let R be a right noetherian ring and let $\{M_i : i \in I\}$ be a direct system of right R-modules, with direct limit M. If $\text{id}(M_i) \leq n$ for all $i \in I$, show that $\text{id}(M) \leq n$. (**Sketch.** Let E be the injective hull of $\bigoplus R/\mathfrak{A}$, where $\mathfrak{A}$ ranges over all right ideals of R. For any N_R, let $\tilde{N} = \text{Hom}_R(N, E)$ and let $F(N)$ be the injective module $\prod_{\tilde{N}} E$. By the choice of E, the natural R-homomorphism $N \to F(N)$ is an embedding. Note that this construction is "functorial" in N, thus leading to a "canonical" injective resolution of N. Now take the direct limit of these resolutions for the M_i's, and use (3.36)(1).)

27. (Osofsky) For any right noetherian ring R, prove the following injective dimension analogue of (5.51): r.gl.dim $R = \sup\{\mathrm{id}(C)\}$, where C ranges over all cyclic right R-modules. (**Note.** Osofsky has also shown that, for any right noetherian ring R, r.gl.dim $R = \sup\{\mathrm{id}(\mathfrak{A})\}$, where $\mathfrak{A}$ ranges over all the right ideals of R.)

Chapter 3

More Theory of Modules

In this Chapter, we shall cover some other aspects of the theory of modules that were not yet touched upon in the first two chapters. In contrast to Ch. 1 and Ch. 2, the module theory presented in the three sections of this Chapter is essentially non-homological in nature. Nevertheless, the idea of injective modules and essential extensions plays a discernible role, especially in the first and last sections of the Chapter.

Section 6 is devoted to several important topics in module theory developed in the 1960s (and thereafter). First and foremost is the general theory of *uniform dimensions* of modules initiated by A. W. Goldie, which occupies §6A. The notion of *complements* and the equivalent notion of *essentially closed submodules* are introduced and discussed in detail in §6B. Another subsection, §6D, focuses on a class of modules called CS modules, and presents two applications of this notion to the study of injective modules. Finally, various finiteness conditions on rings and their relationships are studied, and there is a closing subsection introducing the notion of quasi-injective (QI) modules. This notion has gained considerable popularity in recent years, so a good discussion of it is worthwhile. The class of QI modules lies between that of injective modules and that of CS modules. It will be seen that, in certain theorems, the natural class to look at is that of QI modules, rather than injective modules; some specific instances of this are the later theorem (13.1) on the structure of endomorphism rings, and the double annihilator theorem (13.5) of Johnson and Wong.

In §7, we turn to R. E. Johnson's theory of *singular submodules*, and introduce the important notion of (right or left) nonsingular rings. Iterating the formation of singular submodules, we arrive at Goldie's definition of the "closure" of a submodule, which leads directly to the idea of the (Goldie) "reduced rank", a variant of the uniform dimension. As an excursion, we also introduce and study Rickart rings and Baer rings (and their $*$-analogues) as special cases of nonsingular rings. These classes of rings arise naturally and play a substantial role in the theory of operator algebras in functional analysis. The final section, §8, is occupied with the theory of *dense submodules* and *rational completions* of modules, after Findlay-Lambek and Utumi. We also introduce the important notion of (right or

left) Kasch rings, which will show up again in the later sections on Frobenius and quasi-Frobenius rings.

The main material covered in §§6–8 in this Chapter is, to some extent, in preparation for the theory of rings of quotients and Goldie rings in Chapters 4 and 5. For instance, semiprime right Goldie rings, which occupy center stage in §11, are examples of right nonsingular rings, and of rings of finite (right) uniform dimension. The notion of reduced rank will find natural applications in the problem of characterizing right noetherian rings that are right orders in right artinian rings. Furthermore, the rational completion $\tilde{E}(R)$ of the right regular module R_R (over any ring R) will be shown to have a natural ring structure extending its inherent right R-module structure. The ring $\tilde{E}(R)$, called the *maximal right ring of quotients of* R, turns out to provide the best and the most convenient setting in which to study the theory of rings of quotients over R.

Aside from these applications, however, the material developed in this chapter is also important in its own right. The notions of uniform dimensions, reduced ranks, CS modules, singular submodules, dense submodules, and rational completions apply not only to the right regular module R_R, but also to arbitrary modules over a general ring, and thus provide powerful tools for studying the theory of modules in general.

In writing this chapter, I have consulted extensively the works of Goldie [64], [72], Lambek [66], and Goodearl [76], among other sources.

§6. Uniform Dimensions, Complements, and CS Modules

§6A. *Basic Definitions and Properties*

This subsection is a quick introduction to the theory of *uniform dimensions* of modules due to A. Goldie. The basic idea of this theory is that one measures the "size" of a module M by finding out how big a direct sum of nonzero submodules M can contain.

The notion of an essential submodule plays a crucial role in this theory. It is perhaps useful to point out that, before the term "essential extension" was firmly established in the literature, "algebraic extension" had been used by some authors at one time. Accordingly, injective modules were called "algebraically closed" modules, and injective hulls were regarded as "algebraic closures". In Goldie's theory of uniform dimensions, one seeks, within a given module M, a submodule

$$N = N_1 \oplus \cdots \oplus N_k \qquad (N_i \neq 0)$$

with the *largest possible* k. Assuming that such an N exists, then $N \subseteq_e M$, and M is assigned uniform dimension k (see (6.6)). If we think of M as an "algebraic extension" of N, then we find a useful analogy between the notion of *uniform dimension* and the notion of *transcendence degree* for a field extension.

The central fact which makes the notion of uniform dimensions possible is the following parody of a familiar fact in linear algebra. (Keep in mind that, in the context of vector spaces, a uniform module is just a "line".)

(6.1) The "Steinitz Replacement Theorem". *Let $U = U_1 \oplus \cdots \oplus U_m$ and $V = V_1 \oplus \cdots \oplus V_n$ be essential submodules of a right module M over a ring R, where the U_i's and V_j's are uniform modules. Then $m = n$.*

Proof. We may assume that $n \geq m$. *We claim that $\hat{U} := U_2 \oplus \cdots \oplus U_m$ intersects some V_j trivially.* For, if otherwise, we would have $\hat{U} \cap V_j \subseteq_e V_j$ $(1 \leq j \leq n)$ since V_j is uniform, and (3.38) would give

$$(\hat{U} \cap V_1) \oplus \cdots \oplus (\hat{U} \cap V_n) \subseteq_e V_1 \oplus \cdots \oplus V_n = V,$$

and hence also $\hat{U} \cap V \subseteq_e V \subseteq_e M$. By "transitivity" (cf. (3.27)(2)), this implies $\hat{U} \subseteq_e M$, a contradiction. Therefore, after relabeling the V_j's, we may assume that $\hat{U} \cap V_1 = 0$. Let $U' = \hat{U} \oplus V_1$. We must then have $U' \cap U_1 \neq 0$ (otherwise $U_1 + U_2 + \cdots + U_m + V_1$ would be a direct sum, contradicting $U \subseteq_e M$), and so

$$(U' \cap U_1) \oplus U_2 \oplus \cdots \oplus U_m \subseteq_e U_1 \oplus \cdots \oplus U_m \subseteq_e M.$$

Since the LHS is contained in U', it follows as before that $U' \subseteq_e M$. Going from U to U', we have thus "replaced" the summand U_1 by V_1. Repeating this process (and relabeling $V_2, \ldots, V_n$ if necessary), we can pass from U' to some

$$U'' = V_1 \oplus V_2 \oplus U_3 \oplus \cdots \oplus U_m \subseteq_e M.$$

After m steps, we'll arrive at

$$U^{(m)} = V_1 \oplus \cdots \oplus V_m \subseteq_e M.$$

On the other hand, we have $V = V_1 \oplus \cdots \oplus V_n \subseteq_e M$, so we must have $m = n$! □

(6.2) Definition. We shall say that an R-module M_R has *uniform dimension n* (written u.dim $M = n$) if there is an essential submodule $V \subseteq_e M$ that is a direct sum of n uniform submodules. (By (6.1), u.dim M is well-defined.) If, on the other hand, no such integer n exists, we write u.dim $M = \infty$. (The meaning of u.dim $M = \infty$ will be further clarified in Proposition 6.4 below.)

Another name used for the uniform dimension is Goldie dimension, named after its discoverer. We prefer the term "uniform dimension" since uniform modules play a key role in its definition. Also, we want to avoid the term "Goldie dimension" so that there is no confusion with Goldie's "reduced rank" to be introduced later in §7C.

It is easy to check from the definition that u.dim $M = 0$ iff $M = 0$, and u.dim $M = 1$ iff M is a uniform module. For modules over a division ring, of course uniform dimension is just the usual vector space dimension as defined in linear algebra.

The following result offers a useful piece of information for modules of finite uniform dimension (over any ring).

(6.3) Proposition. *Suppose* u.dim $M = n < \infty$. *Then, any direct sum of nonzero submodules* $N = N_1 \oplus \cdots \oplus N_k \subseteq M$ *has* $k \leq n$ *summands.*

Proof. Let $V \subseteq_e M$ be as in Definition (6.2). Then

$$N_i' := N_i \cap V \neq (0) \quad \text{and} \quad V \supseteq N_1' \oplus \cdots \oplus N_k'.$$

Thus, we may as well assume that $M = V$, say $M = V_1 \oplus \cdots \oplus V_n$, where the V_i 's are uniform. Let $\hat{N} = N_2 \oplus \cdots \oplus N_k$. Arguing as in the first part of the proof of (6.1), we see that, after relabeling the V_i 's, we may assume that $\hat{N} \cap V_1 = 0$.[48] Projecting M modulo V_1 onto $V_2 \oplus \cdots \oplus V_n$, we have then an embedding of $\hat{N}$ into $V_2 \oplus \cdots \oplus V_n$. Invoking an inductive hypothesis (on n) at this point, we'll have $k - 1 \leq n - 1$, and therefore $k \leq n$. (One may start the induction from $n = 0$ or from $n = 1$, whichever one prefers.) □

Next we give a nice interpretation for infinite uniform dimension.

(6.4) Proposition. u.dim $M = \infty$ *iff* M *contains an infinite direct sum of nonzero submodules.*

Proof. The "if" part is clear from (6.3). For the "only if" part, let us assume that M does not contain an infinite direct sum of nonzero submodules. We claim that:

(6.5) *Any nonzero submodule* $N \subseteq M$ *contains a uniform submodule.*

In fact, if this is false, then N is surely not uniform, so it contains some $A_1 \oplus B_1$ with $A_1 \neq (0) \neq B_1$. Then B_1 is again not uniform, and contains some $A_2 \oplus B_2$ with $A_2 \neq (0) \neq B_2$. Continuing this process, we'll get an *infinite* direct sum $A_1 \oplus A_2 \oplus A_3 \oplus \cdots \subseteq M$, a contradiction. Having now proved our claim (6.5), pick a uniform $V_1 \subseteq M$. If V_1 is not essential in M, we'll have $M \supseteq V_1 \oplus V_2$ for some $V_2 \neq 0$, which we may assume to be uniform, by virtue of (6.5). If $V_1 \oplus V_2$ is still not essential in M, we can similarly find $V_1 \oplus V_2 \oplus V_3 \subseteq M$ for a suitable uniform submodule V_3. But by assumption this process cannot be continued indefinitely, so we are bound to arrive at some $V_1 \oplus \cdots \oplus V_n \subseteq_e M$ with all V_i 's uniform. By definition, we have then u.dim $M = n$. □

(6.6) Corollary. u.dim M *is the supremum of the set*

$$\{k : M \text{ contains a direct sum of } k \text{ nonzero submodules}\}.$$

[48] Note that, for that part of the proof of (6.1), we did not need the U_i 's (or the N_i 's here) to be uniform.

Proof. Let $k_0 \leq \infty$ be this supremum. If $k_0 = \infty$, then by (6.3), u.dim M must be ∞ too, and we have u.dim $M = k_0$ in this case. Next, assume $k_0 < \infty$. By (6.4), we see that u.dim M must be *finite*, and by (6.3), we deduce easily that u.dim $M = k_0$. □

(6.7) Corollary. (1) *If M_R is either a noetherian or an artinian module, then* u.dim $M < \infty$.
(2) *Suppose M has finite composition length n. Then* u.dim $M \leq n$, *with equality iff M is semisimple.*

Proof. (1) Either chain condition will rule out the existence of an infinite direct sum (of nonzero submodules) in M.
(2) If M contains $N_1 \oplus \cdots \oplus N_k$ where $N_i \neq 0$, then

$$k \leq \sum_{i=1}^{k} \text{length}(N_i) \leq \text{length}(M) = n\,. \tag{6.8}$$

By (6.6), this implies that u.dim $M \leq n$. If M is semisimple, it is clear from Def. (6.2) that u.dim $M = n$. Conversely, if u.dim $M = n$, then M contains some $N_1 \oplus \cdots \oplus N_n$ with $N_i \neq 0$, and (6.8) above for $k = n$ implies that

$$M = N_1 \oplus \cdots \oplus N_n$$

with length$(N_i) = 1$ for every i. Thus, M is semisimple. □

(6.9) Remarks.
(a) For $R = \mathbb{Z}$, if p is any prime, $M = \mathbb{Z}/p^r\mathbb{Z}$ $(r \geq 1)$ has composition length r, but u.dim $M = 1$. More generally, for $M = \mathbb{Z}/m\,\mathbb{Z}$ $(m > 0)$, u.dim M is given by the number of distinct prime divisors of m, while length(M) is the number of distinct prime-power divisors of m (not counting 0-th powers).

(b) If M_R is only f.g. (instead of being noetherian), u.dim M need not be finite. For instance, the right regular module over $R = \mathbb{Z} \times \mathbb{Z} \times \cdots$ contains the infinite direct sum of ideals $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$, so u.dim$(R_R) = \infty$, although R_R is a cyclic module.

(c) Regarding the results (6.4) and (6.6), the following curious situation is noteworthy. We know from these results that, if, for any k, a module M contains a direct sum of k nonzero modules, then M contains in fact an infinite direct sum of nonzero modules. Now try to "dualize" this statement: if, for any k, the module M has a quotient which is a direct sum of k nonzero modules, does it follow that M has a quotient which is an infinite direct sum (or product) of nonzero modules? A moment's thought shows that the answer is "no"! For instance, over the ring $\mathbb{Z}$, the module $\mathbb{Z}$ maps onto

$$\mathbb{Z}/p_1 \cdots p_k\mathbb{Z} \cong \textstyle\bigoplus_{i=1}^{k} \mathbb{Z}/p_i\mathbb{Z}$$

(for any distinct primes $p_1, \ldots, p_k$), but clearly $\mathbb{Z}$ does not map onto any infinite direct sum (or product) of nonzero abelian groups! This is an interesting example

which serves to show that we should not assume that "dualizing" a correct module-theoretic statement should always give another correct statement.

(6.10) Corollary. (1) u.dim$\left(\bigoplus_{i=1}^{k} M_i\right) = \sum_{i=1}^{k}$ u. dim M_i *(with the usual conventions about* ∞*).*
(2) *Suppose* $N \subseteq M$*. Then* (a) u.dim $N \leq$ u. dim M*, with equality when* $N \subseteq_e M$.
(b) *If* N *is not essential in* M*, then* u. dim $N <$ u. dim M*, unless* u. dim $N =$ u. dim $M = \infty$.

Proof. (1) is obvious from Def. (6.2) and (6.4). (2)(a) follows quickly from (6.6). For (2)(b), assume N is *not* essential in M. If u.dim $N = \infty$, then u.dim $M = \infty$ too, so we may focus on the case u.dim $N = n < \infty$. Here, N contains $N_1 \oplus \cdots \oplus N_n$ with $N_i \neq 0$, and since $N \subseteq M$ is not essential, M contains

$$N' \oplus N = N' \oplus N_1 \oplus \cdots \oplus N_n$$

for some submodule $N' \neq 0$. By (6.3), u.dim $M \geq n + 1$. $\square$

(6.11) Caution. Unlike certain other kinds of dimension or rank, uniform dimension is *not* additive over short exact sequences. For instance, over $R = \mathbb{Z}$, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0,$$

but all three modules have uniform dimension 1, as we have pointed out in (6.9). More surprisingly, the existence of a surjection $M \twoheadrightarrow \overline{M}$ need not imply u.dim $M \geq$ u. dim $\overline{M}$. For instance, over $R = \mathbb{Z}$ again, if $M = \mathbb{Z}$ and $N = p_1 \ldots p_k\mathbb{Z}$ where the p_i 's are distinct primes, then

$$\overline{M} := M/N \cong \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k\mathbb{Z}$$

has uniform dimension k, whereas u.dim $M = 1$. Even more spectacularly, u.dim $\mathbb{Q} = 1$, but, since $\mathbb{Q}/\mathbb{Z}$ is an infinite direct sum (of its primary components), u.dim $\mathbb{Q}/\mathbb{Z} = \infty$! Fortunately, all of this can be explained, in theory. We shall return to give an analysis of the behavior of uniform dimension with respect to short exact sequences in §6C.

Now let us relate uniform dimensions to some other kinds of measure of the size of a module. The next Proposition interprets u.dim M via the "decomposition length" of the injective hull $E(M)$; Proposition (6.14) below compares u.dim M with the "rank" of M for torsion-free modules M over commutative domains.

(6.12) Proposition. *For a natural number* n*, we have* u. dim $M = n$ *iff* $E(M)$ *is a direct sum of* n *indecomposable injective modules.*

Proof. Since u.dim $M =$ u. dim $E(M)$ by (6.10)(2a), we may assume that M is injective. If $M = M_1 \oplus \cdots \oplus M_n$ where the M_i 's are indecomposable injectives, then, since each M_i is uniform by (3.52), we have u.dim $M = n$. Conversely, if

u.dim $M = n$, then M contains an essential submodule $V = V_1 \oplus \cdots \oplus V_n$, where the V_i 's are uniform. From (3.33)(2) and (3.39), it follows that

$$M = E(V) = E(V_1 \oplus \cdots \oplus V_n) = E(V_1) \oplus \cdots \oplus E(V_n),$$

and by (3.52), each $E(V_i)$ is indecomposable, as desired. □

(6.13) Example. For the commutative ring $R = \mathbb{Q}[u, v]$ defined by the relations

$$u^2 = v^2 = uv = 0,$$

R_R is essential over $u\mathbb{Q} \oplus v\mathbb{Q}$ where $u\mathbb{Q}, v\mathbb{Q}$ are minimal ideals. Thus, u.dim $R = 2$. The injective hull $E(R)$ is $\hat{R} \oplus \hat{R}$ where $\hat{R} = \mathrm{Hom}_{\mathbb{Q}}(R, \mathbb{Q})$ is the unique indecomposable injective R-module; see (3.69) (with $n = 1$).

(6.14) Proposition. *Let R be a commutative domain with quotient field K. For any torsion-free module M_R, we have*

(6.15) $$\text{u.dim}\, M = \dim_K(M \otimes_R K).$$

Proof. Since M is torsion-free, we may think of M as embedded in $M^K := M \otimes_R K$. First assume u.dim $M = \infty$. In this case, M contains $U_1 \oplus U_2 \oplus \cdots$ with $(U_i)_R \neq 0$, so M^K contains $U_1^K \oplus U_2^K \oplus \cdots$; hence $\dim_K(M^K) = \infty$ also. Next, assume that u.dim $M = n < \infty$. The same argument as above gives $\dim_K(M^K) \geq n$. If this is a strict inequality, there would exist a direct sum

$$V_1 \oplus \cdots \oplus V_{n+1} \subseteq M^K,$$

where the V_i 's are nonzero K-subspaces. By "clearing denominators", we see that the $M \cap V_i$'s are nonzero R-submodules of M, and

$$M \supseteq (M \cap V_1) \oplus \cdots \oplus (M \cap V_{n+1})$$

gives a contradiction. Therefore, $\dim_K(M^K) = n$. □

Remark. Of course, the formula (6.15) is no longer true if M is not torsion-free. For instance, in the "extreme" case when M is a *torsion* R-module, we have $\dim_K(M^K) = 0$, but u.dim M can be ∞ or any finite number. For the correct generalization of (6.15) to any M, see Exercise 10.

In closing this subsection, we should point out that the theory of uniform dimensions has a dual version, which is also worthy of consideration. The dual of uniform dimension is called *co-uniform dimension*, or *dual Goldie dimension*. The dual of the notion of a uniform module is that of a *hollow module*: a module M is hollow if $M = X + Y$ (for submodules X and Y) implies that $X = M$ or $Y = M$. These are the modules with co-uniform dimension 1. For lack of space, we shall not go into the details of this dual theory here, but shall merely refer the reader to its original source, namely, Varadarajan [79]. (In this paper, Varadarajan used the term "co-rank" for the dual Goldie dimension.)

§6B. *Complements and Closed Submodules*

To better understand the meaning of uniform dimensions, we need to introduce the notion of complements in a module.

(6.16) Definition. Let S be a submodule of an R-module M_R. A submodule $C \subseteq M$ is said to be a *complement*[49] *to* S *(in* M*)* if C is maximal with respect to the property that $C \cap S = 0$. By Zorn's Lemma, any submodule S has a complement; in fact, any submodule C_0 with $C_0 \cap S = 0$ can be enlarged into a complement to S in M.

(6.17) Examples.

(1) If $M = C \oplus S$, then C is a complement to S. We speak of C as a "direct complement" in this case. Note that the isomorphism type of C is uniquely determined by S, since $C \cong M/S$ as R-modules.

(2) If $S \subseteq_e M$, then (0) is the only complement to S.

(3) In the commutative ring $R = \mathbb{Q}[u, v]$ discussed in (6.13), $\mathbb{Q} \cdot (u + av)$ (for any $a \in \mathbb{Q}$) is a complement to $\mathbb{Q}\, v$.

(4) In $M = \mathbb{Z} \oplus \mathbb{Z}$ over $R = \mathbb{Z}$, $C = (0) \oplus \mathbb{Z}$ is a complement to any S of the form $n\,\mathbb{Z} \oplus (0)$ $(n \neq 0)$.

(5) The isomorphic type of a complement to S need not be determined by S. For instance, over $R = \mathbb{Z}$, consider the module $M = C \oplus S$ where $C = \langle c \rangle$ has order 8 and $S = \langle s \rangle$ has order 2. While C is a (direct) complement to S, the subgroup $C' = \langle c' \rangle$ of order 4 generated by $c' := 2c + s$ is *also* a complement to S. (For otherwise $C' \subsetneq N$ for some N with $N \cap S = 0$. We must have $M = N \oplus S$ so $N \cong \mathbb{Z}/8\,\mathbb{Z}$ and $c' \in 2N \subseteq 2M = 2C$, a contradiction.) Therefore, *S has two complements of different cardinalities* (and, in particular, different isomorphism types).

(6.18) Proposition. *Let C, S be submodules of any module M_R, with $C \cap S = 0$. Write "bar" for "image in M/C". Then C is a complement to S iff $\overline{S} \subseteq_e \overline{M}$.*

Proof. This follows easily from definition, since the nonzero submodules of $\overline{M}$ are of the form $\overline{D}$ where $D \supsetneq C$. □

(6.19) Remark. If C is a complement to S, then we have $C \oplus S \subseteq_e M$ (for otherwise we would have a direct sum $X \oplus C \oplus S$ with $X \neq 0$). However, $C \oplus S \subseteq_e M$ is in general not enough to guarantee that C be a complement to S.

[49] In the literature, this is often called an "intersection complement", as opposed to another kind of complement called "addition complement". In this book, "complement" shall always mean intersection complement. Addition complements will only be briefly considered in an exercise in §19; see Exercise (19.34).

For instance, if $C_0 \subseteq_e C$ with $C_0 \neq C$, then

(6.20) $$C \oplus S \subseteq_e M \implies C_0 \oplus S \subseteq_e M,$$

but surely C_0 cannot be a complement to S.

(6.21) Definition. We say that a submodule $C \subseteq M_R$ is a *complement in M* (written $C \subseteq_c M$) if there exists a submodule $S \subseteq M$ such that C is a complement to S in M.

We can think of $C \subseteq_c M$ as a *weakening* of the condition that C be a direct summand of M. Note that the module $C' \subseteq M$ constructed in Example (6.17)(5) is a complement in M, without being a direct summand thereof.

With the notion of $C \subseteq_c M$ defined in (6.21), we can now rectify the situation in Remark (6.19).

(6.22) Proposition. *Suppose $C \subseteq_c M$, and T is a submodule of M such that $C \cap T = 0$. Then C is a complement to T iff $C \oplus T \subseteq_e M$.*

Proof. We need only prove the "if" part, so assume $C \oplus T \subseteq_e M$. Say C is complement to S, as in Def. (6.20). To show that C is maximal with respect to having zero intersection with T, consider any submodule $D \supseteq C$ with $D \cap T = 0$. Then

$$(C + T) \cap (D \cap S) = ((C + T) \cap D) \cap S = C \cap S = 0.$$

Since $C + T \subseteq_e M$, we have $D \cap S = (0)$ and therefore $D = C$. This shows that C is a complement to T in M. □

(6.23) Corollary. *Suppose $C \subseteq_c M$. Let T be a complement to C in M (which always exists). Then C is a complement to T.*

Proof. Since T is a complement to C, we have $T \oplus C \subseteq_e M$ by (6.19). But then by (6.22) this implies that C is also a complement to T. □

The next two results, (6.24) and (6.28), describe some basic properties of complements.

(6.24) Proposition. (1) *Suppose $C \subseteq N \subseteq M$. Then $C \subseteq_c M \implies C \subseteq_c N$.*
(2) *(Transitivity) $C \subseteq_c N$ and $N \subseteq_c M \implies C \subseteq_c M$.*

Proof. (1) If C is a complement to, say, X in M, it follows easily from definition that C is a complement to $X \cap N$ in N. Therefore $C \subseteq_c N$.

(2) Suppose C is a complement to S in N, while N is a complement to T in M. *We claim that C is a complement to $S \oplus T$ in M.* Indeed, consider any submodule $D \supsetneq C$ in M. We wish to show that $D \cap (S \oplus T) \neq 0$. We may assume that

$D \cap N = C$ (otherwise $D \cap N \cap S \neq (0)$ already). Then, there is an element $d \in D \backslash N$. We have $(N + dR) \cap T \neq 0$, so there exists an equation

$$(6.25) \qquad n + dr = t \neq 0, \qquad \text{where } n \in N,\ r \in R,\ t \in T.$$

If $n \in C$, then $n + dr \in D$ and we are done. If $n \notin C$, then, as above, we get an equation

$$(6.26) \qquad c + nr' = s \neq 0, \qquad \text{where } c \in C,\ r' \in R,\ s \in S.$$

Subtracting r' times (6.25) from (6.26), we obtain

$$c - drr' = s - tr' \in (D \cap (S \oplus T)) \backslash \{0\}.$$

□

(6.27) Caution. In spite of (6.24)(1), we have in general

$$(6.27(a)) \qquad C \subseteq M,\ C' \subseteq_c M \not\Longrightarrow C \cap C' \subseteq_c C,$$

even in the case when C is a direct summand of M. In fact, let M and C, C' be as in Example (6.17)(5). Then $C' \subseteq_c M$, but $C \cap C' = 4C$ is *not* a complement in C. This example also suffices to show that:

$$(6.27(b)) \qquad C \subseteq_c M,\ C' \subseteq_c M \not\Longrightarrow C \cap C' \subseteq_c M.$$

(6.28) Proposition. *Let $C \subseteq L \subseteq M$ be R-modules. Then:*

(1) $L \subseteq_c M \Longrightarrow L/C \subseteq_c M/C$.
(2) *If $C \subseteq_c M$, then $L \subseteq_c M \Longleftrightarrow L/C \subseteq_c M/C$, so there is a one-one correspondence between the complements in M/C and the complements in M containing C.*

Proof. Let us write "bar" for "image in M/C". For (1), fix a submodule $V \subseteq M$ to which L is a complement. Applying (6.18), it is straightforward to check that $\overline{L}$ is a complement to $\overline{V}$ in $\overline{M}$, so $\overline{L} \subseteq_c \overline{M}$. For "$\Longleftarrow$" in (2) (which is all we need), fix a submodule S such that C is a complement to S in M. Assume that $\overline{L} \subseteq_c \overline{M}$, say $\overline{L}$ is a complement to $\overline{Q}$ where $Q \supseteq C$. Then

$$(Q \cap S) \cap L = S \cap (Q \cap L) = S \cap C = 0.$$

Enlarge L to L' which is a complement to $Q \cap S$ in M. Then

$$\begin{aligned} (L' \cap Q) \cap S = (0) &\Longrightarrow L' \cap Q = C \\ &\Longrightarrow \overline{L'} \cap \overline{Q} = 0 \\ &\Longrightarrow \overline{L} = \overline{L'} \\ &\Longrightarrow L = L' \subseteq_c M. \end{aligned}$$

□

In the next few results, we shall explore the relationship between complements and uniform dimensions. The first result about modules of finite uniform dimension is an analogue of (6.3).

(6.29) Proposition. *Suppose* $\text{u.dim}\, M = n < \infty$. *Then, in* M, *any chain of complements has length* $\leq n$. *More precisely, if we have* $C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_k$ *where each* $C_i \subseteq_c M$, *then* $k \leq n$.

Proof. By (6.24)(1), we know that $C_{i-1} \subseteq_c C_i$, say, C_{i-1} is a complement to S_i in C_i $(1 \leq i \leq k)$. Since $C_{i-1} \neq C_i$, $S_i \neq 0$. Now we have $S_1 \oplus \cdots \oplus S_k \subseteq M$, so (6.3) gives $k \leq n$. (**Note:** Of course we also have $C_0 \oplus S_1 \oplus \cdots \oplus S_k \subseteq M$. But we cannot deduce $k+1 \leq n$ from this, as C_0 might be zero!) □

Next we present the analogue of (6.4).

(6.30) Proposition. *For any module* M_R, *the following are equivalent:*

(1) $\text{u.dim}\, M = \infty$.
(2) *There exists an infinite strictly ascending chain of complements in* M.
(3) *There exists an infinite strictly descending chain of complements in* M.

Proof. (1) $\Longrightarrow$ (2). By (6.4), we know that M contains $U_1 \oplus U_2 \oplus \ldots$, where each $U_i \neq 0$. Enlarge U_1 into a complement to $U_2 \oplus U_3 \oplus \cdots$, say, C_1. Then enlarge $C_1 \oplus U_2$ into a complement to $U_3 \oplus U_4 \oplus \cdots$, say C_2. In this way, we get an ascending chain $C_1 \subseteq C_2 \subseteq \cdots$, where each $C_i \subseteq_c M$. Since C_i contains U_i and $U_i \cap C_{i-1} = 0$, we have $C_{i-1} \neq C_i$ for each i.

(2)$\Longrightarrow$(3). Say we have $C_0 \subsetneq C_1 \subsetneq \cdots$, where each $C_i \subseteq_c M$. As in the proof of (6.29), C_{i-1} is a complement to some $S_i \neq 0$ in C_i. Enlarge $S_1 \oplus S_2 \oplus \cdots$ into a complement to C_0, say, Y_1. Then, working in Y_1, enlarge $S_2 \oplus S_3 \oplus \cdots$ into a complement to S_1 in Y_1, say Y_2. By the *Transitivity Property* (6.24)(2), $Y_2 \subseteq_c Y_1 \subseteq_c M$ implies that $Y_2 \subseteq_c M$. Also $Y_1 \supsetneq Y_2$ since Y_1 contains S_1 while $Y_2 \cap S_1 = 0$. Continuing in this way, we get a strictly descending chain of complements $Y_1 \supsetneq Y_2 \supsetneq \cdots$ in M.

(3)$\Longrightarrow$(1) follows from (6.29). □

Negating the three statements in (6.30), we get the following equivalent result.

(6.30)′ Proposition. *For any* M_R, *the following are equivalent:*

(1) $\text{u.dim}\, M < \infty$.
(2) *The complements in* M *satisfy* ACC.
(3) *The complements in* M *satisfy* DCC.

Finally, we have the following analogue of (6.6), the proof of which we can safely omit.

(6.31) Proposition. u. dim M *is the supremum of the set*

$$\{k : \ M \text{ contains a chain of complements of length } k\}.$$

To conclude the present subsection, we shall give another important characterization of complement submodules. Let us say that a submodule $C \subseteq M$ is *essentially closed* in M if C has no proper essential extension within M. To simplify language, we shall sometimes drop the word "essentially", and just refer to such C as a *closed submodule* of M.

(6.32) Proposition. *For any submodule $C \subseteq M$, the following are equivalent:*

(1) $C \subseteq_c M$.
(2) *C is (essentially) closed in M.*
(3) *$C = X \cap M$ for some direct summand X of the injective hull $E(M)$.*

Proof. (1) $\Longrightarrow$ (2). Say C is a complement to a submodule $S \subseteq M$. If $C \subseteq_e C' \subseteq M$, then $C \cap (C' \cap S) = 0$ implies that $C' \cap S = 0$. Hence $C = C'$, so C is closed in M.

(2) $\Longrightarrow$ (3). Since $C \subseteq M \subseteq E(M)$, $E(M)$ contains a copy of the injective hull of C, say $E(C)$. Then $C \subseteq_e E(C) \cap M$ implies that $C = E(C) \cap M$ since C is closed in M. Now the injectivity of $X := E(C)$ implies that X is a direct summand of $E(M)$, so we have proved (3).

(3)$\Longrightarrow$(1). Say $E(M) = X \oplus Y$, and let $S = M \cap Y$. We are done if we can show that *C is a complement to S in M*. The property $C \cap S = 0$ is, of course, clear. Now consider a submodule D of M properly containing C, say, with $d \in D \backslash C$. From $(X + dR) \cap Y \neq 0$, we have an equation

$$(6.33) \qquad x + dr = y \neq 0, \quad \text{where } x \in X, \ y \in Y, \text{ and } r \in R.$$

From $M \subseteq_e E(M)$, we have $0 \neq yr' \in S$ for some $r' \in R$. Multiplying (6.33) by r', we get

$$0 \neq yr' = xr' + drr',$$

from which we see that $xr' \in X \cap M = C$, and consequently that $D \cap S$ contains the nonzero element yr'. □

If the reader does not wish to go through the condition (3), it is equally easy to see directly that (2) $\Longrightarrow$ (1). Say C is closed in M and let $T \subseteq M$ be a complement to C. We finish by showing that C is a complement to T. To see this, let $B \supseteq C$ be such that $B \cap T = 0$. Since T is a complement to C, we have $C \oplus T \subseteq_e M$, and hence (by Exercise (3.7)) $(C \oplus T) \cap B \subseteq_e B$; that is, $C \subseteq_e B$, and so $B = C$.

It is indeed fortunate that the words "complement" and "closed" start with the same letter, so we can now regard "$C \subseteq_c M$" as referring to either one. Note that the new criteria in (6.32) do make it easier sometimes to check that a given

submodule is a complement. For instance, a direct application of (6.32)(2) shows the following.

(6.34) Example. *If M is a module over a commutative domain, then the torsion submodule of M is a complement in M.*

§6C. Exact Sequences and Essential Closures

We are now in a good position to study the behavior of uniform dimension with respect to short exact sequences. We first prove the following general result on "subadditivity".

(6.35) Theorem. *Let A be a submodule of M. Then*

$$\text{u.dim}\, M \leq \text{u.dim}\, A + \text{u.dim}\, M/A, \tag{6.36}$$

with the usual conventions on the symbol ∞. If $A \subseteq_c M$, then equality holds.

Proof. Fix a complement T to A in M. Then $A \oplus T \subseteq_e M$, so

$$\text{u.dim}\, M = \text{u.dim}\, (A \oplus T) = \text{u.dim}\, A + \text{u.dim}\, T$$

by (6.10). Since we may view $T \subseteq M/A$, we have u.dim $T \leq \text{u.dim}\, M/A$, so we get the inequality (6.36). Now assume $A \subseteq_c M$. From the argument given in the paragraph following the proof of (6.32) (or else from (6.23)), we see that the image of T in M/A is in fact essential in M/A. Hence u.dim $T = \text{u.dim}\, M/A$, so the proof above yields equality in (6.36). □

While (6.36) fails to be an equality in general (as we have already observed), we can obtain, in some sense, a measure of how much the two sides in (6.36) can differ. As a special case of this, we can deduce that, under a finiteness assumption on u.dim M, equality in (6.36) can occur (if and) only if A is a *closed* submodule of M.

To carry out this analysis, we introduce the idea of essential closures. For a given submodule $A \subseteq M$, consider the family of essential extensions of A inside of M. By Zorn's Lemma, this family has a maximal member, say C. Such a maximal member is by no means unique, but it is clearly closed in M; we call it an *essential closure* of A. Using such an essential closure C, we can formulate the following more precise version of (6.36) due to Camillo and Zelmanowitz [78]. The proof given below is believed to be new.

(6.37) Theorem. *Let $A \subseteq C \subseteq M$ be as above. Then*

$$\text{u.dim}\, A + \text{u.dim}\, M/A = \text{u.dim}\, M + \text{u.dim}\, C/A, \tag{6.38}$$

with the usual conventions on the symbol ∞. If $\text{u.dim}\, M < \infty$, *we have*

$$\text{u.dim}\, M = \text{u.dim}\, A + \text{u.dim}\, M/A \quad \textit{iff} \quad A \subseteq_c M;$$

and u. dim C/A *is independent of the choice of* C *(as an essential closure of* A*) in* M.

Proof. The fact that $A \subseteq_e C$ implies that u.dim $A =$ u. dim C. Now consider the following two exact sequences:

$$0 \to C \to M \to M/C \to 0 \quad \text{and} \quad 0 \to C/A \to M/A \to M/C \to 0.$$

By (6.28)(1), $C \subseteq_c M$ implies that $C/A \subseteq_c M/A$. Therefore, applying the last part of (6.35) to the two exact sequences above, we have

$$\begin{aligned} \text{u.}\dim A + \text{u.}\dim M/A &= \text{u.}\dim C + \text{u.}\dim M/A \\ &= \text{u.}\dim C + \text{u.}\dim M/C + \text{u.}\dim C/A \\ &= \text{u.}\dim M + \text{u.}\dim C/A, \end{aligned}$$

proving (6.38). If $A \subseteq_c M$, then $C = A$, in which case (6.38) gives back the equation

$$\text{u.}\dim A + \text{u.}\dim M/A = \text{u.}\dim M.$$

Conversely, if this latter equation holds *and* u. dim $M < \infty$, then (6.38) implies that u.dim $C/A = 0$, which in turn implies that $C = A$, i.e. $A \subseteq_c M$. The independence of u.dim C/A on the choice of the essential closure C (in case u.dim $M < \infty$) is immediate from the equation (6.38). □

Note that in (6.38), u.dim C/A emerges as the "error term" for the failure of additivity of uniform dimension over short exact sequences. In general, we have little control over this error term, as the examples in (6.11) showed already. A good illustration for the nonuniqueness of the essential closures C of A (*and* for the independence of u.dim C/A on C) is given by Example (6.17)(5) for $\mathbb{Z}$-modules. In the notations there, let $A = 4C \subseteq M$. Since C and C' are both complements to S in M, they give two nonisomorphic essential closures of A. Though $C/A \cong \mathbb{Z}_4$ and $C'/A \cong \mathbb{Z}_2$ are also nonisomorphic, each has uniform dimension 1, and (6.38) checks out as

$$\text{u.}\dim A + \text{u.}\dim M/A - \text{u.}\dim M = 1 + 2 - 2 = 1.$$

Note that the assumption u.dim $M < \infty$ is essential for the "only if" statement in the second conclusion of (6.37). If u.dim $M = \infty$, the condition

$$\text{u.}\dim M = \text{u.}\dim A + \text{u.}\dim M/A$$

only means that one of u.dim A, u.dim M/A is ∞. Of course, we do not expect such a weak piece of information to yield the conclusion $A \subseteq_c M$.

We close this subsection with the following refinement of the second part of (6.37).

(6.39) Corollary. *Let* $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n+1} = M$ *be a filtration in* M*. If each* $A_i \subseteq_c M$*, then*

$$\text{u.dim } M = \sum_{i=0}^{n} \text{u.dim } A_{i+1}/A_i. \tag{6.40}$$

Conversely, if $\text{u.dim } M < \infty$ *and* (6.40) *holds, then each* $A_i \subseteq_c M$*.*

Proof. The equation (6.40) follows from (6.24)(1), the second part of (6.37), and induction on n. Conversely, assume (6.40) holds and that u.dim $M < \infty$. Fix any $j \in [1, n]$. By (6.35) and induction, we have

$$\begin{aligned} \text{u.dim } M &\leq \text{u.dim } A_j + \text{u.dim } M/A_j \\ &\leq \sum_{i=0}^{j-1} \text{u.dim } A_{i+1}/A_i + \sum_{i=j}^{n} \text{u.dim } A_{i+1}/A_i \\ &= \sum_{i=0}^{n} \text{u.dim } A_{i+1}/A_i. \end{aligned}$$

In view of (6.40), all inequalities above must be equalities. In particular,

$$\text{u.dim } M = \text{u.dim } A_j + \text{u.dim } M/A_j.$$

By the second part of (6.37), we conclude that $A_j \subseteq_c M$. Of course, $A_0 = 0$ and $A_{n+1} = M$ are closed in M as well. □

Now a final remark. In spite of the rather precise results in this subsection, the failure of "u.dim" to be additive over *arbitrary* short exact sequences remains a serious drawback for the theory of uniform dimensions. As a remedy for this, Goldie has introduced a modified version of uniform dimension which has come to be known as the "reduced rank". This is an invariant of modules that is bounded by the uniform dimension (hence its name), and enjoys the full additivity property over short exact sequences. The treatment of the reduced rank will be given in §7C below.

§6D. CS Modules: Two Applications

In the previous subsection, we have introduced the notion of an essential closure of a submodule. For two modules $A \subseteq M$, an essential closure of A in M means a *maximal essential extension* C of A in M. Such a C always exists; incidentally, C is also minimal among closed submodules of M containing A. However, as we have observed before, C is in general not unique.

The lack of uniqueness of essential closures is largely a result of the fact that, in general, the intersection of two (essentially) closed submodules of a module need not be closed. Nevertheless, the formation of essential closures turns out to be a useful tool, as we have already seen, for instance, in the formulation and proof of (6.37). In the following we shall give some further applications of essential

closures. (Another kind of closure called the Goldie closure, defined by using the idea of singular submodules, will be introduced later in §7C.)

We begin with the following easy observation.

(6.41) Lemma. *For any (right) R-module M, the following are equivalent:*

(1) *Every complement (i.e., closed submodule) in M is a direct summand.*
(2) *For every submodule $A \subseteq M$, there exists a direct summand C of M such that $A \subseteq_e C$.*

Proof. (2) $\Longrightarrow$ (1) is trivial. (1) $\Longrightarrow$ (2) follows by taking C to be an essential closure of A in M. □

Definition. If M_R satisfies (1), (2) above, we say that M is a CS *module* ("complements are summands"). If all quotients of M are CS, we call M a *completely* CS *module.*

(6.42) Examples.

(0) *A semisimple module is completely* CS.

(1) *Any uniform module is* CS. (So for instance, any subgroup of $\mathbb{Q}$ is a CS module over $\mathbb{Z}$.)

(2) *Any injective module is* CS. (More generally, any "quasi-injective" module, to be defined in §6G, is already CS; see (6.80).)

(3) *A closed submodule N of a* CS *module M is always CS.* In fact, let $C \subseteq_c N$. By (6.24)(2), we have $C \subseteq_c M$. Since M is CS, C is a direct summand of M, and hence of N.

(4) *Over a Prüfer domain R, any f.g. torsionfree module N_R is* CS. To see this, let $C \subseteq_c N$. Then N/C is torsionfree. For, if otherwise, there would exist $x \in N \backslash C$ such that $xr \in C$ for some nonzero $r \in R$. But then $C \subsetneq_e C + xR$, a contradiction. Since N/C is f.g., it must therefore be projective by (2.31). Hence $0 \to C \to N \to N/C \to 0$ splits, so E is a direct summand of N.

(5) By (4) and (1) above, $\mathbb{Z}^n$ and $\mathbb{Z}/p^m\mathbb{Z}$ (for any prime p) are CS modules over $\mathbb{Z}$. However, the $\mathbb{Z}$-module $M = \mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in (6.17)(5) turns out to be *not* CS. (In the notation of (6.17)(5), the submodule C' is easily seen to be closed in M, but it is not a direct summand.) Thus, *the direct sum of two CS modules may not be CS.* For a complete determination of the f.g. CS modules over $\mathbb{Z}$, see Exercise 19D.

The following property of a CS module is noteworthy.

(6.43) Lemma. *A* CS *module M_R has* u. dim $M < \infty$ *iff M is a finite direct sum of uniform modules.*

Proof. ("Only If") Assume $M \neq 0$ and u.dim $M < \infty$. By (6.2), M contains a uniform submodule A. By (6.41), there is a decomposition $M = C \oplus M'$ where $A \subseteq_e C$. Using (6.24)(2), it is easy to see that M' is also CS. Since u.dim $M' <$ u.dim M, the proof proceeds by induction. □

In this subsection, we shall prove a couple of interesting results about CS modules. These are deeper results, the proofs of which require more substantial work. If the reader so wishes, he or she can skip these harder proofs for a first reading, and go on with the rest of §6 without suffering from any discontinuity.

The first result we propose to prove is a powerful theorem of B. Osofsky and P. F. Smith [91] which guarantees that certain classes of CS modules are of finite uniform dimension. For any ring R, consider a family $\mathcal{P}$ of f.g. right R-modules with the following properties:

(P_1) *If N is isomorphic to a quotient of some module in $\mathcal{P}$, then $N \in \mathcal{P}$.*

(P_2) *If $X \in \mathcal{P}$ and Y is any $\mathcal{P}$-submodule of a quotient module of X, then there is a $\mathcal{P}$-submodule $Y' \subseteq X$ that projects onto Y. (Here and in the following, "$\mathcal{P}$-submodule" means "submodule belonging to $\mathcal{P}$".)*

For instance, if $\mathcal{P}$ is any family of f.g. modules closed under submodules, quotient modules (and isomorphic copies), then $\mathcal{P}$ clearly satisfies (P_1) and (P_2). Thus, the family of f.g. semisimple right R-modules satisfies (P_1) and (P_2). The family of cyclic (resp. f.g.) R-modules also satisfies (P_1) and (P_2).

We can now state the following main theorem of Osofsky and Smith (*loc. cit.* [91]).

(6.44) Theorem. *Let $\mathcal{P}$ be a family of right R-modules satisfying (P_1) and (P_2), and let $M \in \mathcal{P}$. If every $\mathcal{P}$-submodule of M is completely* CS, *then M is a finite direct sum of uniform modules.*

Since the formulation of this theorem is rather abstract, it behooves us to state a few special cases of the theorem in the form of corollaries.

(6.45) Corollary. *Let R be a ring such that every cyclic (resp. f.g.) right R-module is CS. Then every cyclic (resp. f.g.) right R-module is a finite direct sum of uniform modules.*

Proof. Apply the theorem to the class $\mathcal{P}$ of cyclic (resp. f.g.) right R-modules. □

(6.46) Corollary. *Let N be any right R-module such that every quotient of every cyclic submodule of N is injective. Then N is semisimple.*

Proof. It suffices to show that every cyclic submodule $M \subseteq N$ is semisimple. Again, let $\mathcal{P}$ be the class of all cyclic R-modules. Consider any quotient A/B of a $\mathcal{P}$-submodule $A \subseteq M$. By assumption, A/B is injective, and therefore CS.

Thus, A is completely CS, and M satisfies the hypothesis of (6.44). By (6.44), $M = M_1 \oplus \cdots \oplus M_n$, where the M_i 's are uniform. For any $0 \neq m_i \in M_i$, $m_i R$ is injective, so $m_i R$ is a direct summand of M_i. This implies that $m_i R = M_i$. It follows that each M_i is simple, and therefore M is semisimple, as desired. □

The following result was stated without proof in *FC*–p. 30. It was first proved by B. Osofsky in her Rutgers thesis in 1964. We can now deduce it easily from (6.46).

(6.47) Corollary. *Let R be a ring such that every cyclic right R-module is injective. Then R is a semisimple ring.*

Proof. By (6.46), every module N_R is semisimple. □

In appreciation of this Corollary, we remark that, under the hypothesis that right cyclic R-modules are injective, every principal right ideal aR is a direct summand of R_R, and this amounts to saying that R is a von Neumann regular ring (see *FC*–(4.23)). And of course, R is right self-injective. However, all this is still a far cry from the semisimplicity of R, for which there is no "easy" proof. For a nice direct application of (6.47), see (7.52)(2) below.

Having seen some of the interesting applications of Theorem (6.44), let us now embark upon its proof, following closely the arguments given by Osofsky and Smith.

Proof of (6.44). Let M and $\mathcal{P}$ be as in (6.44). By (6.43), it suffices to show that u.dim $M < \infty$. Assume, instead, u.dim $M = \infty$. Then

$$M \supseteq M_1 \oplus M_2 \oplus \cdots ,$$

where $M_i \neq 0$. Since M is CS, $M = A_1 \oplus B_1$, where $M_1 \subseteq_e A_1$. After applying the projection map π_1 from M to B_1 (with kernel A_1)[50] we may assume that $\bigoplus_{i \geq 2} M_i \subseteq B_1$. Repeating the construction, we have $B_1 = A_2 \oplus B_2$, with $M_2 \subseteq_e A_2$, and we may assume as before that $\bigoplus_{i \geq 3} M_i \subseteq B_2$. This process leads to nonzero submodules $\{A_i, B_i\}$ such that, for any n,

$$M = A_1 \oplus \cdots \oplus A_n \oplus B_n .$$

In particular, $A_i, B_i \in \mathcal{P}$. Since any module in $\mathcal{P}$ is f.g., each A_i contains a maximal submodule C_i. Let S_i be the simple module A_i/C_i, and let $\overline{M} = M/\bigoplus_{i=1}^{\infty} C_i$, which is CS. For any n, we have:

$$(*) \qquad \overline{M} = \frac{A_1 \oplus \cdots \oplus A_n \oplus B_n}{C_1 \oplus \cdots \oplus C_n \oplus \cdots} = S_1 \oplus \cdots \oplus S_n \oplus (\cdots).$$

[50]Note that π_1 is injective on $\bigoplus_{i \geq 2} M_i$, since $A_1 \cap \bigoplus_{i \geq 2} M_i = 0$.

Let S be the semisimple module $\bigoplus_{i=1}^{\infty} S_i \subseteq \overline{M}$, and let N be an essential closure of S in $\overline{M}$. Since $\overline{M}$ is CS, N is a direct summand of $\overline{M}$. We note the following properties of N:

(A) *Being isomorphic to a quotient of* M, $N \in \mathcal{P}$ *and* N *is completely* CS.

(B) *Any* $\mathcal{P}$*-submodule* $A \subseteq N$ *is completely* CS. (By the property (P_2), $A \subseteq \overline{M}$ is the image of a $\mathcal{P}$-submodule $A' \subseteq M$, and by assumption A' is completely CS.)

(C) *Any f.g. submodule* $T \subseteq S$ *is a direct summand of* N. (This follows from $(*)$ above since T is a direct summand of $S_1 \oplus \cdots \oplus S_n$ for some n.)

Now write $\mathbb{N} = \{1, 2, \dots\}$ as a disjoint union $\bigsqcup_{j=1}^{\infty} \mathbb{N}_j$ where each $\mathbb{N}_j \subset \mathbb{N}$ is (countably) infinite. Let $S_j' = \bigoplus_{i \in \mathbb{N}_j} S_i$, and let N_j be an essential closure of S_j' in N. By (A), N_j is a direct summand of N and so $N_j \in \mathcal{P}$. Also, we have $S = \bigoplus_{j \geq 1} S_j'$ and $N_j \cap S = S_j'$ for all j. Since N/S is also CS by (A), it has a direct summand, say N'/S, with

$$\left(\bigoplus_{j\geq 1} N_j\right)/S \subseteq_e N'/S.$$

(Here, the sum $\sum_{j\geq 1} N_j$ is direct by Exercise (3.8).) From $N \in \mathcal{P}$, we deduce that $N/S \in \mathcal{P}$ and $N'/S \in \mathcal{P}$, so by (P_2), N contains a $\mathcal{P}$-submodule A with $A + S = N'$. We claim that:

(D) $A \cap S_j' \neq 0$ *for every* j.

To see this, fix a submodule V in the semisimple module S such that $S = V \oplus (A \cap S)$, and consider the projection π from $A \oplus V$ $(= A + S)$ to V. If $A \cap S_j' = 0$, we will have

$$S_j' \cong \pi(S_j') \subseteq \pi(N_j) \subseteq V.$$

But $N_j \in \mathcal{P}$ implies that N_j is f.g., so $\pi(N_j)$ is of finite length, in contradiction to the definition of S_j'. This proves (D); in particular, there exists a simple module $T_j \subseteq A \cap S_j'$, for each j. Since A is CS (by (B)), A has a direct summand Y with $\bigoplus_{j=1}^{\infty} T_j \subseteq_e Y$. The latter implies that $Y \not\subseteq S$, since A and hence Y are both f.g. But by (C),

$$S \cap Y \cap \bigoplus_{j=1}^{n} N_j = \bigoplus_{j=1}^{n} T_j \qquad (n \geq 1)$$

is a direct summand of N. Since $S \subseteq_e N$, we must have

$$Y \cap \bigoplus_{j=1}^{n} N_j = \bigoplus_{j=1}^{n} T_j$$

for all n, and hence $Y \cap \bigoplus_{j=1}^{\infty} N_j \subseteq S$. This contradicts $\left(\bigoplus_{j=1}^{\infty} N_j\right)/S \subseteq_e N'/S$, since $Y \subseteq N'$ but $Y \not\subseteq S$. ∎

Though the proof of Theorem (6.44) is rather long, the strength of the result and the richness of its corollaries amply justify its inclusion. For instance, after we introduce the notion of singular modules in §7, it will be clear that the class $\mathcal{P}$ of cyclic (resp. f.g.) *singular* right modules over a ring R also has the two properties (P_1) and (P_2). Thus, (6.44) applies to $\mathcal{P}$. For many other applications of (6.44) in a similar spirit, we refer the reader to the paper of Osofsky and Smith [91].

We now go on to give a second application of the notion of CS modules. This is a remarkable result of Y. Utumi which is a byproduct of his work on self-injective rings in the mid-1960s (see Utumi [65]).

(6.48) Theorem. *Let R be a ring such that both R_R and ${}_RR$ are* CS *modules. Then R is Dedekind-finite.*

In fact, the following special case of the theorem is already noteworthy.

(6.49) Corollary. *Any self-injective ring is Dedekind-finite.*

This follows from the Theorem since, for a self-injective ring R, R_R and ${}_RR$ are injective modules, and hence CS modules. The Corollary does require R to be *both* left and right self-injective, since we have shown earlier (in (3.74B)) that the ring R of linear endomorphisms of an infinite-dimensional left vector space over a division ring is left but not right self-injective — and we know that R is *not* Dedekind-finite!

Proof of (6.48). Let R be as in (6.48), and assume that $ab = 1 \neq ba$ for some $a,\ b \in R$. Then, by a standard observation of Jacobson (FC-(21.26)), the elements $e_{ij} = b^i(1 - ba)a^j$ $(i,\ j \geq 1)$ are nonzero elements of R satisfying the matrix units' equations: $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$. Since ${}_RR$ is CS,

$$\text{(A)} \qquad \sum_i Re_{ii} \subseteq_e Rg$$

for some idempotent g. (Note that any direct summand of ${}_RR$ is generated by an idempotent.) Similarly, since R_R is CS,

$$\text{(B)} \qquad \sum_{j>1} g(e_{1j} + e_{jj})\,R \subseteq_e fR$$

for some idempotent f. Having chosen the idempotents g and f, we proceed in the following sequence of steps.

Step 1. *If $a \in Rg$ is such that $ae_{ii} = 0$ $(\forall i)$, then $a = 0$.* Indeed, let

$$A = \{x \in Rg : \ xe_{ii} = 0\ (\forall\, i)\},$$

which is a left ideal in Rg. We have $A \cap \sum_i Re_{ii} = 0$. (For, if $a = \sum_i x_i e_{ii} \in A$, then $0 = ae_{jj} = x_j e_{jj}$ $(\forall\, j)$, and so $a = 0$.) Since $\sum_i Re_{ii} \subseteq_e Rg$, it follows that $A = 0$.

Step 2. *We claim that* $g = fg$. Indeed, if otherwise, $(1-f)g \in Rg \setminus \{0\}$, so by (A), we have

$$0 \neq x(1-f)g \in \sum_{i=1}^{n} Re_{ii} \quad \text{(for some } x \in R,\ \ n \geq 1).$$

From this, we have $x(1-f)g\, e_{n+1,n+1} = 0$. Left multiplying (B) by $x(1-f)$, we get

$$x(1-f)g(e_{1j} + e_{jj}) = 0 \ \ (\forall\, j > 1).$$

Comparing this (for $j = n+1$) with the last equation, we get $x(1-f)ge_{1,n+1} = 0$. Right multiplying this by $e_{n+1,j}$, we get $x(1-f)ge_{1j} = 0$ $(\forall\, j \geq 1)$, and so $x(1-f)ge_{jj} = 0$ for all $j > 1$ (as well as for $j = 1$). By Step 1, we get $x(1-f)g = 0$, a contradiction.

Step 3. Note that, from (A), $e_{ii}g = e_{ii}$ for all i. In particular, $e_{11}ge_{11} = e_{11} \neq 0$ guarantees that $ge_{11} \neq 0$. Now by Step 2, $ge_{11} = fge_{11} \in fR$, so (B) implies that

$$\text{(C)} \qquad 0 \neq ge_{11}z = \sum_{j=2}^{m} g(e_{1j} + e_{jj})\, z_j$$

for some $z,\ z_j \in R$, and some $m \geq 2$. Left multiplying (C) by e_{jj} $(j > 1)$, we have $e_{jj}z_j = 0$, and hence $e_{1j}z_j = e_{1j}e_{jj}z_j = 0$. Now the RHS of (C) is 0, a contradiction. □

In closing this subsection, we note that the notion of CS modules has also found other applications in ring theory, for instance in the study of simple rings and noetherian rings. In a recent paper of Huynh, Jain, and López-Permouth [96], it is shown that, if every cyclic singular right module over a simple ring R is CS, then R must be a right noetherian ring. In another paper of Huynh, Rizvi, and Yousif [96], it is shown that, if every f.g. right R-module is CS, then R must again be a right noetherian ring.

Some remarks on terminology are also in order. The property (6.41)(2) on a module M, namely, that any submodule of M is essential in some direct summand, is often known in the literature as the *extending property.* Accordingly, the CS modules defined by the (equivalent) properties in (6.41) are also known as *extending modules* (especially in the work of M. Harada, B. Müller, and others). Extending modules (and some subclasses of them, notably "quasi-injective modules", "continuous modules", and "quasi-continuous modules") have gained considerable popularity since the 80s, and were extensively studied in a monograph by that name by Dung, Huynh, Smith and Wisbauer [94], as well as in the earlier book by Mohamed and Müller [90]. While we have no space for continuous and quasi-continuous modules (except in a few exercises), we shall return in §6G to give a quick exposition on quasi-injective modules. If the reader so wishes, he or she can proceed directly to §6G at this point, and come back to work on §6E and §6F later.

§6E. Finiteness Conditions on Rings

Viewing a ring R as a right module over itself, we have an invariant u.dim R_R of the ring, which is either a natural number, or the symbol ∞. According to (6.30)′, the finiteness condition u.dim $R_R < \infty$ amounts to ACC on the complements in R_R, or equivalently, DCC on such complements. On the other hand, the two finiteness conditions

$$\text{(6.50)} \qquad \text{u.dim}\, R_R < \infty, \qquad \text{u.dim}\, {}_R R < \infty$$

turn out to be completely independent. For instance, in Exercise 14, a ring R is exhibited with the property that u.dim ${}_R R = 2$, but u.dim $R_R = \infty$. Later (in §10), it will also be seen that, for any domain R which is left Ore but not right Ore, we have u.dim ${}_R R = 1$, but u.dim $R_R = \infty$.

Many types of finiteness conditions in ring theory can be formulated in terms of ACC or DCC on suitable classes of 1-sided ideals. Upon fixing attention on a specific type of 1-sided ideals, we can investigate if there is a relation between ACC and DCC on such, or if there is a relation between the "left" condition and the "right" condition. And then we can investigate if there is a relation between one finiteness condition and another. Traditionally, concrete results on such relations (or explicit counterexamples showing the lack of them) are of great interest to ring theorists.

In *FC*, we have considered several types of finiteness conditions. Here, we have introduced new ones in (6.50) via uniform dimensions. In this subsection, we shall explain how the different types of finiteness conditions are interrelated.

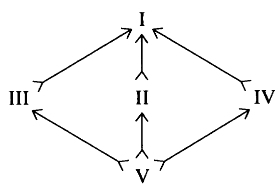
To make our discussions more systematic, we introduce the following notational system (where R is a ring):

$$\text{(6.51)} \qquad \begin{cases} \text{I} & = \text{ right (left) ideals in } R, \\ \text{II} & = \text{ principal right (left) ideals in } R, \\ \text{III} & = \text{ right (left) complements in } R, \\ \text{IV} & = \text{ right (left) annihilators in } R, \\ \text{V} & = \text{ right (left) direct summands in } R. \end{cases}$$

Here, I, II, III, and V should be self-explanatory. In IV, we mean by "right (resp. left) annihilator" a subset in R of the form $\text{ann}_r(A)$ (resp. $\text{ann}_\ell(A)$), where A is a subset of R. Note that a right (resp. left) annihilator is always a right (resp. left) ideal.

The inclusion relations between the different families (indicated by the inclusion arrow "$\rightarrowtail$") are summarized in the following chart:

(6.52)



Here, only the three lower inclusion arrows need to be explained. To be specific, let us work with right ideals. If $\mathfrak{A} \subseteq R_R$ is a right direct summand, then $\mathfrak{A} = eR$ for some idempotent e (see FC–Exercise 1.7). Therefore, $\mathfrak{A}$ is a principal right ideal. We can check easily that $eR = \text{ann}_r(R(1-e))$, so $\mathfrak{A}$ is also a right annihilator. Finally, of course, $\mathfrak{A}$ is a right complement (to $(1-e)R$) in R.

Continuing to work with *right* ideals, let us introduce the notations $\text{ACC}_{\text{I}}, \ldots,$ ACC_{V} for the ACC conditions on the five families of *right* ideals, and $\text{DCC}_{\text{I}}, \ldots,$ DCC_{V} for the corresponding DCC conditions. (Putting the numerals I, ... , V on the right suggests working with right ideals.) In view of the inclusion relations in (6.52), we obtain immediately the following implication chart:

(6.53)
$$\begin{array}{ccccc} & & \text{DCC}_{\text{I}} & & \\ & \swarrow & \Downarrow & \searrow & \\ \text{DCC}_{\text{III}} & & \text{DCC}_{\text{II}} & & \text{DCC}_{\text{IV}} \\ & \searrow & \Downarrow & \swarrow & \\ & & \text{DCC}_{\text{V}} & & \end{array}$$

and a similar one for ACC. Note that DCC_{I} is just "right artinian", DCC_{III} is just "(right) finite-dimensional" (u.dim $R_R < \infty$), and DCC_{II} amounts to "left perfect" (by Bass' Theorem: FC–(23.20))[51]. However, there are no special names in the literature for DCC_{IV} and DCC_{V}. (A nice interpretation for DCC_{V} will be given later in (6.59).)

Of course, we can also work with *left* ideals. In that case, we arrive at the finiteness conditions ${}_{\text{I}}\text{ACC}, \ldots, {}_{\text{V}}\text{ACC}$, and ${}_{\text{I}}\text{DCC}, \ldots, {}_{\text{V}}\text{DCC}$, *and charts similar to* (6.53) *hold for these.*

Next, we shall go through each of the families in (6.51), and investigate the relations between "left" and "right", ACC and DCC. Beginning with the family I, we have the following relations according to the Hopkins-Levitzki Theorem FC–(4.15):

(6.54)
$$\begin{array}{ccc} {}_{\text{I}}\text{DCC} & & \text{DCC}_{\text{I}} \\ \Downarrow & & \Downarrow \\ {}_{\text{I}}\text{ACC} & & \text{ACC}_{\text{I}} \end{array}$$

Here, as in the following, if two conditions are *not* connected by an implication arrow, then there is (usually) none. Thus, in (6.54), left noetherian does not imply left artinian, nor does it imply right noetherian, etc.

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[51]There is no misprint: we did mean to say "left perfect" here. The switch from "right" to "left" is part of the nature of Bass' Theorem.

For the family II, we consider principal 1-sided ideals. Here, Jonah [70] has proved the following remarkable "criss-cross" implications:

$$(6.55)\qquad \begin{array}{ccc} {}_{\mathrm{II}}\mathrm{DCC} & & \mathrm{DCC}_{\mathrm{II}} \\ & \searrow\!\!\!\!\!\!\swarrow & \\ {}_{\mathrm{II}}\mathrm{ACC} & & \mathrm{ACC}_{\mathrm{II}} \end{array}$$

These implications are, of course, not reversible, as the example $\mathbb{Z}$ shows. Since ${}_{\mathrm{II}}\mathrm{DCC}$ means "right perfect" and $\mathrm{DCC}_{\mathrm{II}}$ means "left perfect", the example in *FC*–(23.22) of a right perfect ring which is not left perfect shows that ${}_{\mathrm{II}}\mathrm{DCC}$ and $\mathrm{DCC}_{\mathrm{II}}$ are independent conditions. We leave it to the reader to check that in fact, there are no horizontal or vertical implications at all in (6.55): this is a rather tricky "extra credit" problem (Exercise 24).

For the family III, we consider 1-sided complements and uniform dimensions. Here we get more implication arrows. According to what we said at the beginning of this subsection, we have the following chart:

$$(6.56)\qquad \begin{array}{ccc} {}_{\mathrm{III}}\mathrm{DCC} & & \mathrm{DCC}_{\mathrm{III}} \\ \Updownarrow & & \Updownarrow \\ {}_{\mathrm{III}}\mathrm{ACC} & & \mathrm{ACC}_{\mathrm{III}} \end{array}$$

The examples on uniform dimensions cited at the beginning of §6E show, however, that no implication is possible horizontally.

Next we come to the family IV. Here we consider 1-sided annihilators. Note that a right ideal $\mathfrak{A} \subseteq R$ is a right annihilator iff $\mathfrak{A} = \mathrm{ann}_r(\mathrm{ann}_\ell \mathfrak{A})$. Suppose R satisfies ${}_{\mathrm{IV}}\mathrm{ACC}$, and consider a descending chain of *right* annihilators $\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \supseteq \cdots$. Taking left annihilators, we get $\mathrm{ann}_\ell(\mathfrak{A}_1) \subseteq \mathrm{ann}_\ell(\mathfrak{A}_2) \subseteq \cdots$. By ${}_{\mathrm{IV}}\mathrm{ACC}$, this chain stabilizes. Taking right annihilators again, we see that the original chain $\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \supseteq \cdots$ stabilizes. Therefore, we have ${}_{\mathrm{IV}}\mathrm{ACC} \Longrightarrow \mathrm{DCC}_{\mathrm{IV}}$, and the converse can be proved similarly. Thus we have

$$(6.57)\qquad \begin{array}{ccc} {}_{\mathrm{IV}}\mathrm{DCC} & & \mathrm{DCC}_{\mathrm{IV}} \\ & \nwarrow\!\!\!\!\!\!\nearrow \\ & \swarrow\!\!\!\!\!\!\searrow & \\ {}_{\mathrm{IV}}\mathrm{ACC} & & \mathrm{ACC}_{\mathrm{IV}} \end{array}$$

We leave it to the reader to confirm (Exercise 23) that there are no horizontal (equivalently, vertical) implication arrows. For *commutative* rings, of course, ACC and DCC for annihilator ideals are equivalent.

Finally, we come to the family V. Here we consider 1-sided direct summands (eR 's and Re 's for $e = e^2 \in R$). The following turns out to be true:

$$(6.58)\qquad \begin{array}{ccc} {}_{\mathrm{V}}\mathrm{DCC} & \Longleftrightarrow & \mathrm{DCC}_{\mathrm{V}} \\ \Updownarrow & & \Updownarrow \\ {}_{\mathrm{V}}\mathrm{ACC} & \Longleftrightarrow & \mathrm{ACC}_{\mathrm{V}} \end{array}$$

This follows from the following proposition (and left-right symmetry).

(6.59) Proposition. *For any ring R, the following are equivalent:*

(1) *R satisfies ACC on right direct summands.*
(2) *R satisfies DCC on left direct summands.*
(3) *R has no infinite set of nonzero orthogonal idempotents.*[52]

Proof. (1)$\Longleftrightarrow$(2). Suppose $eR \subsetneq e'R$, where e, e' are idempotents. Taking left annihilators, we have $R(1-e) \supsetneq R(1-e')$. Here the inclusion remains strict, for otherwise we can take right annihilators again and get the contradiction $eR = e'R$. This observation clearly gives the equivalence of (1) and (2).

(1)$\Longrightarrow$(3). Assume R has an infinite orthogonal set of nonzero idempotents $\{e_1, e_2, \dots\}$. Let $c_n = e_1 + \cdots + e_n$ for $n \geq 1$; these are easily checked to be idempotents. Also,

$$c_{n+1}c_n = (e_1 + \cdots + e_n + e_{n+1})(e_1 + \cdots + e_n) = c_n^2 = c_n,$$
$$c_n c_{n+1} = c_n \neq c_{n+1}.$$

These imply that $c_n R \subsetneq c_{n+1} R$ for all n, so (1) fails.

(3)$\Longrightarrow$(2). Assume there exist $R = B_0 \supsetneq B_1 \supsetneq \dots$ where each B_n is a direct summand of ${}_RR$. Then $B_{n-1} = A_n \oplus B_n$ for suitable nonzero left ideals $\{A_n : n \geq 1\}$. Write $1 = e_1 + f_1$ with $e_1 \in A_1$, $f_1 \in B_1$, and write $f_1 = e_2 + f_2$ with $e_2 \in A_2$, $f_2 \in B_2$, and so on. Then $A_n = Re_n$ so $e_n \neq 0$ for each $n \geq 1$. Moreover,

$$1 = e_1 + f_1 = e_1 + e_2 + f_2 = \cdots = e_1 + \cdots + e_n + f_n$$

is the decomposition of 1 with respect to the direct sum expression $R = A_1 \oplus \cdots \oplus A_n \oplus B_n$. It follows that $e_1, e_2, \dots$ are mutually orthogonal (nonzero) idempotents (cf. *FC*–Exercise 1.7). □

Using the five families I, …, V, we have generated altogether 20 finiteness conditions in (6.54)–(6.58). In the following Proposition, we shall show that *any* of these implies two other familiar finiteness conditions.

(6.60) Proposition. *Assume that R satisfies any of the 20 finiteness conditions specified above. Then*

(1) *R_R (resp. ${}_RR$) is a finite direct sum of indecomposable right (resp. left) ideals (or, equivalently, 1 is a sum of a finite number of mutually orthogonal primitive idempotents).*
(2) *R is Dedekind-finite.*

Proof. In view of the hierarchy in (6.52), R satisfies each of the conditions in (6.59).

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[52]Two idempotents $e, e' \in R$ are called *orthogonal* if $ee' = e'e = 0$.

(1) This follows by an easy application of the König Tree Lemma. To give a more self-contained proof, we can proceed as follows. By (6.59)(1) (left ideal version), there exists a left ideal $A \subseteq R$ maximal with respect to the following properties:

(a) A is a direct summand of ${}_RR$.
(b) A is a finite direct sum of indecomposable left ideals.

Let $R = A \oplus B$, where B is a left ideal. If $B \neq 0$, then by (6.59)(2), B contains an indecomposable left ideal C that is a direct summand of ${}_RR$. But then $A \oplus C$ also has the properties (a), (b) above, which contradicts the choice of A. Therefore, $B = 0$ and $R = A$ has the property (b), as desired.

(2) The proof of this part is a famous observation of Jacobson. If R fails to be Dedekind-finite, let us fix $a, b \in R$ such that $ab = 1 \neq ba$. According to Jacobson (cf. *FC*–(21.26)),

$$\{e_i = b^i(1 - ba)a^i \;:\; 1 \leq i < \infty\}$$

is an infinite orthogonal set of nonzero idempotents in R. Thus R does not satisfy (3) in (6.59). Alternatively, we can proceed as follows. If R is not Dedekind-finite, ${}_RR = A_1 \oplus B_1$ for some $B_1 \neq 0$, $A_1 \cong {}_RR$ (cf. Exercise 1.8). Again, we can decompose A_1 into $A_2 \oplus B_2$ for some $B_2 \neq 0$, $A_2 \cong {}_RR$, and so forth. This leads to an infinite chain of direct summands $A_1 \supsetneq A_2 \supsetneq \dots$ in ${}_RR$, so R does not satisfy (2) in (6.59). □

§6F. *Change of Rings*

Let "P" be any of the finiteness conditions introduced in §6E. It is natural to ask:

(1) *Is the property "P" on a ring R inherited by subrings of R?*
(2) *Does the property "P" on R "go up" to the polynomial ring $R[x]$? Or the power series ring $R[[x]]$?*
(3) *Ditto for the matrix rings $\mathbb{M}_n(R)$.*

For instance, one of the most famous theorems in ring theory, the Hilbert Basis Theorem, says precisely that if R satisfies ACC_l, then $R[x]$ also does. But then if R satisfies DCC_l, $R[x]$ never does (unless $R = 0$). For the matrix ring $S = \mathbb{M}_n(R)$, it is quite easy to see that, if R satisfies ACC_l or DCC_l, then S also does, and conversely. On the other hand, according to Exer. (1.18), it is possible for a ring R to be Dedekind-finite, and a matrix ring $S = \mathbb{M}_n(R)$ to be not Dedekind-finite. Generally speaking, the answer to the questions (1), (2), and (3) above is: it all depends!

There is much to be done if we want to answer all of these questions fully, but we have neither the time nor the space here. In any case, Question (3) above is best dealt with later in the context of the Morita Theory for categories of modules. We shall return to this point in Chapter 7.

Instead of trying to answer the questions (1)–(3) in full, let us just make a few comments on some important sample cases here. We'll start with (1): the question on inheritance of the various properties to subrings. It is true that most of

the conditions we considered are *not* inherited by subrings. However, some are. For instance, the "subscript V" conditions (on direct summands) in (6.58), *when interpreted in the form* (6.59)(3), are obviously inherited by subrings. In the same spirit, Dedekind-finiteness (and stable finiteness) is inherited by subrings. More substantially, the "subscript I" conditions (ACC and DCC on 1-sided ideals) are inherited by subrings under suitable conditions, according to the Eakin-Nagata-Eisenbud Theorem (3.91) and the Eisenbud-Robson Theorem (3.93). As for the chain conditions on annihilators, we have the following proposition which is often greeted by beginning ring theory students with surprise and delight.

(6.61) Proposition. *Let "P" be any of the conditions in* (6.57) *for annihilators. If R has the property "P", so does any subring $S \subseteq R$.*

Proof. Consider, say, the case $P = \mathrm{ACC}_{\mathrm{IV}}$. Let $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$ be a chain of right annihilators in S, and let $\mathfrak{B}_i = \mathrm{ann}_\ell^S(\mathfrak{A}_i)$. (The superscript "$S$" suggests that the annihilators are taken in S.) Then $\mathfrak{B}_1 \supseteq \mathfrak{B}_2 \supseteq \cdots$, and we have

$$\mathrm{ann}_r^R(\mathfrak{B}_1) \subseteq \mathrm{ann}_r^R(\mathfrak{B}_2) \subseteq \ldots .$$

By $\mathrm{ACC}_{\mathrm{IV}}$ on R, this chain stabilizes. Since

$$S \cap \mathrm{ann}_r^R(\mathfrak{B}_i) = \mathrm{ann}_r^S(\mathfrak{B}_i) = \mathfrak{A}_i,$$

the chain $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \ldots$ also stabilizes. □

Although (6.61) is basically an easy result, its significance is not to be underestimated. It implies, for instance, that *any subring of a right noetherian ring satisfies* $\mathrm{ACC}_{\mathrm{IV}}$ *(on right annihilators)*. As we shall see in §11, this observation plays a key role in formulating the definition of a (right) Goldie ring.

In contrast to (6.61), $\mathrm{ACC}_{\mathrm{IV}}$ *does not go up to matrix rings*. In Exercise (1.18), we have mentioned Shepherdson's example of a domain R for which $S = \mathbb{M}_2(R)$ is not Dedekind-finite. By (6.60), S does not satisfy any of the 20 finiteness conditions in (6.54)–(6.58). But of course R satisfies ACC and DCC on both left and right annihilators, since R is a domain!

Returning now to uniform dimensions which are the main concern of §6, we shall conclude the present subsection with some results on the behavior of "u.dim" vis-à-vis the types of change of rings in Questions (1)–(3) above. For *subrings*, not too much can be said. If $R \subseteq S$ are rings, u.dim $S_S < \infty$ need not imply u.dim $R_R < \infty$. For instance, the free algebra $R = \mathbb{Q}\langle x, y\rangle$ has infinite right uniform dimension (it contains $\bigoplus_{i=1}^{\infty} x^i yR$); yet it can be embedded in a division ring S (*FC*–(14.25)), which has right uniform dimension 1. For matrix rings, the situation is much more amenable, as the following result shows.

(6.62) Proposition. *Suppose* u. dim $R_R = d$, *and let* $S = \mathbb{M}_n(R)$. *Then* u. dim $S_S = nd$.

Proof. Let $M = R^n$. Viewing the elements of R^n as row vectors, we can make M into a right S-module by matrix multiplication. For any right ideal $\mathfrak{A} \subseteq R$, define

(6.63) $$\alpha(\mathfrak{A}) = \{(a_1, \ldots, a_n) \in M : a_1, \ldots, a_n \in \mathfrak{A}\},$$

which is an S-submodule in M. Conversely, for any S-submodule $N \subseteq M$, let

(6.64) $$\beta(N) = \{a \in R : (a, 0, \ldots, 0) \in N\},$$

which is a right ideal in R. It is easy to check that α and β define mutually inverse 1–1 correspondences between the right ideals of R and the S-submodules of M. Moreover, α and β both preserve direct sums. It follows immediately that u.dim $M_S = \text{u.dim}\, R_R = d$. Since $S_S \cong n \cdot M_S$, we have u.dim $S_S = nd$. □

For polynomial extensions $S = R[x]$, "(right) finite dimensionality" (ACC_{III}) is preserved as well. This may be viewed as an analogue of the Hilbert Basis Theorem. However, while other kinds of dimensions (Krull dimension, global dimension, ...) tend to get bigger under a polynomial extension, the right uniform dimension does not, as the following result of R. C. Shock [72] shows.

(6.65) Theorem. *For* $S = R[x]$, $\text{u.dim}\, S_S = \text{u.dim}\, R_R$.

The proof of this result is preceded by a few lemmas. *The notation* $S = R[x]$ *will be fixed below.*

For convenience, let us say that a polynomial $f \in S$ is *good* if $f \neq 0$, and all nonzero coefficients of f have the same right annihilator in R. Note that if f is good, then for any $d \in R$, fd remains good, unless it is zero.

(6.66) Lemma. *For any* $f \in S \backslash \{0\}$, *there exists* $d \in R$ *such that* fd *is good.*

Proof. We induct on the number k of nonzero coefficients of f, the case $k = 1$ being clear. Assume $k \geq 2$, and that f is not yet good. Then f has two coefficients $a,\ a' \neq 0$ with $\text{ann}_r^R(a) \neq \text{ann}_r^R(a')$. We may assume there exists $b \in R$ such that $ab = 0 \neq a'b$. Then $fb \neq 0$ and has fewer than k nonzero coefficients. By the inductive hypothesis, $(fb)c$ is good for some $c \in R$, so we can pick $d = bc$.

□

(6.67) Lemma. *Let* $f(x) = a_n x^n + \cdots + a_0 \in R[x]$ *be good, where* $a_n \neq 0$. *Let* $\mathfrak{B} = \text{ann}_r^R(a_n)$. *Then* $\text{ann}_r^S(f) = \mathfrak{B}[x]$.

Proof. The inclusion $\text{ann}_r^S(f) \supseteq \mathfrak{B}[x]$ is clear. If equality does not hold, pick $g \in \text{ann}_r^S(f) \backslash \mathfrak{B}[x]$ of the least degree m, say,

$$g(x) = b_m x^m + \cdots + b_0.$$

Then $a_n b_m = 0$ implies that $b_m \in \mathfrak{B}$. But then

$$b_{m-1} x^{m-1} + \cdots + b_0 \in \text{ann}_r^S(f) \backslash \mathfrak{B}[x],$$

a contradiction. $\square$

(6.68) Lemma. *If $\mathfrak{A}$ is a uniform right ideal in R, then $\mathfrak{A}[x]$ is a uniform right ideal in $S = R[x]$.*

Proof. Suppose $\mathfrak{A}[x]$ is *not* uniform in S_S. Then there exist $f,\ g \in \mathfrak{A}[x]\backslash\{0\}$ such that $fS \cap gS = 0$. We may assume that this pair is chosen such that $\deg f + \deg g$ is as small as possible. In view of (6.66), we may assume (after a right scaling) that f and g are both *good*, say,

$$f = a_n x^n + \cdots + a_0 \ (a_n \neq 0), \quad g = b_m x^m + \cdots + b_0 \ (b_m \neq 0), \ \text{ and } m \geq n.$$

Since $\mathfrak{A}_R$ is uniform, $a_n c = b_m d \neq 0$ for suitable $c,\ d \in R$. After replacing f and g by fc and gd (which remain good as noted earlier), we may assume that $a_n = b_m$. From (6.67), it follows that $\text{ann}_r^S(f) = \text{ann}_r^S(g)$. Now consider $g' := g - fx^{m-n}$, which has degree $< m$, and is not zero (for otherwise $0 \neq g \in fS \cap gS$). Choose $e \in R$ such that $g'e$ is good. By the minimal choice of $m+n$, $fS \cap g'eS \neq 0$, so there exist $h,\ k \in S$ such that

$$(6.69) \qquad 0 \neq fh = g'ek = (g - fx^{m-n})ek.$$

But then $f(h + x^{m-n}ek) = gek$ implies that $gek = 0$. Since $\text{ann}_r^S(f) = \text{ann}_r^S(g)$, we must also have $fek = 0$. Now the RHS of (6.69) is zero, a contradiction! $\square$

With the aid of the above lemma, we can now prove Shock's Theorem.

Proof of (6.65). If $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$ is a direct sum of right ideals in R, clearly $\mathfrak{A}[x] = \mathfrak{A}_1[x] + \cdots + \mathfrak{A}_n[x]$ is a direct sum of right ideals in S. This gives the desired conclusion if $\text{u.dim } R_R = \infty$. Now assume $\text{u.dim } R_R = n < \infty$. Pick *uniform* right ideals $\{\mathfrak{A}_i\}$ such that

$$\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n \subseteq_e R_R.$$

By (6.68), each $\mathfrak{A}_i[x]$ is uniform in S_S. But by Exercise (3.30),

$$\mathfrak{A}_1[x] \oplus \cdots \oplus \mathfrak{A}_n[x] = \mathfrak{A}[x] \subseteq_e S_S.$$

Therefore, $\text{u.dim } S_S = n$. $\square$

By an easy induction, it follows that, for $S = R[X]$ where X is any finite set of commuting indeterminates, we have $\text{u.dim } S_S = \text{u.dim } R_R$. In fact, as Shock has pointed out, the same conclusion holds for *any* (possibly infinite) set X of commuting indeterminates.[53] We invite the reader to verify this fact in Exercise 26.

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[53]The commutativity of the set X is essential here, as the case of free algebras in more than one variable shows.

For the power series extension $S = R[[x]]$, however, "(right) finite-dimensionality" is no longer preserved, in general. In (10.31A), we shall provide an example of a (noncommutative) domain R with u.dim ${}_RR =$ u. dim $R_R = 1$ (a so-called Ore domain) but with u.dim $S_S = \infty$ for the power series ring $S = R[[x]]$.

§6G. *Quasi-Injective Modules*

In this subsection, we discuss in some detail the notion of a quasi-injective module which generalizes that of an injective module. At first, this material might look a little out of place since the present section (§6) is supposed to be on uniform dimensions, closed submodules, and the like. However, in studying the theory of quasi-injective modules, it turns out that we will have occasion to use some of the notions developed earlier in this section such as complements, essential closures, and CS modules, and so forth. Thus, it makes sense to present the material on quasi-injective modules here and showcase it as an interesting application of the earlier material in §6.

(6.70) Definition. A module M_R over an arbitrary ring R is said to be *quasi-injective* (QI) if, for any submodule $L \subseteq M$, any $f \in \text{Hom}_R(L, M)$ can be extended to an endomorphism of M.

The definition of a quasi-injective module was given by R. E. Johnson and E. T. Wong in 1961. With such a definition, several thoughts immediately come to mind.

(6.71) Remarks.

(1) Clearly, *any injective module is always* QI. The converse is not true in general; see (6.72) below.

(2) *If* M_R *contains a copy of the right regular module* R_R, *then* M *is* QI *iff it is injective.* (The nontrivial direction follows from Baer's Criterion (3.7), since, upon viewing R_R as a submodule of M, any homomorphism from a right ideal to M extends to an endomorphism of M, which then restricts to a homomorphism from R_R to M.) The following are some quick consequences of this fact:

(2A) *Over a domain R, any torsionfree* QI *module M is injective.* (If $0 \neq m \in M$, then $M \supseteq m \cdot R \cong R$.)

(2B) *For any module N, $R_R \oplus N$ is* QI *iff R_R and M are both injective.* In particular, R_R is QI iff R is right self-injective.

(3) To check that M is QI, it suffices to check, in the notation of (6.70), the extendibility of $f \in \text{Hom}_R(L, M)$ in the case when $L \subseteq_e M$. Once this is checked, the case of a general submodule $L \subseteq M$ follows from an easy application of Zorn's Lemma.

(4) Of course, there also exists the obvious dual notion of a *quasi-projective* module: a module P is quasi-projective if, for any quotient module Q of P, any

$g \in \mathrm{Hom}_R(P, Q)$ can be "lifted" to an endomorphism of P. We shall, however, limit ourselves to a discussion of quasi-injectives, and shall not go into quasi-projectives in this text.

In mathematics, we often generalize notions that we have defined earlier. Whether a particular generalization is worthwhile should be judged, among other factors, by whether the generalization allows for nice new examples, and whether it leads to a mathematically interesting theory. To justify the notion of QI modules, we should therefore first look at some nontrivial (i.e., non-injective) examples.

(6.72) Examples.

(1) *Any simple module M_R is obviously* QI, since the only submodules of M are (0) and M. Of course, a simple module need not always be injective (unless R is a so-called right V-ring; see §3H).

(2) *More generally, any semisimple module M is* QI. Given $f \in \mathrm{Hom}_R(L, M)$ where $L \subseteq M$, we can extend f to $g \in \mathrm{End}_R(M)$ by letting g be zero (for instance) on a direct complement of L in M.

(3) *Let R be a commutative* PID. *Then any cyclic module $M \not\cong R$ is* QI. To see this, write $M = R/A$, where A is a nonzero ideal, and consider any submodule $L = B/A$, where $B \supseteq A$ is an ideal. Say, $B = bR$, and $A = bcR$ $(c \neq 0)$. For any $f \in \mathrm{Hom}_R(L, M)$, write $f(\bar{b}) = \bar{x}$, where the "bars" mean modulo A. Since $0 = f(\overline{bc}) = \bar{x}c$, we have $xc \in bcR$, whence $x = br$ for some $r \in R$. Therefore, f extends to the endomorphism of M given by multiplication by r, and we have checked the quasi-injectivity of M. (A slightly different proof for this can be given using the self-injectivity of R/A; see Exercise 27B.)

(4) Let R be a finite-dimensional algebra over a field k such that any simple right R-module has endomorphism ring k. Then any module M_R of length ≤ 2 is QI. It suffices to check the case of a nonsemisimple M of length 2. In this case $\mathrm{soc}(M)$ is a simple module S. Consider $f \in \mathrm{Hom}_R(L, M)$ where $L \subseteq M$. In order to extend f, we may assume that L has length 1. But then $L = S$, and we must have $f(L) \subseteq \mathrm{soc}(M) = L$. By assumption, f is a scalar multiplication on L by some $a \in k$, so obviously it extends to an endomorphism of M. Thus, for instance, in the 3-dimensional k-algebra $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, any proper right ideal (and every proper homomorphic image of R_R) is QI.

(6.73) Proposition. *(1) A direct summand of a* QI *module is always* QI. *(2) In general, a direct sum of two* QI *modules need not be* QI.

Proof. (1) Suppose $N = M \oplus M'$ is QI. To see that M also is, consider $f \in \mathrm{Hom}_R(L, M)$, where L is any submodule of M. We may view f as in $\mathrm{Hom}_R(L, N)$ and extend it to some $g \in \mathrm{End}_R(N)$. If π is the projection map from N to M (with kernel M'), clearly $\pi \circ g$ restricted to M extends f. Hence M is QI.

(2) We give here a nice (and quick) counterexample due to B. Osofsky. Over the ring $R = \mathbb{Z}$, let $M = \mathbb{Q}$ and $M' = \mathbb{Z}_n$ for any natural number n. By (6.71)(1) and (6.72)(2), these are QI $\mathbb{Z}$-modules. However, $N = M \oplus M'$ is *not* QI. In fact, let $L = \mathbb{Z} \oplus (0) \subseteq N$, and take $f \in \text{Hom}_{\mathbb{Z}}(L, N)$ such that f takes $\mathbb{Z}$ to $\mathbb{Z}_n$ by the natural projection map. This f clearly cannot be extended to an endomorphism of N, since $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_n) = 0$ ($\mathbb{Q}$ being divisible). Therefore, N fails to be QI! Another example is $N = M \oplus M'$ where M, M' are the QI modules $\mathbb{Z}_m$ and $\mathbb{Z}_n$, with $n|m$ and $1 < n < m$. An argument similar to that above shows that N is not QI. (This observation, plus (6.77) below, leads quickly to a complete determination of the f.g. abelian groups which are QI as $\mathbb{Z}$-modules; see Exercise 28.) □

Next, we give an interesting characterization of QI modules M in terms of the injective hull $E(M)$.

(6.74) Theorem. *A module M_R is* QI *iff M is fully invariant in $E(M)$ (i.e., iff M is stabilized by every endomorphism of $E(M)$).*

Proof. First suppose M is fully invariant. For any $f \in \text{Hom}_R(L, M)$ where $L \subseteq M$, we can extend f to an endomorphism g of $E(M)$ (since $E(M)$ is injective). By assumption, $g(M) \subseteq M$, so $g|_M \in \text{End}_R(M)$ extends f. This checks that M is QI. Conversely, assume M is QI, and let $f \in \text{End}_R(E(M))$. It is easy to check that $L = \{m \in M : f(m) \in M\}$ is an R-submodule of M. Since we have $f : L \longrightarrow M$, there exists $g \in \text{End}_R(M)$ such that $f|_L = g|_L$. Without complicating the notation, we may assume that $g \in \text{End}_R(E(M))$ (since $E(M)$ is injective). Suppose for the moment that $(g - f)M \neq 0$. Then $M \cap (g - f)M \neq 0$, so $(g - f)m = m' \neq 0$ for some $m, m' \in M$. Now $f(m) = g(m) - m' \in M$ implies that $m \in L$, and $f|_L = g|_L$ leads to $m' = 0$, a contradiction. Therefore, we must have $(g - f)M = 0$, and hence $f(M) = g(M) \subseteq M$. This checks that M is fully invariant in $E(M)$. □

(6.75) Remark. Let $S = \text{End}_R(E(M))$ operate on the left of $E(M)$, so that $E(M)$ is an (S, R)-bimodule. From this perspective, we can rephrase (6.74) by saying that *M is* QI *iff it is an (S, R)-subbimodule of $E(M)$.*

Let us state some useful consequences of (6.74). The first one is obvious from the proof given above.

(6.76) Corollary. *If M_R is* QI, *there exists a natural surjection*

$$\alpha : \text{End}_R(E(M)) \to \text{End}_R(M),$$

defined by restriction of endomorphisms.

In general, of course, the surjection α is not an isomorphism. For instance, consider the simple (and hence QI) module $M = \mathbb{Z}_p$ over $R = \mathbb{Z}$. Here, $E(M)$ is

the Prüfer p-group C_{p^∞}, whose endomorphism ring is the ring of p-adic integers. This maps onto $\text{End}_{\mathbb{Z}}(M)$ which is the ring $\mathbb{Z}_p$. The kernel of α is the unique maximal ideal of the ring of p-adic integers. However, in some important special cases, α is an isomorphism; see, for instance, Exercise (7.32), where it is claimed that α is an isomorphism when M is a "nonsingular" QI module.

(6.77) Corollary. *If a module M is* QI, *so is any finite direct sum M^n.*

Proof. Let $E = E(M)$, and $S = \text{End}_R(E)$. For $N = M^n$, we have $E(N) = E^n$, and so $\text{End}_R(E(N)) = \mathbb{M}_n(S)$, operating on the right of N by "matrix multiplication". Given any row vector $(m_1, \ldots, m_n) \in N$, right multiplication by a matrix in $\mathbb{M}_n(S)$ results in another row vector in N, since M is invariant under any entry of the matrix. This checks that N is fully invariant in $E(N)$, so by (6.74), N is QI. □

Another application of (6.74) is to the notion of a *quasi-injective envelope* (QI envelope, or QI hull) of a module. The definition is the obvious one: a QI envelope of M is a QI module $Q \supseteq M$ which is minimal with respect to these properties (i.e., Q does not properly contain a QI module $\supseteq M$).

(6.78) Corollary. *Any module M has a* QI *envelope.*

Proof. Let $S = \text{End}_R(E(M))$ as before, and let Q be the smallest (S, R)-sub-bimodule of $E(M)$ containing M. Since $E(Q) = E(M)$, we see that Q is fully invariant in $E(Q)$. Therefore, Q is QI, and clearly Q has the required minimal property of a QI envelope. □

Let us denote the Q constructed above by $E_q(M)$ (the subscript "q" standing for "quasi"). It is natural to ask if, up to an isomorphism over M, $E_q(M)$ is the *only* QI hull of M. This turns out to be the case; however, it is not quite obvious. The main point here is that we have by construction $M \subseteq_e E_q(M)$, but it is not clear that another QI hull will *also* be essential over M. To see that this is indeed the case, we shall need the machinery of closed submodules and complements developed earlier in this section. The key step is to observe the following crucial "cutting property" in the injective hull of a QI module.

(6.79) Proposition. *Let M_R be a* QI *module, and let $E(M) = \bigoplus_{i \in I} X_i$ be a direct sum decomposition of $E(M)$. Then $M = \bigoplus_{i \in I} X_i'$ for $X_i' = M \cap X_i$.*

Proof. Let π_i be the projection map from $E(M)$ to X_i with respect to the given decomposition. If $m = \sum_{i \in I} x_i \in M$ (finite sum, with $x_i \in X_i$ for each i), then, by (6.74), $x_i \in \pi_i(m) \in M \cap X_i = X_i'$, so we have $M = \bigoplus_{i \in I} X_i'$. □

The "cutting property" in (6.79) was first proved by Goel and Jain in a slightly more general context. In fact, they showed that this property characterizes a notion

called "π-injectivity" which is a generalization of the notion of quasi-injectivity of M. For more details on this, see Exercise 37 below.

Recall that a CS module is one in which any closed submodule is a direct summand. We have remarked before (in (6.42)(2)) that any injective module is CS. The Proposition above allows us to extend this fact to quasi-injective modules, after Faith and Utumi.

(6.80) Corollary. *Any* QI *module M is* CS *(but not conversely).*

Proof. Let $N \subseteq_c M$ (i.e., N is a closed submodule of M), and take a complement T for N in M. Then $N \oplus T \subseteq_e M$, so $E(M) = E(N) \oplus E(T)$. By (6.79),

$$M = (M \cap E(N)) \oplus (M \cap E(T)).$$

Since $N \subseteq_c M$ and $N \subseteq_e M \cap E(N)$, we have $N = M \cap E(N)$. Thus, $M = N \oplus (M \cap E(T))$, proving that M is a CS module. On the other hand, a CS module need not be QI. For instance, over the ring $R = \mathbb{Z}$, the module R_R is CS, but is clearly not QI. Another example is $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ (for any prime p): this is CS by Exercise 19C, but is not QI by what we said in the proof of (6.73)(2). □

We can now use Corollary (6.80) to prove the uniqueness of a QI envelope.

(6.81) Corollary. *Up to an isomorphism over M, $E_q(M)$ is the only* QI *envelope of M. Any* QI *module containing M contains a copy of $E_q(M)$ (over M).*

Proof. It suffices to prove the second statement. Let Q be a QI module containing M, and let $N \supseteq M$ be an essential closure of M in Q. By (6.80), N is a direct summand of Q, so by (6.73)(1), N is also QI. We finish by showing that N contains a copy of $E_q(M)$ over M. Since $M \subseteq_e N$, we may assume that $N \subseteq E(M)$. But then N is fully invariant in $E(N) = E(M)$, so $N \supseteq E_q(M)$ (since $E_q(M)$ is the smallest fully invariant submodule of $E(M)$ containing M). □

(6.82) Examples. Let us compute a few quick examples of QI hulls over $R = \mathbb{Z}$. Changing notations, let us write C_n for the cyclic group of order n. First let $M = \mathbb{Q} \oplus C_{p^k}$ (p a prime), which has injective hull $E(M) = \mathbb{Q} \oplus C_{p^\infty}$, where C_{p^∞} denotes the Prüfer p-group. Proper submodules of $E(M)$ containing M are $\mathbb{Q} \oplus C_{p^i}$ (for $i \geq k$), none of which is QI by the proof of (6.73)(2). Therefore, $E_q(M) = E(M) = \mathbb{Q} \oplus C_{p^\infty}$. Now let us try another M, say, $M = C_m \oplus C_n$ where $n|m$ and $1 < n < m$. Note that M is contained in $H := C_m \oplus C_m$, which is QI by (6.77). Any proper submodule of H containing M cannot be QI (by what we said in the proof of (6.73)(2)), so we must have $E_q(M) = H$.

In (6.47), we proved Osofsky's result that a ring R must be semisimple if all cyclic right R-modules are injective. Taking full advantage of this result, we obtain almost for free the following interesting quasi-injective analogue.

(6.83) Theorem. *For any ring R, the following are equivalent:*

(1) *R is a semisimple ring;*
(2) *every right R-module is* QI*;*
(3) *every f.g. right R-module is* QI*;*
(4) *every right R-module generated by two elements is* QI.

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are trivial, so we need only show (4) $\Longrightarrow$ (1). Let N be any cyclic right R-module. By (4), $R_R \oplus N$ is QI, so by (6.71)(2), N must be injective. Therefore, by Osofsky's result (6.47), R is a semisimple ring. □

(6.84) Remark. The condition: (5) *Every cyclic right R-module is* QI is, in general, weaker than the four conditions in the Theorem. For instance, the self-injective ring $R = \mathbb{Q}[x]/(x^2)$ is not semisimple, but clearly satisfies (5).

The notion of QI modules has several interesting generalizations, the most notable among which are the notions of continuous and quasi-continuous modules. It would take us too far afield to develop these notions in any detail in the text, so we shall only include the beginning part of this development in a few exercises below (see Ex.'s 36-39).

Exercises for §6

1. Recall that a module M_R is called *Dedekind-finite* (cf. Exercise 1.8) if $M \oplus N \cong M$ (for some module N) implies that $N = 0$. Show that any module M with u.dim $M < \infty$ is Dedekind-finite.

2. Show that $|\text{Ass}(M)| \leq$ u.dim M for any module M.

3. Give an example of a f.g. module M over a commutative noetherian ring such that $|\text{Ass}(M)| = 1$, and u. dim $M = n$ (a prescribed integer).

4. Let M_R be any module with u.dim $M = n < \infty$. Show that there exist closed submodules $M_i \subseteq_c M$ $(1 \leq i \leq n)$ with the following properties:

   (1) Each M/M_i is uniform.
   (2) $M_1 \cap \cdots \cap M_n = 0$.
   (3) $E(M) \cong \bigoplus_{i=1}^n E(M/M_i)$.
   (4) $\text{Ass}(M) = \bigcup_{i=1}^n \text{Ass}(M/M_i)$.

5. Let $R = k[x, y]/(x, y)^n$ where k is a field. Show that u.dim$(R_R) = n$.

6. Give an example of a module of finite uniform dimension that is neither noetherian nor artinian.

7. For any submodules A, B of a module M, show that

$$\text{u.dim}\, A + \text{u.dim}\, B \leq \text{u.dim}(A \cap B) + \text{u.dim}(A + B).$$

8. Let $C \subseteq D$ be modules such that u.dim $D < \infty$ and u.dim $D/C = \infty$, and let $C' = \{(c, -c) : c \in C\} \subseteq D \oplus D$, $M = (D \oplus D)/C'$.
(1) Show that u.dim $M = \infty$.
(2) Let $A = \overline{D \oplus 0} \subseteq M$ and $B = \overline{0 \oplus D} \subseteq M$. Show that $A \cong B \cong D$ and $A + B = M$. (Thus, it is possible for u.dim A, u.dim B to be finite and u.dim$(A + B) = \infty$ in Exercise 7.)

9. Show that an abelian group $M \neq 0$ is a uniform $\mathbb{Z}$-module iff either $M \subseteq \mathbb{Q}$, or $M \cong \mathbb{Z}/p^n\mathbb{Z}$, or $M \cong \varinjlim \mathbb{Z}/p^n\mathbb{Z}$, where p is a prime. Generalize this to a (commutative) PID.

10. Let R be a commutative domain with quotient field K. For any R-module M with torsion submodule $t(M)$, show that
$$\dim_K(M \otimes_R K) = \text{u.}\dim M/t(M);$$
this number is called the "torsion-free rank" of M. If u. dim $t(M) < \infty$, show that the torsion-free rank of M is given by u. dim $M -$ u. dim $t(M)$.

11. Show that a module M_R is noetherian iff every essential submodule of M is f.g.

12. For any module M_R, let soc(M) (the *socle* of M) be the sum of all simple submodules of M (with soc$(M) = 0$ if there are no simple submodules). Show that:
(1) $M \cdot \text{soc}(R_R) \subseteq \text{soc}(M)$;
(2) $\text{soc}(M) = \bigcap\{N : N \subseteq_e M\}$;[54]
(3) for any submodule $N \subseteq M$, $\text{soc}(N) = N \cap \text{soc}(M)$;
(4) if $N \subseteq_e M$, then $\text{soc}(N) = \text{soc}(M)$;
(5) a maximal submodule $N \subseteq M$ is essential in M iff $N \supseteq \text{soc}(M)$;
(6) $\text{soc}\left(\bigoplus_{i \in I} M_i\right) = \bigoplus_{i \in I} \text{soc}(M_i)$;
(7) for any idempotent $f \in R$, $\text{soc}(fR) = f \cdot \text{soc}(R_R)$.

13. If R is a semisimple ring with Wedderburn decomposition $\mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_r}(D_r)$ where $D_1, \ldots, D_r$ are division rings, show that
$$\text{u.}\dim R_R = n_1 + \cdots + n_r.$$

14. Let $S \subseteq R$ be fields such that $\dim_S R = \infty$. Let T be the triangular ring $\begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$. By *FC*–(1.22), T is left artinian but not right noetherian. Show that u.dim$({}_T T) = 2$ and u.dim$(T_T) = \infty$.

15. Let R be a commutative PID, and $C \subseteq M$ be right R-modules. Show that $C \subseteq_c M$ iff, for every nonzero prime element $p \in R$, $C \cap Mp = Cp$. Using this, show that any pure submodule of M is a complement.

[54]This description of soc(M), due to Kasch and Sandomierski, may be viewed as dual to the description of the radical of M as the sum of the "small" submodules of M (cf. *FC*–(24.4)).

16. (1) Give an example of a complement $C \subseteq_c M$ (over a commutative PID if possible) such that C is not a pure submodule of M. (2) Give an example of a pure submodule $C \subseteq M$ (over some ring R) such that C is not a complement in M.

17. Decide which of the following statements is true:

    (1) If T is a direct summand of a module M_R, then any submodule $N \subseteq M$ with $N \cap T = 0$ can be enlarged to a direct complement to T in M.
    (2) Let $f \in \text{Hom}_R(M, M')$. Then $L \subseteq_c M$ implies that $f(L) \subseteq_c f(M)$.

18. Show that a subgroup C of a divisible abelian group M is a direct summand iff $C \cap Mp = Cp$ for every prime p.

    The following four exercises are intended to give a complete determination of all f.g. abelian groups that are CS *modules over the ring of integers* $\mathbb{Z}$.

19A. Show that a free abelian group F is CS as a $\mathbb{Z}$-module iff F has finite rank.

19B. Let M be a f.g. abelian group of rank $n \geq 1$. Show that M is a CS module over $\mathbb{Z}$ iff $M \cong \mathbb{Z}^n$.

19C. Let p be any prime, and $r \geq 1$.

    (1) Show that $\mathbb{Z}_{p^r} \oplus \mathbb{Z}_{p^{r+i}}$ $(i \geq 2)$ is not CS as a $\mathbb{Z}$-module.
    (2) Show that $(\mathbb{Z}_{p^r})^k \oplus (\mathbb{Z}_{p^{r+1}})^\ell$ is CS as a $\mathbb{Z}$-module.

19D. Show that a f.g. abelian group M is a CS module over $\mathbb{Z}$ iff either $M \cong \mathbb{Z}^n$ for some n, or M is finite and for any prime p, the p-primary part M_p of M is of the form $(\mathbb{Z}_{p^r})^k \oplus (\mathbb{Z}_{p^{r+1}})^\ell$ for some r, k, and ℓ (depending on p).

20. Show that, if R is a von Neumann regular ring, then the 20 finiteness conditions formulated in §6E are each equivalent to R being semisimple.

21. Show that R satisfies DCC on right annihilators iff, for any set $S \subseteq R$, there exists a *finite* subset $S_0 \subseteq S$ such that $\text{ann}_r(S) = \text{ann}_r(S_0)$.

22. Show that R *does not* satisfy ACC on right (resp. left) annihilators iff there exist elements $s_i, t_i \in R$ $(i = 1, 2, \ldots)$ such that $s_i t_i \neq 0$ for all i and $s_i t_j = 0$ for all $i > j$ (resp. for all $i < j$).

23. Show that "ACC on right annihilators" and "DCC on right annihilators" are independent conditions (for noncommutative rings).

24. (Extra Credit) Find examples to show that there cannot be any horizontal or vertical implications in (6.55) for chain conditions on principal 1-sided ideals. (**Hint.** Search far and wide!)

25. Show that for any ring R, the set A of right annihilator ideals in R is a complete lattice with respect to the partial ordering given by inclusion. Show that A is anti-isomorphic to the lattice A' of left annihilator ideals in

R. (A *complete lattice* is a partially ordered set in which any subset has a greatest lower bound, or equivalently, any subset has a least upper bound. Here, of course, "subset" means "possibly empty subset".)

26. (Shock) Let $S = R[X]$, where X is any (possibly infinite) set of commuting indeterminates. Show that u.dim $S_S = \text{u.dim } R_R$.

The remaining exercises in this section are devoted to the study of the properties of QI *(quasi-injective) modules and their generalizations. Exercise* 27A *is to be compared with Exercises* (3.28) *and* (3.29) *in* §3.

27A. For any module M_R and an ideal $J \subseteq R$, let $P = \{m \in M : mJ = 0\}$.

(1) If M is a QI R-module, show that P is a QI R/J-module.
(2) If $MJ = 0$, show that M is a QI R-module iff it is a QI R/J-module.

27B. For an ideal J in a ring R, show that the cyclic right R-module R/J is QI iff R/J is a right self-injective ring.

28. Show that a f.g. abelian group M is QI as a $\mathbb{Z}$-module iff $M \cong (\mathbb{Z}_n)^k$ for some natural numbers n and k. (This is to be compared with Exercise 19D, where we determined all f.g. abelian groups that are CS as $\mathbb{Z}$-modules.)

29. Let M_R be a QI module, and let A be an R-submodule of $E(M)$ isomorphic to a subquotient (quotient of a submodule) of M. Show that $A \subseteq M$.

30. (Ravel) Let M, N be QI modules with $E(M) \cong E(N)$. Show that $M \oplus N$ is QI iff $M \cong N$.

31. For any module M_R, consider the following conditions:

(1) M is Dedekind-finite;
(2) $E(M)$ is Dedekind-finite;
(3) for any nonzero module X, $X \oplus X \oplus \cdots$ cannot be embedded into $E(M)$;
(4) for any nonzero module X, $X \oplus X \oplus \cdots$ cannot be embedded into M.

Show that (2) $\iff$ (3) $\implies$ (4) $\implies$ (1), and that all four conditions are equivalent in case M is QI. In general, show that (1) does not imply (2), (3) or (4).

32. For any QI module M_R, show that the following are equivalent:

(1) M is uniform;
(2) M is indecomposable;
(3) $\text{End}(M_R)$ is a local ring;
(4) $E(M)$ is uniform;
(5) $E(M)$ is indecomposable;
(6) $\text{End}(E(M)_R)$ is a local ring.

Under these assumptions, show that the unique maximal (left, right) ideal of $\text{End}(M)$ is $\{f \in \text{End}(M) : \ker(f) \neq 0\}$. Is this the same set as $\{f \in \text{End}(M) : f(M) \neq M\}$?

33. Over a right artinian ring R, show that any faithful QI module M_R is injective. (**Hint.** Show that R_R embeds into M^n for some natural number n, and use (6.71)(2).)

34. (L. Fuchs) For any module M_R, show that the following are equivalent:
(1) M is QI;
(2) for any submodule $L \subseteq M$ contained in a cyclic submodule of M, any $f \in \mathrm{Hom}_R(L, M)$ extends to an endomorphism of M;
(3) for any B_R such that $\forall\, b \in B$, $\exists\, m \in M$ with $\mathrm{ann}(m) \subseteq \mathrm{ann}(b)$, any R-homomorphism from a submodule of B to M extends to B;
(4) ("Quasi Baer's Test") For any right ideal $J \subseteq R$, any R-homomorphism $g : J \to M$ whose kernel contains $\mathrm{ann}(m)$ for some $m \in M$ extends to R_R.

(**Hint.** The most efficient route is (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) $\Longrightarrow$ (3).)

35. An exercise in a ring theory monograph asked the reader to prove the equivalence of the following two conditions on a right ideal $I \subseteq R$: (a) $I = eR$ for some idempotent $e \in R$; (b) I is isomorphic to a direct summand of R. Provide some counterexamples to this alleged equivalence.

36. For any module M_R, consider the following conditions, where the word "summand" means throughout "direct summand":
(C_1) M is CS (i.e., any $N \subseteq_c M$ is a summand);
(C_2) If $K \subseteq M$ is isomorphic to a summand A of M, then K itself is a summand of M;
(C_3) If A, B are summands of M and $A \cap B = 0$, then $A + B$ is a summand of M.

Show that $(C_2) \Longrightarrow (C_3)$, and that any QI module M satisfies (C_1), (C_2), and (C_3). In the literature, M is called *continuous* if it satisfies (C_1), (C_2), and *quasi-continuous* if it satisfies (C_1), (C_3). With this terminology, we have the following basic implications:

Injective $\Longrightarrow$ QI $\Longrightarrow$ continuous $\Longrightarrow$ quasi-continuous $\Longrightarrow$ CS.

37. (Goel-Jain) For any module M_R, show that the following are equivalent:
(1) M is quasi-continuous (i.e., M satisfies (C_1) and (C_3));
(2) any idempotent endomorphism of a submodule of M extends to an idempotent endomorphism of M;
(3) any idempotent endomorphism of a submodule of M extends to an endomorphism of M;
(4) M is invariant under any idempotent endomorphism of $E(M)$;
(5) if $E(M) = \bigoplus_{i \in I} X_i$, then $M = \bigoplus_{i \in I} (M \cap X_i)$;
(6) if $E(M) = X \oplus Y$, then $M = (M \cap X) \oplus (M \cap Y)$.

Note. In the literature, the property (3) above is often referred to as the "*π-injectivity*" of the module M ("π" here stands for "projection").

38. For any von Neumann regular ring R, show that the following are equivalent:

    (a) R_R is continuous;
    (b) R_R is quasi-continuous;
    (3) R_R is CS.

    (A von Neumann regular ring R is said to be *right continuous* if it satisfies these equivalent conditions. For instance, any right self-injective von Neumann regular ring is right continuous.)

39. Show that each of the four implications listed at the end of Exercise 36 is irreversible.

40. For any QI module M_R, let $S = \text{End}(M_R)$, and $m \in M$. If $m \cdot R$ is a simple R-module, show that $S \cdot m$ is a simple S-module. From this fact, deduce that $\text{soc}(M_R) \subseteq \text{soc}({}_SM)$.

§7. Singular Submodules and Nonsingular Rings

§7A. *Basic Definitions and Examples*

In this subsection, we introduce the notion of the *singular submodule* of a module, due to R. E. Johnson. This leads us to the notions of nonsingular modules and (right or left) nonsingular rings which will be needed later in developing the theory of rings of quotients and the theory of semiprime Goldie rings in Chapter 4. In a later subsection (§7C), we shall deal with Goldie's work on the closure of submodules and Goldie's definition of the reduced rank.

(7.1) Definition. Let M be a right module over a ring R. An element $m \in M$ is said to be a *singular element* of M if the right ideal $\text{ann}(m)$ is essential in R_R. The set of all singular elements of M is denoted by $\mathcal{Z}(M)$.

(7.2) Lemma.

(1) *$\mathcal{Z}(M)$ is a submodule, called the singular submodule of M.*
(2) *$\mathcal{Z}(M) \cdot \text{soc}(R_R) = 0$, where $\text{soc}(R_R)$ denotes the socle of R_R.*
(3) *If $f : M \to N$ is any R-homomorphism, then $f(\mathcal{Z}(M)) \subseteq \mathcal{Z}(N)$.*
(4) *If $M \subseteq N$, then $\mathcal{Z}(M) = M \cap \mathcal{Z}(N)$.*

Proof. (1) If $m_1, m_2 \in \mathcal{Z}(M)$, then $\text{ann}(m_i) \subseteq_e R_R$ $(i = 1, 2)$ imply that

$$\text{ann}(m_1) \cap \text{ann}(m_2) \subseteq_e R_R.$$

Since $\text{ann}(m_1 + m_2)$ contains the LHS, it follows that $m_1 + m_2 \in \mathcal{Z}(M)$. It remains to prove that

$$(7.3) \qquad \text{ann}(m) \subseteq_e R_R \Longrightarrow \text{ann}(mr) \subseteq_e R_R \quad (\forall r \in R).$$

For this we apply the criterion for essential extensions in (3.27)(1). Given any element $s \in R \backslash \text{ann}(mr)$, we have $m(rs) \neq 0$, so from $\text{ann}(m) \subseteq_e R_R$, we see that $m(rst) = 0$ for some $t \in R$ such that $rst \neq 0$. Now we have $0 \neq st \in \text{ann}(mr)$ which yields the desired conclusion $\text{ann}(mr) \subseteq_e R_R$.

(2) For any $m \in \mathcal{Z}(M)$, $\text{ann}(m) \subseteq_e R_R$, so by Exercise (6.12)(2), $\text{ann}(m) \supseteq \text{soc}(R_R)$. This shows that $m \cdot \text{soc}(R_R) = 0$, as desired.

(3) follows from the fact that $\text{ann}(m) \subseteq \text{ann}(f(m))$ for any $m \in M$, and (4) follows directly from definition. □

(7.4) Corollary. (1) *$\mathcal{Z}(R_R)$ is an ideal in R, called the right singular ideal of R. (The left singular ideal is similarly defined to be $\mathcal{Z}({}_RR)$.)* (2) *If $R \neq 0$, then $\mathcal{Z}(R_R) \neq R$.*

Proof. (1) In view of (7.2)(1), we need only show that $m \in \mathcal{Z}(R_R)$ and $s \in R$ imply that $sm \in \mathcal{Z}(R_R)$. This is clear from the fact that $\text{ann}_r(sm) \supseteq \text{ann}_r(m)$. (2) $\text{ann}_r(1) = 0$ cannot be essential in R_R, unless $R = 0$. □

(7.5) Definition. We say that M_R is a *singular* (resp. *nonsingular*) module if $\mathcal{Z}(M) = M$ (resp. $\mathcal{Z}(M) = 0$).[55] In particular, we say that R is a *right nonsingular ring* if $\mathcal{Z}(R_R) = 0$ (or equivalently, any right ideal $\mathfrak{A} \subseteq_e R_R$ has $\text{ann}_\ell(\mathfrak{A}) = 0$). Left nonsingular rings are defined similarly, and "nonsingular ring" shall mean a ring that is both right and left nonsingular.

(7.6) Examples.

(0) *Any simple ring is nonsingular*: this follows readily from (7.4).

(1) Let R be a commutative domain. Then all nonzero ideals of R are essential. Therefore, for any M_R,

$$\mathcal{Z}(M) = \{m \in M : \text{ann}(m) \neq 0\}$$

is just the torsion submodule of M. In particular, *M is singular iff M is torsion, and M is nonsingular iff M is torsion-free.* (For a module M over an *arbitrary* ring, it is perhaps not inappropriate to think of $\mathcal{Z}(M)$ as a kind of substitute for the torsion submodule of a module over a commutative domain.)

(2) Let $M \subseteq N$ be R-modules. *If N is nonsingular, so is M, and the converse holds if $M \subseteq_e N$.* The first part follows from (7.2)(4). For the second part, assume that $\mathcal{Z}(M) = 0$ and $M \subseteq_e N$. Then, by (7.2)(4), $\mathcal{Z}(N) \cap M = \mathcal{Z}(M) = 0$ and so $\mathcal{Z}(N) = 0$. (In particular, we see that M is nonsingular iff its injective hull $E(M)$ is nonsingular.)

(3) *An R-module M_R is singular iff there exist R-modules $A \subseteq_e B$ such that $M \cong B/A$.* Indeed, suppose such A, B exist, and identify M with $\overline{B} := B/A$. For

[55]**Caution:** "Nonsingular" $\neq$ "not singular"! But of course, if a module is both singular and nonsingular, it must be zero.

any element $\overline{b} \in \overline{B}\backslash\{0\}$, we have to show that

$$(*) \qquad \operatorname{ann}(\overline{b}) = \{x \in R : bx \in A\} \subseteq_e R_R.$$

Let $y \in R\backslash\operatorname{ann}(\overline{b})$. Then $by \notin A$, so from $A \subseteq_e B$, we see that $byz \in A\backslash\{0\}$ for some $z \in R$. Therefore, $0 \neq yz \in \operatorname{ann}(\overline{b})$, proving $(*)$. We have now established the "if" part of (3). The "only if" part is left to the reader; see Exercise 2(a) below.

(4) Let R be the $\mathbb{Z}$-algebra generated by x, y with relations $yx = y^2 = 0$. We have used R in *FC*–(1.26) as an example of a left noetherian ring that is not right noetherian. We claim that *R is left nonsingular but not right nonsingular*. For the latter, let us show that $y \in \mathcal{Z}(R_R)$. Since $\operatorname{ann}_r(y) \supseteq xR + yR$, we need only show that $xR + yR \subseteq_e R_R$. Consider any nonzero element

$$\alpha := f(x) + g(x)y \in R = \mathbb{Z}[x] \oplus \mathbb{Z}[x]y.$$

If $f(x) = 0$, then

$$0 \neq \alpha \cdot 1 = g(x)y \in xR + yR.$$

If $f(x) \neq 0$, then $\alpha x = xf(x) \in xR\backslash\{0\}$. This shows that $xR + yR \subseteq_e R_R$. Now let us show that $\mathcal{Z}({}_RR) = (0)$. Say,

$$\beta = p_1(x) + p_2(x)y \in \mathcal{Z}({}_RR).$$

Then $\operatorname{ann}_\ell(\beta) \subseteq_e {}_RR$, so for the element $x \in R$, there would exist $h(x)$, $k(x) \in \mathbb{Z}[x]$ such that

$$0 \neq (h(x) + k(x)y)x = h(x)x \in \operatorname{ann}_\ell(\beta).$$

This means that $h(x)xp_i(x) = 0$, so $p_i(x) = 0$ for $i = 1, 2$, which shows that $\beta = 0$.

(5) The triangular ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$ is left and right noetherian (see *FC*–(1.22)). This implies, as we shall see in (7.15)(1), that $\mathcal{Z}(R_R)$ and $\mathcal{Z}({}_RR)$ are nil ideals, so they are contained in $\begin{pmatrix} 0 & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{pmatrix}$. For the matrix $x = \begin{pmatrix} 0 & \overline{1} \\ 0 & 0 \end{pmatrix}$, an easy computation shows that

$$\operatorname{ann}_\ell(x) = \begin{pmatrix} 2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}, \qquad \operatorname{ann}_r(x) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{pmatrix}.$$

The former is essential in ${}_RR$, so $\mathcal{Z}({}_RR) = \{0, x\}$. The latter is *not* essential in R_R, so $\mathcal{Z}(R_R) = 0$. Therefore, *R is a noetherian ring that is right nonsingular but not left nonsingular*.

(6) The ring $R = \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} \\ 0 & \mathbb{Z}/4\mathbb{Z} \end{pmatrix}$ considered in (3.45) is finite and hence left and right noetherian. We leave it to the reader to show that

$$\mathcal{Z}({}_RR) = \mathcal{Z}(R_R) = \operatorname{rad}(R) = \begin{pmatrix} 2\mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} \\ 0 & 2\mathbb{Z}/4\mathbb{Z} \end{pmatrix},$$

where $\text{rad}(R)$ is the Jacobson radical of R. In particular, R is neither left nor right nonsingular.

(7) Any semisimple ring is nonsingular. More precisely, *a ring R is semisimple iff every right R-module M_R is nonsingular.* Indeed, if R is semisimple, and $m \in \mathcal{Z}(M)$, then $\text{ann}(m) \subseteq_e R_R$ implies that $\text{ann}(m) = R$, so $m = 0$. Conversely, suppose every M_R is nonsingular. For any right ideal $\mathfrak{A} \subseteq R$, let $\mathfrak{B}$ be a complement, so that $\mathfrak{A} \oplus \mathfrak{B} \subseteq_e R_R$. Then $R/(\mathfrak{A} \oplus \mathfrak{B})$ is a singular right module (by (7.6)(3)), and so $\mathfrak{A} \oplus \mathfrak{B} = R$. Thus, every right ideal is a direct summand in R_R so R is a semisimple ring.

(8) (A generalization of (7).) *If R is a ring whose principal right ideals are all projective,*[56] *then R is right nonsingular.* In fact, for any $m \neq 0$ in R, consider the exact sequence:

$$0 \longrightarrow \text{ann}_r(m) \longrightarrow R \stackrel{f}{\longrightarrow} mR \longrightarrow 0 \quad \text{where } f(x) = mx.$$

Since $(mR)_R$ is projective, this sequence splits. Thus, $\text{ann}_r(m)$ is a proper direct summand of R_R, and cannot be essential in R_R. Hence, $\mathcal{Z}(R_R) = 0$.

(9) *There exist primitive rings that are neither left nor right nonsingular.* This is by no means easy: the first such examples were constructed by Osofsky and Lawrence. For more information in this direction, see (11.21)(4) below.

(7.7) Corollary. *Any right semihereditary ring is right nonsingular. In particular, any von Neumann regular ring is nonsingular.*

More examples of nonsingular rings are provided by the next three or four results.

(7.8) Lemma. *Let R be a reduced ring (i.e., R has no nonzero nilpotent elements). Then R is right (and also left) nonsingular.*

Proof. It suffices to show that, for any $x \in R$, $\text{ann}_r(x) \cap xR = 0$ (for this shows that $\text{ann}_r(x)$ cannot be essential in R_R if $x \neq 0$). For any $y \in \text{ann}_r(x) \cap xR$, write $y = xz$ where $z \in R$. We have $(yx)^2 = y(xy)x = 0$, so $yx = 0$. But then $y^2 = y \cdot xz = 0$ gives $y = 0$. □

There is a good analogue of (7.8) for rings with involution. Recall that an additive homomorphism $*: R \to R$ is called an *involution on R* if $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in R$.

(7.9) Lemma. *Let $(R, *)$ be a ring with involution such that $y^*y = 0 \Rightarrow y = 0$ in R. Then R is right (and also left) nonsingular.*

[56] Such a ring is known as a *right PP-ring*, or a *right Rickart ring*. For more details, see §7D.

Proof. As before, it suffices to show that, for any $x \in R$, $\text{ann}_r(x) \cap x^*R = 0$. For any $y \in \text{ann}_r(x) \cap x^*R$, write $y = x^*z$. Then

$$y^*y = (x^*z)^*y = z^*xy = 0,$$

so $y = 0$. (The argument for proving that R is left nonsingular is similar.) □

The lemma enables us to draw many examples of nonsingular rings from analysis. In fact, if R is any ring of bounded operators on a Hilbert space H such that R is closed under the adjoint, then, taking the involution on R to be the adjoint map *, the lemma implies that R is nonsingular. (If $T^*T = 0$ for $T \in R$, then $0 = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle$ implies that $Tv = 0$ for every $v \in H$.)

On the other hand, Lemma (7.9) also leads to a large class of nonsingular rings via the group ring formation.

(7.10) Proposition. *Let $(k, {}^*)$ be a ring with involution such that $\sum \alpha_i^*\alpha_i = 0 \Longrightarrow$ all $\alpha_i = 0$ in k. Then, for any group G, $R = kG$ is a nonsingular ring.*

Proof. We can extend the involution * from k to R by defining $g^* = g^{-1}$ for any $g \in G$. The hypothesis on $(k, {}^*)$ is easily seen to yield that $y^*y = 0 \Rightarrow y = 0$ in R (cf. *FC*–(6.11)). Now apply (7.9). □

From this Proposition, we see, for instance, that kG is nonsingular for any group G and any subring k of a formally real field. However, not all group rings $R = kG$ are nonsingular, even when k is a field. For instance, if $|G| < \infty$ and k is a field, we shall show (see (13.2) and the paragraph thereafter) that $\mathcal{Z}({}_RR) = \mathcal{Z}(R_R) = \text{rad } R$ (the Jacobson radical of R). Thus, if k has positive characteristic dividing $|G|$, Maschke's Theorem (as stated in *FC*–(6.1)) implies that kG is neither left nor right nonsingular.

Our next result demonstrates a large supply of rings that fail to be (left or right) nonsingular.

(7.11) Lemma. *Let x be a central nilpotent element in a ring R. Then $x \in \mathcal{Z}(R_R)$ (and by symmetry, $x \in \mathcal{Z}({}_RR)$ as well).*

Proof. To show that $\text{ann}_r(x) \subseteq_e R_R$, consider any nonzero $y \in R$. There exists a smallest $n \geq 0$ such that $x^{n+1}y = 0$. Then $x^ny \in \text{ann}_r(x)\backslash\{0\}$. Since $x^ny = yx^n$, we have shown that $\text{ann}_r(x) \subseteq_e R_R$. □

From (7.8) and (7.11), we conclude that:

(7.12) Corollary. *A commutative ring is nonsingular iff it is reduced.*

We shall try to construct an example to show that (the "only if" part of) this Corollary is not true in general for noncommutative rings. For this construction, it

is useful to know another description for the right singular ideal in a *right artinian* ring.

(7.13) Proposition. *For any ring R, let $S = \text{soc}(R_R)$ be its right socle (the sum of all minimal right ideals of R). Then $\mathcal{Z}(R_R) \subseteq \text{ann}_\ell(S)$. If R is right artinian, equality holds. If R is semiprime, $\mathcal{Z}(R_R) \cap S = 0$.*

Proof. The first conclusion follows from (7.2)(2), which gives $\mathcal{Z}(R_R) \cdot S = 0$. From this, we have $(\mathcal{Z}(R_R) \cap S)^2 = 0$. If R is semiprime, this implies $\mathcal{Z}(R_R) \cap S = 0$. Now assume R is right artinian. Then $S \subseteq_e R_R$. For, if $\mathfrak{A} \neq 0$ is any right ideal, then $\mathfrak{A}$ contains a minimal right ideal, so $\mathfrak{A} \cap S \neq 0$. Therefore, if $x \in \text{ann}_\ell(S)$, we have $\text{ann}_r(x) \supseteq S$, and hence $\text{ann}_r(x) \subseteq_e R_R$. This gives the inclusion $\text{ann}_\ell(S) \subseteq \mathcal{Z}(R_R)$, and so equality holds. □

(7.14a) Example. Let $(R, \mathfrak{m})$ be a right artinian local ring, with $\mathfrak{m}\,(= \text{rad } R) \neq 0$. *Then R is not right nonsingular.* To see this, recall that $\mathfrak{m}$ is nilpotent (*FC*–(4.12)), and note that $S := \text{soc}(R_R)$ is a nonzero ideal in $\mathfrak{m}$. Choose $n \geq 2$ such that $S^n = 0 \neq S^{n-1}$. Then

$$\mathcal{Z}(R_R) = \text{ann}_\ell(S) \supseteq S^{n-1} \neq 0.$$

For instance, if $R = k[u, v]$ with relations $u^2 = v^2 = uv = 0$, where k is a division ring, then R is local with $\mathfrak{m} = uk \oplus vk$. Since $\mathfrak{m}^2 = 0$, we have $S = \mathfrak{m}$, and $\mathcal{Z}(R_R) = \text{ann}_\ell(\mathfrak{m}) = \mathfrak{m}$ as well.

(7.14b) Example. Let R be the ring of upper triangular $n \times n$ matrices over a semisimple ring k. *We claim that R is nonsingular.* To see this, note first that the right socle $S := \text{soc}(R_R)$ is the left annihilator of rad R, the Jacobson radical of R. (see *FC*–Exercise (4.20)). Since rad R consists of all matrices of R with a zero diagonal, an easy computation shows that S is the ideal of all matrices in R with only nonzero entries on the last column. Identifying such matrices with their last columns, we may view S as k^n. In particular, it follows that $\mathcal{Z}(R_R) = \text{ann}_\ell(S) = 0$. Similarly, we can show that $\text{soc}({}_RR)$ consists of all matrices of R with only nonzero entries on the first row, and consequently that $\mathcal{Z}({}_RR) = \text{ann}_r(\text{soc}({}_RR)) = 0$. This shows that R is a nonsingular ring. However, for $n \geq 2$, R is *not* reduced; in fact, it is not even semiprime.

(7.14c) Example. Let $R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, where k is semisimple, as in (7.14b). Here, again, rad R consists of all matrices in R with a zero diagonal. Computing the left and right annihilators of rad R, we see that

$$\text{soc}(R_R) = \begin{pmatrix} 0 & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}, \quad \text{soc}({}_RR) = \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Computing the left and right annihilators of these, we get, respectively, $\mathcal{Z}(R_R) = 0$ and $\mathcal{Z}({}_R R) = 0$, so again R is a nonsingular ring.

(7.14d) Example. Here is another way by which we can deal with the rings in the two examples above. Let R be a subring of a ring T such that $R_R \subseteq_e T_R$. According to Exercise 33, if T is right nonsingular, so is R. To apply this, let $T = \mathbb{M}_n(k)$, where k is some other ring. Let R be any subring of T containing $k = k \cdot I_n$ and the matrix units $E_{1i}, \ldots, E_{ni}$ for a fixed i. It is easy to check that $R_R \subseteq_e T_R$. If we choose k to be a semisimple ring, then T is also semisimple and hence right nonsingular (by (7.6)(7)), so by what we said above, R is also right nonsingular. This applies well to the ring in (7.14b), and its left analogue applies to the same ring, as well as to the ring in (7.14c).

(7.14e) Example (Shock). *If R is a right nonsingular ring, so is the polynomial ring $R[x]$.* (More generally, see Exercise 35.)

§7B. *Nilpotency of the Right Singular Ideal*

In this subsection, we shall develop some results concerning the nilpotency of the right singular ideal of a ring R. Loosely speaking, if we impose suitable chain conditions on R, then we might expect some nilpotency property for $\mathcal{Z}(R_R)$. The chain conditions will be imposed on the "right annihilators". Recall that a right annihilator is a subset in R (necessarily a right ideal) of the form $\text{ann}_r(X)$, where X is some subset of R. In the case when X is a singleton set $\{x\}$, we speak of $\text{ann}_r(x)$ as a *right annihilator of an element.*

(7.15) Theorem. *Let R be a ring.*

(1) *Assume that R satisfies* ACC *on right annihilators of elements. Then $\mathcal{Z}(R_R)$ is a nil ideal, and $\mathcal{Z}(R_R) \subseteq \text{Nil}_*R$ (the lower nilradical of R).*
(2) (Mewborn-Winton) *Assume that R satisfies* ACC *on right annihilators. Then $\mathcal{Z}(R_R)$ is a nilpotent ideal.*

Proof. (1) Let $x \in \mathcal{Z}(R_R)$. The assumption on R implies that

(7.16) $\quad \text{ann}_r(x^m) = \text{ann}_r(x^{m+1}) = \ldots \quad$ *for some* $m \geq 1$.

We claim that $x^m = 0$. For, if otherwise, $\text{ann}_r(x^m) \cap x^m R$ would contain a nonzero element $x^m y$ ($y \in R$), with $x^m \cdot x^m y = 0$. But then

$$y \in \text{ann}_r(x^{2m}) = \text{ann}_r(x^m)$$

yields $x^m y = 0$, a contradiction. This shows that $\mathcal{Z}(R_R)$ is nil, and *FC*–(10.29) gives $\mathcal{Z}(R_R) \subseteq \text{Nil}_*R$.

(2) is proved by using similar ideas. Let $I := \mathcal{Z}(R_R)$. The assumption in (2) implies that

(7.17) $\quad \text{ann}_r(I^m) = \text{ann}_r(I^{m+1}) = \ldots \quad$ *for some* $m \geq 1$.

We claim that $I^m = 0$. Indeed, assume $I^m \neq 0$. In the family of right annihilators

$$\{\text{ann}_r(z) : z \notin \text{ann}_r(I^m)\}, \tag{7.18}$$

choose a maximal member, say, $\text{ann}_r(x)$, where $I^m x \neq 0$. Consider any $a \in I$. Since $\text{ann}_r(a) \cap xR \neq (0)$, we have $axy = 0$ for some $y \in R$ such that $xy \neq 0$. This shows that $\text{ann}_r(x) \subsetneq \text{ann}_r(ax)$, so the choice of x forces $I^m ax = 0$. Since this holds for any $a \in I$, we get

$$x \in \text{ann}_r(I^{m+1}) = \text{ann}_r(I^m),$$

a contradiction. □

Remark. Under the hypothesis of (1) above, we have in particular $\mathcal{Z}(R_R) \subseteq \text{rad}(R)$ (the Jacobson radical of R). This inclusion, however, does not hold in general, since we have pointed out in (7.6)(9) that it is possible for $\text{rad}(R)$ to be zero without $\mathcal{Z}(R_R)$ being zero.

Let us now record some consequences of (7.15).

(7.19) Corollary. *Let R be a semiprime ring (FC–§10) that satisfies* ACC *on right annihilators of elements. Then R is right nonsingular.*

Proof. This follows from (7.15)(1) since $\text{Nil}_* R = 0$ when R is semiprime. □

(7.20) Corollary. *Let R be a commutative ring that satisfies* ACC *on annihilators of elements. Then $\mathcal{Z}(R) =$ Nil R (the nilradical of R).*

Proof. This follows from (7.15)(1) and (7.11). □

If a commutative ring R does not satisfy the chain condition in (7.20), Nil $R \subseteq \mathcal{Z}(R)$ may be a strict inclusion; see Exercise 9.

The results (7.15) and (7.19) will find applications in the next Chapter where we study the theory of rings of quotients in the noncommutative setting. In that Chapter, the utility of the notion of a right nonsingular ring will become clear. For instance, we shall see that if R is right nonsingular, then the injective hull $E(R_R)$ has a natural (and unique) ring structure that is compatible with its structure as a right R-module; the ring $E(R_R)$ is the so-called "maximal right ring of quotients" of R.

§7C. Goldie Closures and the Reduced Rank

In this subsection, we shall present the theory of Goldie closures of submodules, leading up to the definition of "reduced rank" of a module. Goldie's theory, which originated from Goldie [64], has found many applications in the modern theory of noetherian rings.

For any R-module M_R, we have defined its singular submodule $\mathcal{Z}(M)$, so we can form the quotient module $M/\mathcal{Z}(M)$. It is natural to ask if this will be a nonsingular module. The following result shows that this is indeed the case, if R is a right nonsingular ring.

(7.21) Theorem. *Let R be a right nonsingular ring, and M be any right R-module. Then $\mathcal{Z}(M/\mathcal{Z}(M)) = 0$.*

Proof. Let $m \in M$ be such that $\overline{m} \in \mathcal{Z}(M/\mathcal{Z}(M))$. Then $m\mathfrak{A} \subseteq \mathcal{Z}(M)$ for some right ideal $\mathfrak{A} \subseteq_e R_R$. To show that $m \in \mathcal{Z}(M)$, we must show that $\text{ann}(m) \subseteq_e R_R$. Let $\mathfrak{B} \neq 0$ be any right ideal in R. Fixing a nonzero element $b \in \mathfrak{A} \cap \mathfrak{B}$, we have $mb \in m\mathfrak{A} \subseteq \mathcal{Z}(M)$, so $mb\mathfrak{C} = 0$ for some right ideal $\mathfrak{C} \subseteq_e R_R$. But $b\mathfrak{C} \neq 0$, for otherwise $b \in \mathcal{Z}(R_R)\backslash\{0\}$. Therefore $bc \neq 0$ for some $c \in \mathfrak{C}$, and $mbc = 0$ implies that $bc \in \text{ann}(m) \cap \mathfrak{B}$. This shows that $\text{ann}(m) \subseteq_e R_R$, as desired. □

Without the assumption that R be right nonsingular, however, the quotient module $M/\mathcal{Z}(M)$ is not necessarily nonsingular, as the following example shows. (With due apology to the reader, we shall work with *left* modules in this example.)

(7.22) Example. Let R be the triangular ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$. According to (7.6)(5), R is right nonsingular, though not left nonsingular, with $\mathcal{Z}({}_RR) = \{0, x\}$ where $x = \begin{pmatrix} 0 & \overline{1} \\ 0 & 0 \end{pmatrix}$. We claim that $R/\mathcal{Z}({}_RR)$ *is not nonsingular as a left R-module*; indeed, the image of $m = \begin{pmatrix} 0 & 0 \\ 0 & \overline{1} \end{pmatrix}$ in $R/\mathcal{Z}({}_RR)$ turns out to belong to its singular submodule. To see this, note that $\overline{m} \in R/\mathcal{Z}({}_RR)$ has annihilator $\begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{pmatrix}$, which is easily seen to be essential in ${}_RR$ (though not essential in R_R, as noted in (7.6)(5)). Thus, $0 \neq \overline{m} \in \mathcal{Z}(R/\mathcal{Z}({}_RR))$. The curious reader can further verify that, in fact, $\mathcal{Z}(R/\mathcal{Z}({}_RR)) = \{0, \overline{m}\}$.

Coming back to right modules M_R, we must therefore exercise caution in dealing with $M/\mathcal{Z}(M)$, for it may not be nonsingular. This remark suggests the idea of *iterating* the construction of singular submodules. To formalize this idea, we proceed as follows.

(7.23) Definition. Let N be any submodule of M_R. We define N^* to be the (unique) submodule of M containing N such that $N^*/N = \mathcal{Z}(M/N)$. This process can be repeated, so we can define N^{**}, N^{***}, and so forth. Of course, 0^* is just $\mathcal{Z}(M)$.

Elsewhere in the text, N^* is used to denote the R-dual of the right module N. In this section, however, we will not be concerned with the R-dual, so the notation N^* will be used only in the sense of (7.23). To facilitate working with N^*, let us

introduce one more piece of notation. For any element $y \in M$, we shall write

(7.24) $$y^{-1}N = \{r \in R : \ yr \in N\}.$$

This is a right ideal of R, which is nothing but the annihilator of the element $\bar{y} \in M/N$. (Another possible notation for $y^{-1}N$ is $(N : y)$. However, $y^{-1}N$ is the better notation, especially in the case when M is a right ideal in R.) With the notation in (7.24), we have, for instance:

(7.25) $$N^* = \{y \in M : \ y^{-1}N \subseteq_e R_R\}$$

(7.26) $$N^{**} = \{y \in M : \ y^{-1}N^* \subseteq_e R_R\}, \quad \text{etc.}$$

Using (7.25), we see immediately that:

(7.27) Lemma. *If $L \subseteq N \subseteq M$, then $L^* \subseteq N^*$. In particular, $\mathcal{Z}(M) = 0^* \subseteq N^*$.*

At first sight, the idea of forming repeated "stars" may not look promising, for apparently one has to deal with a messy chain $N^* \subseteq N^{**} \subseteq \ldots$ in M. However, the following nice result of Goldie saves the day.

(7.28) Theorem. *For any submodule $N \subseteq M$, $N^{***} = N^{**}$. In other words, although M/N^* may not be nonsingular, M/N^{**} is always nonsingular.*

We shall now develop the necessary tools to prove this wonderful theorem of Goldie. First, we need a slight generalization of the idea of an essential extension. Following Goldie, we say that two submodules $S, T \subseteq M$ are *related* (written $S \sim T$) if, for any submodule $X \subseteq M$, $X \cap S \neq 0$ iff $X \cap T \neq 0$. Clearly, "$\sim$" is an equivalence relation on the submodules of M. In the special case when $S \subseteq T$, $S \sim T$ simply boils down to $S \subseteq_e T$.

A few basic properties of the equivalence relation "$\sim$" are collected in the following.

(7.29) Proposition. *Let N, S, T be submodules of M_R. Then:*

(1) $N + 0^* \sim N^*$.
(2) $N^* \sim N^{**}$.
(3) $S \sim N \Longrightarrow S \subseteq N^*$.
(4) $S \sim T \Longrightarrow \text{u.dim}\, S = \text{u.dim}\, T$.

Proof. (1) Our job is to check that $N + 0^* \subseteq_e N^*$. Let X be a submodule of N^* such that $X \cap (N + 0^*) = 0$. For any $x \in X$ there is a right ideal $\mathfrak{A} \subseteq_e R_R$ such that $x\mathfrak{A} \subseteq N$. Then $x\mathfrak{A} \subseteq X \cap N = 0$ implies that $x \in X \cap 0^* = 0$, and hence $X = 0$.

(2) Replacing N by N^* in (1), we get

$$N^{**} \sim N^* + 0^* = N^* \quad (\text{i.e., } N^* \subseteq_e N^{**}).$$

(3) Let $s \in S$ where $S \sim N$. We must show that $s^{-1}N \subseteq_e R_R$. Consider any $r \in R \backslash s^{-1}N$. Then $sr \notin N$. Since $S \sim N$, we have $srR \cap N \neq 0$, so $srr' \in N \backslash \{0\}$ for some $r' \in R$. But then $rr' \in s^{-1}N \backslash \{0\}$. This checks that $s^{-1}N \subseteq_e R_R$.

(4) follows easily from (6.6). □

We are now ready to tackle (7.28).

Proof of (7.28). Let $N \subseteq M$ be given. Replacing N by N^* in (7.29)(2), we deduce that

$$N^{***} \sim N^{**} \sim N^*.$$

Applying (7.29)(3) with N there replaced by N^* and with $S = N^{***}$, we get $N^{***} \subseteq N^{**}$, and hence $N^{***} = N^{**}$. □

(7.30) Corollary. *For any $N \subseteq M$, $N^{**} \subseteq_c M$ (i.e., N^{**} is a complement in M).*

Proof. By (6.32), it suffices to show that N^{**} is essentially closed in M. Consider any submodule Y such that $N^{**} \subseteq_e Y \subseteq M$. Then, by (7.29)(3), $Y \sim N^{**}$ implies that $Y \subseteq N^{***} = N^{**}$, so $Y = N^{**}$. □

(7.31) Definition. For any $N \subseteq M$, we write $\mathrm{cl}(N) = N^{**}$, and call this the (Goldie) *closure* of N (in M).[57] This is a reasonable terminology since

(7.32) $$\mathrm{cl}(\mathrm{cl}(N)) = (N^{**})^{**} = N^{**} = \mathrm{cl}(N).$$

Note that in the special case when R is a right nonsingular ring, $\mathrm{cl}(N) = N^*$ by (7.21) (applied to M/N).

(7.33) Example. For the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$ and the *left* regular module ${}_RR$, we have

$$\mathcal{Z}({}_RR) = 0^* = \begin{pmatrix} 0 & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{pmatrix}, \quad \text{and } \mathrm{cl}(0) = 0^{**} = \begin{pmatrix} 0 & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$$

by the calculations in (7.22).

We are now in a position to define the notion of the reduced rank of a module, due to A. Goldie.

(7.34) Definition. For any module M_R, define the (Goldie) *reduced rank* of M by $\mathrm{rank}(M) = \mathrm{u.dim}\ M/0^{**}$. Note that, since $0^{**} \subseteq_c M$ and $0^* \sim 0^{**}$, (6.35) and

[57] This is to be distinguished from an "essential closure" of N. Note that although N^{**} is essentially closed, it need not be essential over N. Thus, the (Goldie) closure need not be an essential closure, and conversely, an essential closure need not be a (Goldie) closure. In the rest of this book, "closure" will always mean Goldie closure, unless otherwise specified.

(7.29)(4) imply that

$$\text{(7.35)} \qquad \begin{aligned} \text{u.dim } M &= \text{rank}(M) + \text{u.dim } 0^{**} \\ &= \text{rank}(M) + \text{u.dim } 0^{*}. \end{aligned}$$

In particular, we have always $\text{rank}(M) \leq \text{u.dim } M$, which perhaps explains why $\text{rank}(M)$ is called the reduced rank of M.

(7.36) Examples.

(a) For the ring R in (7.33), the reduced rank of the left regular module ${}_RR$ is 1, since ${}_RR/0^{**} \cong \mathbb{Z}$ with the left R-action given via the natural projection from R to $\mathbb{Z}$. On the other hand, since $0^* \cong \mathbb{Z}/2\mathbb{Z}$, u.dim $0^* = 1$ and so u.dim $({}_RR) = 1 + 1 = 2$ by (7.35).

(b) Let R be any commutative domain with quotient field K. Then, for any M_R, 0^* is just the torsion submodule $t(M)$ of M, and $0^{**} = t(M)$ as well. By Exercise (6.10),

$$\text{(7.37)} \qquad \text{rank}(M) := \text{u.dim } M/t(M) = \dim_K(M \otimes_R K).$$

Thus, the reduced rank of M boils down to what is usually called the "torsion-free rank" of M. After we develop the basic theory of rings of quotients in Chapter 4, we shall see that (7.37) has a nice generalization to right modules over semiprime right Goldie rings; see (11.15)(B).

As we have pointed out before, one serious drawback in the theory of uniform dimensions is the lack of additivity of this invariant over short exact sequences. Since the reduced rank is the correct generalization of the torsion-free rank for modules over commutative domains, we might hope that this new invariant has a better behavior with respect to short exact sequences of modules. Indeed, the reduced rank function enjoys the full additivity property, as the following fundamental result of Goldie shows.

(7.38) Theorem. *For any right R-modules $N \subseteq M$, we have*

$$\text{rank}(M) = \text{rank}(N) + \text{rank}(M/N),$$

with the usual conventions on the symbol ∞.

There are several proofs of this theorem in the literature. However, all these proofs seem to rely on the implicit assumption that the uniform dimensions of all modules involved are finite. In order to prove (7.38) in general, we have to recast some of these earlier arguments. As a preliminary step, we prove the following lemma.

(7.39) Lemma. *Let $C \subseteq_c M$ and let S, T be R-submodules of M containing C. Then*

$$S \sim T \text{ in } M \implies S/C \sim T/C \text{ in } M/C.$$

Proof. Consider any nonzero submodule $X/C \subseteq S/C$. Since $C \subseteq_c M$, we have also $C \subseteq_c X$ by (6.24)(1). Suppose C is a complement to a submodule Y in X. Then $Y \neq 0$ since $X \neq C$. Pick a nonzero element $y \in Y \cap T$ (which exists since $S \sim T$). Then, the image of y in M/C is nonzero and lies in $(X/C) \cap (T/C)$. This shows that $(X/C) \cap (T/C) \neq 0$. By symmetry, we see that $S/C \sim T/C$ in M/C. □

Proof of (7.38). Let $P = \text{cl}(N) = N^{**}$. Then P/N is the closure of (0) in M/N, so by definition

$$\text{(7.40)} \qquad \text{rank}(M/N) = \text{u.dim}\left(\frac{M/N}{P/N}\right) = \text{u.dim}(M/P).$$

On the other hand, using the fact that $N \cap \mathcal{Z}(M) = \mathcal{Z}(N)$ for *any* submodule $N \subseteq M$, we see easily that $N \cap 0^{**}$ is the same as the closure of (0) in N. Therefore,

$$\text{(7.41)} \qquad \begin{aligned} \text{rank}(N) &= \text{u.dim}(N/N \cap 0^{**}) \\ &= \text{u.dim}\,[(N + 0^{**})/0^{**}]. \end{aligned}$$

By (1) and (2) of (7.29), we have $N + 0^* \sim N^{**} = P$. Since $N + 0^{**}$ is between $N + 0^*$ and P, it follows that $N + 0^{**} \sim P$. Now $0^{**} \subseteq_c M$ by (7.30), so we have

$$(N + 0^{**})/0^{**} \sim P/0^{**} \quad \text{in} \quad M/0^{**}$$

by (7.39). Therefore, (7.41) and (7.29)(4) give

$$\text{(7.42)} \qquad \text{rank}(N) = \text{u.dim}(P/0^{**}).$$

Adding (7.40) and (7.42), we get

$$\text{(7.43)} \qquad \text{rank}(N) + \text{rank}(M/N) = \text{u.dim}(P/0^{**}) + \text{u.dim}(M/P).$$

But $P \subseteq_c M$ by (7.30), so $P/0^{**} \subseteq_c M/0^{**}$ by (6.28)(1). From the exact sequence

$$0 \longrightarrow P/0^{**} \longrightarrow M/0^{**} \longrightarrow M/P \longrightarrow 0$$

and (6.35), we see that the RHS of (7.43) is u.dim$(M/0^{**}) = \text{rank}(M)$, as desired. □

The proof above is slightly longer than the proofs that have appeared in the literature. The point is that we have to work harder in order to use only addition, but not subtraction, in the above proof. (Subtraction is troublesome, as we can see from the fact that $m + \infty = n + \infty$ does not imply $m = n$!)

For various applications, it may be necessary to modify the reduced rank invariant some more. For instance, if M is a uniform module, (7.35) implies that rank$(M) \in \{0, 1\}$, which may not be very desirable. For right noetherian rings R with prime radical $\mathfrak{A}$, it turns out to be useful to define another rank function on R-modules M_R, by using the reduced ranks of certain associated $R/\mathfrak{A}$-modules. We shall come back to this matter in §12 when the need for such a modified rank function arises.

To conclude this section, let us make some useful remarks about *nonsingular modules*. If $N \subseteq M$ where M is a general module, then N need not be essential in N^*. For instance, (0) is not essential in 0^*, unless M is nonsingular. In case M is nonsingular, the star operation behaves in a much nicer way. We summarize the relevant facts in the proposition below. Note in particular that, in a nonsingular module, the notions of Goldie closure and essential closure coincide.

(7.44) Proposition. *Let M be a nonsingular right R-module, and let $N \subseteq M$. Then:*

(1) $N \subseteq_e N^*$.
(2) *N^* is the largest submodule $\sim N$.*
(3) *N^* is the smallest essentially closed submodule of M containing N. (In particular, N^* is the unique essential closure of N in M.)*
(4) $N^{**} = N^*$.
(5) *$N = N^*$ iff $N \subseteq_c M$.*
(6) *If $N_i \subseteq_c M$ $(i \in I)$ then $\bigcap_{i \in I} N_i \subseteq_c M$.*

Proof. Since $0^* = 0$, (7.29)(1) gives $N \sim N^*$, proving (1). If $S \sim N$, (7.29)(3) gives $S \subseteq N^*$, proving (2). From (7.29)(2), we have $N^{**} \sim N^* \sim N$ so (2) shows that $N^{**} = N^*$, proving (4). To prove (5), first assume $N = N^*$. Then $N = N^{**} \subseteq_c M$ by (7.30). Conversely, if $N \subseteq_c M$, then (1) implies that $N^* = N$. To prove (3), note that $N^* = N^{**} \subseteq_c M$. Moreover, if $N \subseteq X \subseteq_c M$, then $N^* \subseteq X^* = X$ by (5). For (6), let $N := \bigcap N_i$. We have an injection $M/N \to \bigoplus M/N_i$. Since $N_i^* = N_i$ by (5), each M/N_i is nonsingular. By Exercise 11, $\bigoplus M/N_i$ is nonsingular, and by (7.2)(4), so is M/N. This means that $N = N^*$, so by (5) again, $N \subseteq_c M$. □

Dropping the word "essentially" for the moment, we see from (7.44) that, in a *nonsingular* module M, the Goldie closure operator $N \mapsto \text{cl}(N) = N^{**} = N^*$ assigns to N the smallest closed submodule containing N. The family of closed submodules is closed under intersections by (6) above. This suggests that one can start doing some topology in the module M, although we will not pursue this matter here. If M is not a nonsingular module, however, we knew that even a finite intersection of closed submodules need not be closed; see (6.27)(b). In this case, for $N \subseteq M$, $\text{cl}(N)$ is closed and is equal to $\text{cl}(\text{cl}(N))$; it still contains all submodules $\sim N$ by (7.29)(3), and is essential over N^* (by the first step in the proof of (7.28)), although it may not be essential over N. The description of $\text{cl}(N)$ is not nearly as nice as in the nonsingular case.

We close this subsection with the following well-known result of R. E. Johnson.

(7.44)′ Corollary. *Let $E = E(M)$ where M is a nonsingular right R-module. Then there is a one-one correspondence between the closed submodules of M and those of E.*

Proof. By (7.6)(2), E is also a nonsingular module. To any closed submodule $N \subseteq_c M$, we associate the closed submodule $N^* \subseteq_c E$ (where N^* is formed with respect to E). Since $N^* \cap M = N$, we get a one-one map from closed submodules of M to those of E. Applying (6.32), we see that this map is also onto (with inverse given by contraction). □

For a fuller understanding of (7.44)′, we should recall that closed submodules of E are just direct summands in E; hence, for $N \subseteq M$ (closed or not), N^* is a copy of the injective hull $E(N)$ of N.

§7D. Baer Rings and Rickart Rings

In this subsection, we give a quick introduction to two classes of rings, namely, Baer rings and Rickart rings. There are two reasons why we are interested in these rings. First, they form natural subclasses of the class of nonsingular rings. Second, and more importantly, they play a special role in the theory of rings of operators in functional analysis. Because of limitation of space, however, we shall treat Baer rings and Rickart rings primarily as interesting examples, rather than as main objects for a theoretical study. For a full development of a theory of these rings in the proper context of operator algebras, the reader should consult Kaplansky [68] and Berberian [72].

(7.45) Definition. A ring R is called a *right Baer ring* if every right annihilator in R is of the form eR for some idempotent $e \in R$; R is called a *right Rickart ring* if the right annihilator of any element in R is of the form eR for some idempotent $e \in R$. (Clearly, a right Baer ring is always a right Rickart ring.) Left Baer (resp. Rickart) rings are defined similarly.

For the first notion introduced above, it turns out that there is actually no difference between "left" and "right". We prove this below.

(7.46) Proposition. *A ring R is right Baer iff it is left Baer.*

Proof. Say R is right Baer, and consider a left annihilator $\text{ann}_\ell(S)$, where S is a subset of R. By assumption, $\text{ann}_r(\text{ann}_\ell\, S) = eR$ for some $e = e^2 \in R$. We have then

$$\text{ann}_\ell\, S = \text{ann}_\ell(\text{ann}_r(\text{ann}_\ell\, S)) = \text{ann}_\ell(eR) = R \cdot (1 - e).$$

This checks that R is left Baer, and the converse follows from left-right symmetry. □

Because of (7.46), we are at liberty to drop the adjectives "left", "right" when we talk about Baer rings. As we shall see later, however, the situation is different for left/right Rickart rings.

(7.47) Examples.

(1) Clearly, *any domain is Baer*. Any semisimple ring is also Baer.

(2) *Let V_k be any semisimple right module over a ring k. Then R =End(V_k) is always a Baer ring.* In fact, consider any right annihilator $\text{ann}_r(S)$, where $S \subseteq R$. Let $U = \bigcap_{s \in S} \ker(s)$, and write $V = U \oplus U'$ as k-modules. Define $e = e^2 \in R$ by $e \,|\, U' = 0$ and $e \,|\, U = 1_U$. For any $s \in S$, $(se)(V) = s(U) = 0$. And if $S \cdot f = 0$, then $f(V) \subseteq U$ and so $f = ef$. Therefore, $\text{ann}_r S = eR$, and R is Baer. In particular, if V is a right vector space over a division ring k, End(V_k) is always a Baer ring. This fact was first observed by R. Baer in his book [52] on linear algebra and projective geometry. The term "Baer ring" was coined by I. Kaplansky in his writings on rings of operators.

(3) The same proof as in (2) (with minor modifications) shows that *the ring of bounded operators on a Hilbert space is a Baer ring.* (See also (7.57) below.)

(4) Let $R = \prod_{i \in I} R_i$, where each R_i is a Baer ring. A routine check shows that R is a Baer ring. Thus, for instance, any direct product of domains is a Baer ring. Taking each domain to be $\mathbb{Z}/2\mathbb{Z}$, we see that the Boolean ring of all subsets of a given set is a commutative Baer ring.

We now turn our attention to right Rickart rings. First we give an alternative characterization.

(7.48) Proposition. *A ring R is right Rickart iff every principal right ideal in R is projective (as a right R-module).*

Proof. This follows basically from Exercise (2.2). For the sake of completeness, we include the proof. Assume R is right Rickart. For any principal right ideal aR, we have the exact sequence

$$0 \longrightarrow \text{ann}_r(a) \longrightarrow R \stackrel{f}{\longrightarrow} aR \longrightarrow 0, \tag{7.49}$$

where $f(x) = ax$ for any $x \in R$. Since R is right Rickart, $\text{ann}_r(a) = eR$ for some $e = e^2 \in R$, so the above sequence splits. This implies that aR is projective. Conversely, if aR is projective, the sequence must split, and this implies that $\text{ann}_r(a) = eR$ for some $e = e^2 \in R$. $\square$

Because of this new characterization, right Rickart rings are often called *right PP-rings* ("principal $\Rightarrow$ projective"). As an application of (7.48), we see that *right semihereditary rings are right Rickart.* Recall that, in §2F, we have presented an example, due to Chase, of a ring T that is left semihereditary, but has a principal non-projective right ideal. Therefore, T *gives an example of a left Rickart ring that is not right Rickart.*

For later reference, we record the following chart of basic implications. It is not hard to verify that all implications are irreversible.

(7.50)
$$\begin{array}{ccccccc} & & & & (\text{ Baer }) & \searrow & \\ & & & & \Downarrow & & \\ \begin{pmatrix}\text{von Neumann}\\ \text{regular}\end{pmatrix} & \Longrightarrow & \begin{pmatrix}\text{right}\\ \text{semiherditary}\end{pmatrix} & \Longrightarrow & \begin{pmatrix}\text{right}\\ \text{Rickart}\end{pmatrix} & \Longrightarrow & \begin{pmatrix}\text{right}\\ \text{nonsingular}\end{pmatrix}\end{array}$$

Here, the last horizontal implication follows from (7.6)(8).

Remarkably, if we impose a suitable property on the ring, then all conditions in this chart become equivalent! To see this, let us first make the following general observation on right nonsingular rings.

(7.51) Lemma. *For any set S in a right nonsingular ring R, the right annihilator $A = \text{ann}_r(S)$ is (essentially) closed in R_R.*

Proof. Consider any right ideal B such that $A \subseteq_e B$, and let $b \in B$. By (7.6)(3), $C := \{x \in R : bx \in A\}$ is essential in R_R. From $bC \subseteq A$, we have $SbC = 0$. This implies that $Sb \subseteq \mathcal{Z}(R_R) = 0$, so $b \in A$. We have therefore shown that $A = B$. □

(7.51)′ Corollary. *Let R be any right nonsingular ring with* u. dim $R_R = n < \infty$. *Then any chain of right (resp. left) annihilators in R has length $\leq n$. In particular, R satisfies both* ACC *and* DCC *on right (resp. left) annihilators.*

Proof. This follows from the Lemma in view of (6.31) and the "duality" between left and right annihilators (as expressed in the chart (6.57)). □

Since a right Rickart ring R is always right nonsingular, the lemma applies, in particular, to R. Thus, any right annihilator in R is always a complement, though not necessarily a *direct complement*, of R_R (as would be the case for a Baer ring).

(7.52) Theorem. *Let R be any right self-injective ring. Then:*

(1) *The five notions listed in the chart* (7.50) *are all equivalent.*
(2) *If R is right hereditary, then it must be semisimple.*

Proof. (1) In view of (7.50), it is enough to show that, if R is right nonsingular, then R is Baer, and also von Neumann regular. Let $A = \text{ann}_r(S)$ $(S \subseteq R)$ be any right annihilator. By Lemma (7.51), $A \subseteq_c R_R$. Since R_R is injective, (6.32) implies that A is a direct summand. Hence $A = eR$ for some $e = e^2 \in R$, so R is Baer. Now consider any element $a \in R$. Since R is right Rickart, aR is projective, so the sequence (7.49) splits. This implies that aR is also injective (as R_R is). Therefore, aR is a direct summand of R_R. This checks that R is von Neumann regular (see *FC*–(4.23)).

(2) Assume R is right hereditary. Since R_R is injective, (3.22) implies that any quotient module of R_R is also injective. This means that any cyclic right R-module is injective, so by (6.47) R is a semisimple ring. □

The following is just a part of (1) of the theorem we just proved. We state it explicitly for the reason that von Neumann regular rings and Baer rings are both important classes of rings arising in the theory of operator algebras.

(7.53) Corollary. *Any right (or left) self-injective von Neumann regular ring is Baer.*

Readers with a good memory will recall that there exist von Neumann regular rings which are left self-injective but not right self-injective (see (3.74B)).

The fact that right self-injective nonsingular rings are von Neumann regular will be further generalized later in Chapter 5; see (13.2). In that chapter, we shall see that these rings arise naturally as maximal (right) rings of quotients. In fact, for any right nonsingular ring R, the maximal right ring of quotients of R will be a right self-injective von Neumann regular (Baer) ring; see (13.36).

(7.54) Example. Of course, a Baer ring (e.g., a domain) need not be von Neumann regular. *We construct here a ring R that is von Neumann regular, but not Baer.* By (7.53), R is necessarily neither right self-injective nor left self-injective. Also, in view of the implication chart (7.50), *R will provide an example of a right Rickart ring that is not Baer.* Let F be a field, and $A = F \times F \times \cdots$. This ring is commutative, von Neumann regular, and is a Baer ring by (7.47)(4). Now let R be the subring of A consisting of "sequences" $(a_1, a_2, \ldots) \in A$ that are eventually constant. For any $(a_1, a_2, \ldots) \in R$, define $x = (x_1, x_2, \ldots)$ by: $x_n = a_n^{-1}$ if $a_n \neq 0$, and $x_n = 0$ if $a_n = 0$. Then $x \in R$ and $a = axa$. Therefore, R is von Neumann regular. Let $e_i \in R$ denote the ith "unit vector" $(0, \ldots, 1, 0, \ldots)$, and let $S = \{e_1, e_3, e_5, \ldots\}$. Then $\text{ann}^R(S)$ consists of sequences $a = (a_1, a_2, \ldots)$ which are eventually zero, and such that $a_n = 0$ for n odd. Clearly, $\text{ann}^R(S)$ cannot be a f.g. ideal of R since, for any f.g. ideal $\mathfrak{A} \subseteq \text{ann}^R(S)$, there exists a big integer N such that $(a_1, a_2, \ldots) \in \mathfrak{A}$ implies that $a_n = 0$ for $n \geq N$. In particular, R is *not* a Baer ring.

Note that in the above example the ring R has an infinite orthogonal set of nonzero idempotents, namely, $\{e_1, e_2, e_3, \ldots\}$. This turns out to be a feature common to all rings which are right Rickart but not Baer, as the following result of L. Small shows.

(7.55) Theorem. *Let R be a ring that has no infinite orthogonal set of nonzero idempotents. Then the following are equivalent:*

(1) *R is Baer.*
(2) *R is right Rickart.*

(3) *R is left Rickart.*

If any of these conditions holds, then R satisfies ACC *and* DCC *on left (resp. right) annihilators.*

The proof of this result will be preceded by a lemma.

(7.56) Lemma. *Let R be a right Rickart ring. Then any nonzero left annihilator L contains a nonzero idempotent.*

Proof. Let $S = \text{ann}_r(L)$ so that $L = \text{ann}_\ell(S)$. Fix a nonzero element $a \in L$. Since R is right Rickart, $\text{ann}_r(a) = eR$ for some $e = e^2 \neq 1$ in R. But $S \subseteq \text{ann}_r(a) = eR$ implies that $(1-e)S \subseteq (1-e)eR = 0$, so $L = \text{ann}_\ell(S)$ contains the nonzero idempotent $1-e$. □

Proof of (7.55). It suffices to prove (2) ⇒ (1), so assume R is right Rickart. To show that R is (left) Baer, consider any nonzero left annihilator $L = \text{ann}_\ell(S)$, where S is any subset of R. According to (6.59), the hypothesis on R amounts to the fact that direct summands of ${}_RR$ satisfy the DCC. Among all nonzero idempotents in L (which exist by the lemma), choose e with $R(1-e) = \text{ann}_\ell(e)$ minimal. *We claim that $L \cap \text{ann}_\ell(e) = 0$.* Indeed, if

$$0 \neq L \cap \text{ann}_\ell(e) = \text{ann}_\ell(S \cup \{e\}),$$

there would exist a nonzero idempotent f in this left annihilator. Since $fe = 0$, $e' := e + (1-e)f$ is an idempotent in L. Also, $e'e = e$ implies that $e' \neq 0$ and $\text{ann}_\ell(e') \subseteq \text{ann}_\ell(e)$. This inclusion is proper since $fe = 0$ but

$$fe' = f(1-e)f = f \neq 0.$$

This contradicts the choice of e, so we have proved our claim. Thus, for any $x \in L$,

$$x - xe \in L \cap \text{ann}_\ell(e) = 0,$$

and hence $x = xe$. This shows that $L = Re$, so R is (left) Baer. With this conclusion, the last statement in (7.55) now follows from (6.59). □

In closing, let us make some remarks about terminology. The ring-theoretic notions that are really needed in operator theory are not quite Baer rings and left/right Rickart rings, but rather their "$*$-analogues". Let $(R, *)$ be a ring with an involution $*$. An idempotent $e \in R$ with the property that $e^* = e$ is called a *projection*. By definition, $(R, *)$ is a *Baer $*$-ring* if every right annihilator $\text{ann}_r(S)$ $(S \subseteq R)$ has the form eR where e is a projection, and $(R, *)$ is a *Rickart $*$-ring* if every $\text{ann}_r(s)$ $(s \in R)$ has the form eR where e is a projection. Of course, any Rickart $*$-ring is a (right) Rickart ring, and any Baer $*$-ring is a Baer ring.

Note that in the $*$ setting, there is an additional "built-in" good feature. Since R is isomorphic to its opposite ring, there is no need to distinguish "left" from

"right". More explicitly, if $(R, *)$ is a Rickart $*$-ring as defined above, then, for any $t \in R$, $\text{ann}_r(t^*) = eR$ for some projection e and hence

$$\text{ann}_\ell(t) = (\text{ann}_r(t^*))^* = (eR)^* = R^*e^* = Re.$$

The left-right symmetry for Baer $*$-rings is derived similarly! This contrasts with the "non $*$ " case, where "Baer" is left-right symmetric, but "Rickart" is not. In the special case when idempotents in R happen to be all central, a theorem of Endo guarantees that, if R is right Rickart and $\text{ann}_r(t) = eR$ for an idempotent e, then $\text{ann}_\ell(t) = Re$; in particular, R is also left Rickart (cf. Exercise 21). This is as close as one can come to what happens in the $*$ case without assuming the presence of an involution.

As we mentioned above, the motivation for studying Baer $*$-rings and Rickart $*$-rings comes from functional analysis, or more specifically, from the theory of operator algebras on a Hilbert space. In this context, of course, $*$ is given by the adjoint. The major example to keep in mind is the following.

(7.57) Example. *Let R be the ring of all bounded linear operators on a Hilbert space V, and let $*$ be the adjoint involution on R. Then $(R, *)$ is a Baer $*$-ring.* This is checked by using the same argument as that given in Example (7.47)(2). We just replace the V there by a Hilbert space, and work with the ring R of bounded linear operators on V instead. Keeping the notations there, we can define $e \in R$ to be the projection operator on the closed subspace U (taking U' to be the *orthogonal complement* of U). Then $\text{ann}_r(S) = eR$ and $e^2 = e = e^*$, so we have checked that $(R, *)$ is a Baer $*$-ring.

By definition, a *von Neumann algebra* (on the Hilbert space V) is a $*$-invariant subalgebra A of the algebra R above which is closed with respect to the weak operator topology. (An equivalent condition, according to a theorem of von Neumann, is that A has the "double-commutant" property: $A'' = A$, where A' denotes the commutant of A in R.) Example (7.57) can be generalized easily: *Any von Neumann algebra in the above sense is a Baer $*$-ring.*

In operator theory, a C^*-algebra that is a Rickart $*$-ring is called a *Rickart C^*-algebra*, and a C^*-algebra that is a Baer $*$-ring is called a Baer C^*-algebra (or an AW^*-algebra following I. Kaplansky). Generally speaking, the introduction of these classes of C^*-algebras provides an algebraic framework in which to study the theory of von Neumann algebras of operators on Hilbert spaces. For a thorough treatment of these topics, see S. K. Berberian's book [72].

§7E. Applications to Hereditary and Semihereditary Rings

In this subsection, we shall present some applications of the material of §6 and the earlier subsections of §7 to hereditary and semihereditary rings. Other applications of the notion of right nonsingular rings will be reserved for the next chapter on rings of quotients.

The first application below is due to F. L. Sandomierski. It says, in short, that for right hereditary rings, some of the finiteness conditions in §6E coincide. This result does not involve right nonsingular rings in its statement, but its proof relies on a certain property of such rings, and on the notion of singular modules as well.

(7.58) Theorem. *Let R be a right hereditary ring. Then R is right noetherian if (and only if)* $\text{u.dim } R_R < \infty$.

The "only if" part here is clear from (6.7)(1), so the thrust of the theorem is in its "if" part. Even in the case when R is a domain, (7.58) is a highly interesting result. In (10.22), we shall see that, for any domain R:

$$\text{u.dim } R_R < \infty \iff \text{u.dim } R_R = 1 \iff R \text{ is right Ore.}$$

Therefore, (7.58) implies that *any right hereditary Ore domain is right noetherian.* In the commutative case, this boils down to the fact that any hereditary domain (= Dedekind domain) is noetherian: we knew this earlier, as a consequence of (2.17).

To prove (7.58), we need a definition: we say that a module M (over any ring R) is *essentially f.g.* if there exists a f.g. submodule $N \subseteq_e M$. The following lemma shows that this notion is closely related to the finiteness of uniform dimension.

(7.59) Lemma. *A module M_R has* $\text{u.dim } M < \infty$ *iff every submodule of M is essentially f.g.*

Proof. First assume u.dim $M < \infty$. Then, for any submodule $N \subseteq M$, we have also $n := \text{u.dim } N < \infty$, so there exist uniform submodules $U_i \subseteq N$ such that

$$U_1 \oplus \cdots \oplus U_n \subseteq_e N.$$

Fixing a nonzero element $u_i \in U_i$ $(1 \leq i \leq n)$, we have then

$$\sum u_i R \subseteq_e \bigoplus U_i \subseteq_e N,$$

so by (3.27)(2), N is essentially f.g.

Next, assume u.dim $M = \infty$. By (6.4), M contains some $N = M_1 \oplus M_2 \oplus \cdots$ with $M_i \neq 0$. Any f.g. $N' \subseteq N$ is contained in some $M_1 \oplus \cdots \oplus M_r$, so $N' \cap M_{r+1} = 0$. This shows that N' is not essential in N, so N fails to be essentially f.g. □

The proof of (7.58) is based on an interesting property of right nonsingular rings given in the Proposition below.

(7.60) Proposition. *Let R be a right nonsingular ring, and M_R be a projective R-module. Then M is f.g. iff it is essentially f.g. (In particular,* $\text{u.dim } M < \infty \Longrightarrow M$ *is f.g.)*

Proof. ("If" part) Say $N = b_1 R + \cdots + b_n R \subseteq_e M$. We claim that:

(7.61) *For any* $f \in \text{Hom}_R(M, R)$, $f(N) = 0 \Longrightarrow f = 0$.

Indeed, if $f(N) = 0$, then f induces a functional $\overline{f} : M/N \to R$. Since $N \subseteq_e M$, we have $\mathcal{Z}(M/N) = M/N$ by (7.6)(3), and hence (7.2)(3) implies

$$\overline{f}(M/N) = \overline{f}(\mathcal{Z}(M/N)) \subseteq \mathcal{Z}(R_R) = 0.$$

Therefore, $\overline{f} = 0$, proving (7.61). Continuing our proof, let $a_i \in M$, $f_i \in \mathrm{Hom}_R(M, R)$ $(i \in I)$ be as in the Dual Basis Lemma (2.9). For each b_j, $f_i(b_j) = 0$ for almost all i, so there exists a *finite* subset $I_0 \subseteq I$ such that

$$i \notin I_0 \Longrightarrow f_i(b_j) = 0 \quad (\forall j,\ 1 \le j \le n).$$

Now the RHS implies that $f_i(N) = 0$, and hence $f_i = 0$ by (7.61). It follows immediately that $M = \sum_{i \in I_0} a_i R$. □

Proof of (7.58). If R is right nonsingular with u.dim$(R_R) < \infty$, it follows from (7.60) that any projective right ideal of R is f.g. Now let R be any right hereditary ring with u.dim$(R_R) < \infty$. Then R is right nonsingular by (7.7), so we conclude that any right ideal is f.g.; i.e., R is right noetherian. □

Next we shall give some applications of the results in §7C to hereditary and semihereditary rings, due to L. Small. We first state the following result.

(7.62) Theorem. *For any fixed integer $n \ge 1$, a ring R is right (semi) hereditary iff the matrix ring $S = \mathbb{M}_n(R)$ is.*

This result is best proved by using the fact that the right module categories over R and S are "naturally equivalent". This is the beginning point of the Morita Theory of category equivalences which will be presented in Chapter 7. Therefore, instead of giving an ad hoc proof for (7.62) here, we shall postpone its proof altogether, until Morita theory is developed. (See (18.6) below.)

Recall that a right Rickart ring may be characterized by the "right PP" property; that is, R is right Rickart iff every principal right ideal in R is projective (see (7.48)). In particular, a right semihereditary ring is always right Rickart. While the converse of this is not true in general, we do have the following interesting result.

(7.63) Proposition. *A ring R is right semihereditary iff, for every $n \ge 1$, $\mathbb{M}_n(R)$ is right Rickart.*

Proof. The "only if" part is clear from (7.62) and the remarks preceding the Proposition. Conversely, assume $\mathbb{M}_n(R)$ is right Rickart for every $n \ge 1$. Let $I = a_1R + \cdots + a_nR$ be any f.g. right ideal in R. Denote by A the matrix in $S := \mathbb{M}_n(R)$ with first row $(a_1, \ldots, a_n)$, and zeros elsewhere. By assumption,

$$A \cdot S = \begin{pmatrix} I & I & \ldots & I \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

is a projective right S-module. On the other hand, with R embedded in S in the usual way, S is a free right R-module, generated by the n^2 matrix units. Therefore, $A \cdot S$ is also projective as a right R-module. Since $(A \cdot S)_R \cong n \cdot I_R$, it follows that I_R is projective, so R is right semihereditary. $\square$

(7.64) Theorem (Small). *Assume that* $\text{u.dim } R_R < \infty$ *or* $\text{u.dim }_R R < \infty$. *Then the following are equivalent:*

(1) *R is right semihereditary.*
(2) *R is left semihereditary.*

If one of these holds, then $\mathbb{M}_n(R)$ is a semihereditary Baer ring for every $n \geq 1$.

Proof. By symmetry, it is enough to work with the case where u.dim $R_R < \infty$. Let $S = \mathbb{M}_n(R)$. By (6.62),

$$\text{u.dim } S_S = n \cdot \text{u.dim } R_R < \infty,$$

so by (6.59), S has no infinite orthogonal set of nonzero idempotents. Assume (1). By (7.63), S is right Rickart, so by (7.55), S is Baer (for every $n \geq 1$). Applying now the *left* analogue of (7.63), we deduce (2). It now follows from (7.62) that $S = \mathbb{M}_n(R)$ is *(right and left)* semihereditary. The reverse implication (2) $\Longrightarrow$ (1) is proved similarly. $\square$

As a consequence of (7.64), we have the following result which was promised earlier in §2F.

(7.65) Corollary. *A one-sided noetherian ring is right semihereditary iff it is left semihereditary.*

(7.66) Corollary. *Let R be either a* PRID, *or a Prüfer domain (i.e., a commutative semihereditary domain). Then $S = \mathbb{M}_n(R)$ is a semihereditary Baer ring for every $n \geq 1$.*

Proof. To make sure that (7.64) applies, we need only check that u.dim $R_R < \infty$. In the latter case, we have clearly u.dim $R_R = 1$. In the former case, R is right noetherian, so we also have u.dim $R_R < \infty$. (Actually, u.dim $R_R = 1$ in the former case too, but we don't need this information for the proof here.) $\square$

(7.67) Remark. If R is a commutative domain, a sharper statement is possible: *if $n > 1$, $\mathbb{M}_n(R)$ is a Baer ring iff R is a Prüfer domain.* This is a theorem of Wolfson and Yohe; we shall not prove the harder "only if" part here.

Exercises for §7

1. Compute $\mathcal{Z}(R_R)$ for the ring R in Example (7.6)(4).

2. (a) Show that an R-module S is singular iff there exist two R-modules $N \subseteq_e M$ such that $S \cong M/N$. (The "if" part here is already proved in (7.6)(3).)
(b) Let $N \subseteq M$ be two R-modules, where M is R-free. Show that M/N is singular iff $N \subseteq_e M$.

3. For any submodule N in a nonsingular module M, show that M/N is singular iff $N \subseteq_e M$.

4. Show that an R-module M is nonsingular iff, for any singular R-module S, $\text{Hom}_R(S, M) = 0$.

5. Let $N \subseteq M$ be R-modules.

   (a) If N and M/N are both nonsingular, show that M is also nonsingular.
   (b) Does this statement remain true if we replace the word "nonsingular" throughout by "singular"?

6. Let $I \subseteq R$ be any left ideal.

   (a) For any $n \geq 1$, show that $\text{ann}_\ell(I) \subseteq_e R_R$ iff $\text{ann}_\ell(I^n) \subseteq_e R_R$.
   (b) If I is nilpotent, show that $\text{ann}_\ell(I) \subseteq_e R_R$.

7. Let R be a ring for which every ideal right essential in R contains a non left-0-divisor. Show that R must be semiprime.

8. (a) For any central element $x \in R$ and any $n \geq 1$, show that $\text{ann}_\ell(x) \subseteq_e R_R$ iff $\text{ann}_\ell(x^n) \subseteq_e R_R$.
(b) Use (a) to show that the center of a right nonsingular ring is reduced.
(c) Use (a) to show that, for any commutative ring R, $R/\mathcal{Z}(R)$ is a nonsingular ring.

9. Show that, for any commutative ring R, $\text{Nil}(R) \subseteq_e \mathcal{Z}(R)$. Give an example of a commutative ring R for which this inclusion is *not* an equality.

10. Show that a commutative semihereditary ring must be reduced.

11. Show that, for R-modules M_i $(i \in I)$, $\mathcal{Z}(\bigoplus_i M_i) = \bigoplus_i \mathcal{Z}(M_i)$.

12A. Let M_R be a simple R-module, and $S = \text{soc}(R_R)$. Show that

   (a) M is either singular or projective, but not both;
   (b) M is singular iff $M \cdot S = 0$;
   (c) Deduce from (a) that a semisimple module is nonsingular iff it is projective.

12B. Let M_R be an R-module all of whose nonzero quotients have minimal submodules.[58] Show that M is nonsingular iff $P := \text{soc}(M)$ is nonsingular, iff P is projective.

[58] Such a module M is said to be *semi-artinian* in the literature. For instance, an artinian module is always semi-artinian.

12C. Let R be a right self-injective ring, and M_R be a nonsingular module with u.dim$(M) < \infty$. Show that M is f.g. semisimple, and is both projective and injective.

13. Let M_R be any CS module over a ring R.

(1) Show that any surjection from M to a nonsingular module splits.
(2) Show that the Goldie closure 0^{**} (defined in (7.31)) splits in M.

14. (Sandomierski) Let R be a right nonsingular ring, and let N_R be a quotient of a CS module. Show that

(1) $\mathcal{Z}(N)$ splits in N, and deduce that
(2) if N is indecomposable, then it is either singular or nonsingular.

15. Let N_1, N_2 be injective submodules of a nonsingular module M_R. Show that $N_1 + N_2$ is also injective. Give an example to show that this may not be true if M is an arbitrary module over R.

16. Let $S \subseteq R$ be rings such that $S_S \subseteq_e R_S$, and let M, N be right R-modules. If M_S is nonsingular, show that $\text{Hom}_S(N, M) = \text{Hom}_R(N, M)$.

17. Show that "right semihereditary" and "Baer" are independent notions.

18. Let R be any right semihereditary ring and $S \subseteq R$ be a *finite* set. Show that $\text{ann}_r(S) = eR$ for some idempotent $e \in R$.

19A. For any $a \in R$, show that the following are equivalent:

(1) $a = ava$ for some $v \in \text{U}(R)$. (Such a is called a *unit-regular* element of R.)
(2) $a = uf$ for some $f = f^2 \in R$ and $u \in \text{U}(R)$.
(3) $a = eu$ for some $e = e^2 \in R$ and $u \in \text{U}(R)$.
(4) $a = asa$ for some $s \in R$, and $R/aR \cong \text{ann}_r(a)$ as right R-modules.
(5) aR is a direct summand of R_R, and $R/aR \cong \text{ann}_r(a)$ as right R-modules.

19B. Refer to Condition (5) in the list of equivalent conditions in the last exercise. Show that

(A) If R is von Neumann regular, we can drop the first condition in (5).
(B) If R is commutative, we can drop the second condition in (5).
(C) If R is commutative and von Neumann regular, every $a \in R$ is unit-regular.
(D) In general, the two conditions in (5) are independent.

20. Let R be a ring in which all idempotents are central, and let $a \in R$. Show that aR is projective iff $a = be$ where $e = e^2$ and $\text{ann}_r(b) = 0$.

21. (Endo) Let R be a ring in which all idempotents are central. Show that R is right Rickart iff it is left Rickart.

22. (a) Show that a reduced ring is right Rickart iff it is left Rickart.
(b) Name a Rickart ring that is not reduced.

23. Let R be a ring with exactly two idempotents 0 and 1. Show that R is right Rickart iff R is Baer, iff R is a domain.

24. (a) Show that a commutative Rickart ring is always reduced.
(b) Name a commutative reduced ring that is not Rickart.

25. For any domain k, and a fixed integer $n > 1$, let T be the ring of upper triangular $n \times n$ matrices over k. Show that T is a Baer ring iff T is a right Rickart ring, iff k is a division ring.

26. Let R be a Baer ring. Show that the annihilator of a central subset is generated by a central idempotent. State and prove the analogue of this for a Rickart ring.

27. Show that the center of a Baer (resp. Rickart) ring is also a Baer (resp. Rickart) ring.

28. For any Baer ring R, let L be the poset (with respect to inclusion) of principal right ideals of the form eR where $e = e^2$. Show that L is a complete lattice,[59] anti-isomorphic to the complete lattice L' of principal left ideals of the form Re' where $e' = e'^2$.

29. Let $(R, *)$ be a Rickart $*$-ring.
(1) For any $x \in R$, show that $\text{ann}_r(x) \cap x^*R = 0$.
(2) Deduce from (1) that $xx^* = 0 \Longrightarrow x = 0$.

The next two exercises are intended for readers who are familiar with the notion of Boolean algebras. Briefly, a Boolean algebra is a distributive lattice with 0 *and* 1 *in which every element has a complement.*

30. For any ring R, the set $B(R)$ of central idempotents is known to form a lattice under the following (binary) meet and join operations:

$$e \wedge f = ef, \qquad e \vee f = e + f - ef \qquad (e, f \in B(R)).$$

In fact $B(R)$ is isomorphic to the lattice $B'(R)$ of ideal direct summands of R (by the map $e \mapsto eR$), where meet is given by intersection and join is given by sum. ($B(R)$ and $B'(R)$ are both Boolean algebras.) For any Baer ring, show that the lattices $B(R)$ and $B'(R)$ are complete.

31. For any commutative ring R and any ideal $\mathfrak{A} \subseteq R$, recall that $V(\mathfrak{A})$ denotes the Zariski closed set

$$\{\mathfrak{p} \in \text{Spec } R : \ \mathfrak{p} \supseteq \mathfrak{A}\} \subseteq \text{Spec } R.$$

Let $B(\text{Spec } R)$ be the Boolean algebra of clopen (closed and open) sets in Spec R. Show that $\varphi : B(R) \to B(\text{Spec } R)$ defined by $\varphi(e) = V(eR)$ is an anti-isomorphism of Boolean algebras, and $\tilde{\varphi} : B(R) \to B(\text{Spec } R)$ defined by $\tilde{\varphi}(e) = V((1-e)R)$ is an isomorphism of Boolean algebras.

[59] See Exercise (6.25) for definition.

32. (Johnson-Wong) For any nonsingular module M_R, show that

    (1) there is a canonical embedding ε of the ring $\text{End}_R(M)$ into the ring $\text{End}_R(E(M))$; and

    (2) M is QI (quasi-injective) iff ε is an isomorphism.

33. (Johnson-Wong) Let M_R be a nonsingular uniform module, with $E = \text{End}_R(M)$.

    (1) Show that any nonzero $f \in E$ is injective, and deduce that E is a domain.

    (2) If M is also QI, show that E is a division ring.

34. Let R be a subring of a ring T such that $R_R \subseteq_e T_R$. Show that $\mathcal{Z}(R_R) \subseteq \mathcal{Z}(T_T)$. In particular, if T is right nonsingular, so is R.

35. (Shock) Let $S = R[X]$ where X is a set of commuting indeterminates over the ring R. Show that $\mathcal{Z}(S_S) = \mathcal{Z}(R_R)[X]$. (In particular, R is right nonsingular iff S is.)

§8. Dense Submodules and Rational Hulls

§8A. Basic Definitions and Examples

In this subsection we introduce the notions of dense submodules and rational extensions. This theory will be needed in the construction of the maximal right ring of quotients in Chapter 5. Much of the content of the current section comes from the work of Utumi and Findlay-Lambek. Our exposition here follows in part Storrer [72].

The notion of a *dense submodule* is a refinement of that of an essential submodule. To discuss this notion, let us first recall a notation introduced in §7. Let $N \subseteq M$ be right R-modules, and let $y \in M$. In (7.24), we have defined

(8.1) $$y^{-1}N := \{r \in R : \ yr \in N\};$$

this is a right ideal in R. Using this notation, the condition for $N \subseteq_e M$ may be expressed as follows (see (3.27)(1)):

$$y \in M \backslash \{0\} \implies y \cdot (y^{-1}N) \neq (0).$$

By strengthening this condition, we are led to the following new notion.

(8.2) Definition. We say that N is a *dense submodule* of M (written $N \subseteq_d M$) if, for any $y \in M$ and $x \in M \backslash \{0\}$, $x \cdot y^{-1}N \neq 0$ (i.e., there exists $r \in R$ such that $xr \neq 0$, and $yr \in N$). If $N \subseteq_d M$, we also say that M is a *rational extension* of N.

(8.3) Examples.

(1) Of course, $N \subseteq_d M \implies N \subseteq_e M$. The converse is not true in general. For instance, for $R = \mathbb{Z}$, if $M = \mathbb{Z}/p^{n+1}\mathbb{Z}$ and $N = p\mathbb{Z}/p^{n+1}\mathbb{Z}$ ($p =$ prime, $n \geq 1$),

then $N \subseteq_e M$, but N is *not* dense in M. (For $y = \bar{1}$ and $x = \overline{p^n}$, $x(y^{-1}N) = 0$.) A similar example is given by $R = \mathbb{Q}[u, v]$ with relations $u^2 = v^2 = uv = 0$. Taking $M = R_R$ and $N = u\,\mathbb{Q} \oplus v\,\mathbb{Q}$, we have $N \subseteq_e M$, but N is not dense in M. (Here, $u \cdot 1^{-1}N = 0$.)

(2) Let R be a commutative domain with quotient field K. For any torsion-free module N_R, let $M = N \otimes_R K$. Then $N \subseteq_d M$ since $y^{-1}N \neq 0$ for any $y \in M$ and hence $x \cdot y^{-1}N \neq 0$ for any $x \in M\backslash\{0\}$.

(3) *Surprisingly perhaps,* $N_i \subseteq_d M_i$ $(i = 1, 2)$ *does not imply* $N_1 \oplus N_2 \subseteq_d M_1 \oplus M_2$. In fact this implication may fail already in the special case $N_2 = M_2$. For a concrete example over $R = \mathbb{Z}$, take $N_1 = \mathbb{Z} \subseteq_d \mathbb{Q} = M_1$ and $N_2 = M_2 = \mathbb{Z}/n\mathbb{Z}$, where $n > 0$. For $y = (1/n, 0) \in M_1 \oplus M_2$ and $x = (0, \bar{1})$, if $yr \in N_1 \oplus N_2$, we must have $n \mid r$, but then $xr = (0, \bar{r}) = 0$!

(4) For any right ideal $\mathfrak{A} \subseteq R$, we have $\mathfrak{A} \subseteq_d R_R$ iff, for any $y \in R$, $\text{ann}_\ell(y^{-1}\mathfrak{A}) = 0$. If $\mathfrak{A}$ is in fact an ideal, then

$$\mathfrak{A} \subseteq_d R_R \iff \text{ann}_\ell(\mathfrak{A}) = 0,$$

since, in this case, $y^{-1}\mathfrak{A} \supseteq \mathfrak{A}$. (In particular, if R is a commutative ring, $\mathfrak{A} \subseteq_d R$ simply amounts to $\mathfrak{A}$ being *faithful* as an R-module.)

For instance, in Osofsky's example (3.45), the ring $R = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$ has an ideal $\mathfrak{A} = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. Since $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{A} = 0$, $\mathfrak{A}$ is not dense in R_R. In fact, $\mathfrak{A}$ is not even essential in R_R, as the reader may check. On the other hand, in the ring $R = R_1 \times R_2 \times \cdots$ where R_i are arbitrary rings, the ideal $\mathfrak{A} = R_1 \oplus R_2 \oplus \cdots$ in R has $\text{ann}_r(\mathfrak{A}) = \text{ann}_\ell(\mathfrak{A}) = 0$, so $\mathfrak{A}$ is left and right dense in R.

(8.4) Corollary.

(1) *If $a \in R$ is a central element that is not a* 0*-divisor, then $aR \subseteq_d R_R$ (and $Ra \subseteq_d {}_RR$).*
(2) *Let $\mathfrak{A}$ be any right ideal in a ring R, and $z \in R$. Then $\mathfrak{A} \subseteq_d R_R$ implies that $z^{-1}\mathfrak{A} \subseteq_d R_R$.*
(3) *Let R be a prime ring (FC–*(10.15)*). Then any nonzero ideal in $\mathfrak{A}$ is dense in R_R and ${}_RR$.*

Proof. All parts follow quickly from (8.3)(4). For (2), note that $y^{-1}(z^{-1}\mathfrak{A}) = (zy)^{-1}\mathfrak{A}$ for any $y, z \in R$. □

Note that (1) above applies to the polynomial ring $R = S[t]$, if we take $a = t$. Another concrete example is the following, where we construct an ideal that is left dense but not right dense.

(8.5) Example. Let $R = \mathbb{Z}[x] \oplus \mathbb{Z}[x]y$ be the ring in (7.6)(4), where we have the relations $yx = y^2 = 0$. Let $\mathfrak{A} = Rx + Ry$, which is easily checked to be an ideal

of R. Since x is not a left 0-divisor, we have $\text{ann}_r(\mathfrak{A}) = 0$. By (8.3)(4) (left ideal version), $\mathfrak{A} \subseteq_d {}_R R$. On the other hand, since $yRx = 0$ and $yRy = 0$, we have $y \in \text{ann}_\ell(\mathfrak{A})$, so $\mathfrak{A}$ is not dense in R_R. (In fact, a short calculation shows that $\text{ann}_\ell(\mathfrak{A}) = R \cdot y$.) Nevertheless, we have $\mathfrak{A} \subseteq_e R_R$, for, if $f(x) + g(x)y \notin \mathfrak{A}$, then $f(x) \neq 0$ and

$$(f(x) + g(x)y)y = f(x)y \in \mathfrak{A}\backslash\{0\}.$$

The next Proposition provides two more characterizations for dense submodules in terms of the nonexistence of certain types of homomorphisms. Here, $E(M)$ denotes the injective hull of a module M.

(8.6) Proposition. *For right R-modules $N \subseteq M$, the following are equivalent:*

(1) $N \subseteq_d M$.
(2) $\text{Hom}_R(M/N, E(M)) = 0$.
(3) *For any submodule P such that $N \subseteq P \subseteq M$,* $\text{Hom}_R(P/N, M) = 0$.

Proof. (1) $\Longrightarrow$ (2). Assume, instead, that there exists a nonzero R-homomorphism $f : M \to E(M)$ with $f(N) = 0$. Then $M \cap f(M) \neq 0$ so there exist $x, y \in M\backslash\{0\}$ such that $f(y) = x$. By (1), there exists $r \in R$ with $xr \neq 0$ and $yr \in N$. But then

$$0 = f(yr) = f(y)r = xr,$$

a contradiction.

(2) $\Longrightarrow$ (3). Suppose that, for some P as in (3), there exists a nonzero R-homomorphism $g : P/N \to M$. By the injectivity of $E(M)$, we can extend g to a (nonzero) $M/N \to E(M)$.

(3) $\Longrightarrow$ (1). Suppose that $x \cdot y^{-1}N = 0$ for some $y \in M$, $x \in M\backslash\{0\}$. We define $f : N + yR \to M$ by

$$f(n + yr) = xr \qquad (n \in N, \;\; r \in R).$$

This map is well-defined, for, if $n + yr = n' + yr'$, then $n - n' = y(r' - r) \in N$, hence $x(r - r') = 0$. Clearly, f is an R-homomorphism vanishing on N, so by (3), $0 = f(y) = x$, a contradiction. □

Note that conditions (2) and (3) apply very well, for instance, to check the failure of $N \subseteq_d M$ in (8.3)(1). In fact, in both examples there, $\text{Hom}_R(M/N, M) \neq 0$.

(8.7) Proposition.

(1) *If $N \subseteq_d M$, $N' \subseteq_d M$, then $N \cap N' \subseteq_d M$.*
(2) *Let $N \subseteq P \subseteq M$. Then $N \subseteq_d M$ iff $N \subseteq_d P$ and $P \subseteq_d M$.*
(3) *Assume M is a nonsingular module. Then $N \subseteq_d M$ iff $N \subseteq_e M$.*

Proof. (1) Let $x, y \in M$, where $x \neq 0$. There exists $r \in R$ such that $xr \neq 0$, $yr \in N$. There also exists $r' \in R$ such that $xrr' \neq 0$ and $yrr' \in N \cap N'$.

(2) We need only prove the "if" part. Assume that $N \subseteq_d P$ and $P \subseteq_d M$ and let $x, y \in M$, $x \neq 0$. There exists $r \in R$ such that $xr \neq 0$, $yr \in P$. Since $P \subseteq_e M$, there also exists $s \in R$ such that $0 \neq xrs \in P$ (and $yrs \in P$). Finally, use $N \subseteq_d P$ to find $t \in R$ such that $x(rst) \neq 0$ and $y(rst) \in N$.

(3) Again, we need only prove the "if" part, so let us assume $N \subseteq_e M$. Let $x, y \in M$, $x \neq 0$. Consider the R-homomorphism $f : R \to M$ defined by $f(r) = yr$ $(r \in R)$. By Exercise (3.7), $f^{-1}(N) \subseteq_e R_R$, where

(8.8) $$f^{-1}(N) = \{r \in R : \ yr \in N\} = y^{-1}N .$$

Therefore, $x \cdot y^{-1}N \neq 0$, for otherwise $\text{ann}(x) \supseteq y^{-1}N$ would give the contradiction $x \in \mathcal{Z}(M)$. □

(8.9) Corollary. *A ring R is right nonsingular iff every essential right ideal $\mathfrak{A} \subseteq R$ is dense in R_R.*

Proof. The "only if" part follows by applying (8.7)(3) to $M = R_R$. For the "if" part, assume there exists $x \in \mathcal{Z}(R_R)\backslash\{0\}$. Then $\text{ann}_r(x) \subseteq_e R_R$, but $\text{ann}_r(x)$ is not dense in R_R, since $x \cdot 1^{-1}\text{ann}_r(x) = 0$. □

§8B. Rational Hull of a Module

Our next goal is to show that *any module M has a unique maximal rational extension.* We proceed as follows. Let $I = E(M)$, and let $H = \text{End}(I_R)$, operating on the left of I. We define

(8.10) $$\tilde{E}(M) = \{i \in I : \ \forall h \in H, \ h(M) = 0 \Longrightarrow h(i) = 0\} .$$

Clearly, this is an R-submodule of I containing M.

(8.11) Lemma. *Let M' be any submodule of I containing M. Then $M \subseteq_d M'$ iff $M' \subseteq \tilde{E}(M)$.*

Proof. For the "if" part, it suffices to show that $M \subseteq_d \tilde{E}(M)$. We do this by applying (8.6). Consider any R-homomorphism

(8.12) $$h : \tilde{E}(M) \to E(\tilde{E}(M)) = E(M) \ \text{ with } \ h(M) = 0 .$$

After extending the domain of h to $E(M)$, we may assume that $h \in H$. But then (8.10) implies that $h(\tilde{E}(M)) = 0$. Thus, (8.6) yields $M \subseteq_d \tilde{E}(M)$.

For the "only if" part, assume that $M \subseteq_d M'$, and consider $h \in H$ such that $h(M) = 0$. If $h(M') \neq 0$, then

$$0 \neq \text{Hom}_R(M'/M, E(M)) = \text{Hom}_R(M'/M, E(M')) ,$$

contradicting $M \subseteq_d M'$ (by (8.6)). Therefore, $h(M') = 0$, and we have proved $M' \subseteq \tilde{E}(M)$. □

(8.13) Proposition. *Suppose $M \subseteq_d P$. Then there exists a unique R-homomorphism $g : P \to \tilde{E}(M)$ extending the inclusion map $M \hookrightarrow \tilde{E}(M)$. This g is necessarily one-to-one.*

Proof. Since $M \subseteq_e P$, the inclusion $M \to E(M)$ extends to an embedding $g : P \to E(M)$. Clearly $M \subseteq_d g(P)$ so by (8.11), $g(P) \subseteq \tilde{E}(M)$. Now suppose $g_1, g_2 : P \to \tilde{E}(M)$ both extend the inclusion map $M \to \tilde{E}(M)$. Since $M \subseteq_e P$, the g_i 's are monomorphisms. Consider the map $f : g_1(P) \to \tilde{E}(M)$ defined by

$$f(g_1(p)) = g_1(p) - g_2(p) \qquad (p \in P).$$

Since $f(M) = 0$ and $M \subseteq_d \tilde{E}(M)$, we must have $f = 0$ (by (8.6)), so $g_1(p) = g_2(p)$ for all $p \in P$. □

In view of (8.11) and (8.13), $\tilde{E}(M)$ is the unique maximal rational extension of M; we call it the *rational hull* (or *rational completion*) of M. As the notation suggests, $\tilde{E}(M)$ is an object somewhat akin to $E(M)$. However, there is one main difference between $\tilde{E}(M)$ and $E(M)$. For $E(M)$, it is possible to have a non-identity R-automorphism of $E(M)$ that is the identity on M. However, according to (the uniqueness part of) (8.13), any R-endomorphism of $\tilde{E}(M)$ that is the identity on M must be the identity on $\tilde{E}(M)$. The following basic example illustrating this phenomenon should be kept in mind.

(8.14) Example. Let $R = \mathbb{Z}$, and p be a fixed prime. Let $M = C_p$, using the cyclic group notations in (3.37). We have

$$M = C_p \subset C_{p^2} \subset C_{p^3} \subset \cdots \subset C_{p^\infty},$$

where $I = E(M) = C_{p^\infty}$ is the Prüfer p-group. As is well-known, $\text{End}(C_{p^n}) \cong C_{p^n}$ (viewed as a ring), for $n < \infty$. Each element $h \in H = \text{End}(I)$ induces a "compatible" family $\{h_n \in \text{End}(C_{p^n}) : n \geq 1\}$. Therefore,

$$H = \text{End}\big(\varinjlim C_{p^n}\big) \cong \varprojlim C_{p^n}, \tag{8.15}$$

which is isomorphic to $\hat{\mathbb{Z}}_p$, the discrete valuation ring of the p-adic integers. As we can see, there are many endomorphisms (and automorphisms) of I that are not the identity, but restrict to the identity of M. On the other hand, since C_{p^n} is not dense in $C_{p^{n+1}}$ (by (8.3)(1)), we have $\tilde{E}(C_{p^n}) = C_{p^n}$. (Note that if $\tilde{E}(C_{p^n})$ were bigger than C_{p^n}, it would have had nontrivial automorphisms restricting to the identity on C_{p^n}.)

The following Proposition offers a description of $\tilde{E}(M)$ in general which is ostensibly not dependent on the use of the ring $H = \text{End}(E(M)_R)$.

(8.16) Proposition. $\tilde{E}(M) = \{y \in E(M) : \ \forall x \in E(M)\backslash\{0\}, \ x \cdot y^{-1}M \neq 0\}$.

Proof. Let $y \in$ RHS. Let $h \in H$ be such that $h(M) = 0$. If $x := h(y) \neq 0$, there exists $r \in y^{-1}M$ such that $xr \neq 0$. But then

$$xr = h(y)r = h(yr) \in h(M) = 0,$$

a contradiction. Thus, $h(y) = 0$, which shows that $y \in \tilde{E}(M)$. Conversely, assume $y \in \tilde{E}(M)$ and let $x \in E(M)\backslash\{0\}$. Fix an element $r \in R$ such that $0 \neq xr \in \tilde{E}(M)$. Since $M \subseteq_d \tilde{E}(M)$, there exists $s \in R$ such that $(xr)s \neq 0$ and $(yr)s \in M$. Now we have $rs \in y^{-1}M$, and $x \cdot y^{-1}M$ contains $x(rs) \neq 0$. □

(8.17) Definition. An R-module M_R is said to be *rationally complete* if it has no proper rational extensions, or equivalently $\tilde{E}(M) = M$. Note that, by the transitivity property of denseness (cf. (8.7)(2)), *$\tilde{E}(M)$ is always rationally complete.*

(8.18) Examples.

(1) *If M_R is injective, then M is rationally complete.*

(2) *Let $(R, \mathfrak{m})$ be a local ring, and M_R be any R-module that has a simple submodule N. Then M is rationally complete.* To see this, note first that we have an R-isomorphism $N \cong (R/\mathfrak{m})_R$. Let $x \in N$ correspond to $\overline{1}$ under such an isomorphism. Consider any rational extension $M' \supseteq M$, and $y \in M'$. Then there exists $r \in R$ such that $yr \in M$ and $xr \neq 0$. The latter implies that $r \notin \mathfrak{m}$ so r is a unit of R. But then $yr \in M$ implies that $y \in M$. Hence $M' = M$.

(3) As seen in (8.14), over $R = \mathbb{Z}$, any cyclic group C_{p^n} is rationally complete. *In fact, any torsion abelian group M is rationally complete.* To see this, consider $M \subseteq_d M'$, and assume $M \neq M'$, say, $y_0 \in M'\backslash M$. Let n be the least natural number such that $y_0 n \in M$. Write $n = n_0 p$ where p is a prime. Then $y := y_0 n_0 \notin M$ and $yp \in M$. Clearly, y has finite order divisible by p, so a suitable multiple of y will give an element $x \in M'$ of order p. Since $M \subseteq_d M'$, there exists $r \in \mathbb{Z}$ such that $yr \in M$ and $xr \neq 0$. Such an r must be relatively prime to p, but then $yr \in M$, $yp \in M$ imply that $y \in M$, a contradiction.

(4) Let R be an integral domain with quotient field K. For any *torsion-free* module M_R, we have $E(M) = M \otimes_R K$ by (3.35) and $M \subseteq_d M \otimes_R K$ by (8.3)(2). Thus, $\tilde{E}(M) = E(M)$, so *M is rationally complete iff it is already injective. In particular, R_R is rationally complete iff R is a field.*

(5) (Generalizing (4).) Let M be a nonsingular module, over any ring R. Then we have $\tilde{E}(M) = E(M)$. (By (7.6)(2), $E(M)$ is also nonsingular, so by (8.7)(3), $E(M)$ is a rational extension of M.) Therefore, M is rationally complete iff M is injective. Specializing to the case of the right regular module, we see that, *if R is a right nonsingular ring, then $\tilde{E}(R_R) = E(R_R)$, so R_R is rationally complete iff R is right self-injective.* These facts will be of importance in the next chapter when we study the various rings of quotients associated with a given ring.

What can we say about $\tilde{E}(M_1 \oplus M_2)$? As it turns out, this is not always given by $\tilde{E}(M_1) \oplus \tilde{E}(M_2)$! To analyze $\tilde{E}(M_1 \oplus M_2)$, we shall work inside $E(M_1 \oplus M_2)$, which we identify with $E(M_1) \oplus E(M_2)$.

(8.19) Proposition. $\tilde{E}(M_1 \oplus M_2) \subseteq \tilde{E}(M_1) \oplus \tilde{E}(M_2) \subseteq E(M_1) \oplus E(M_2)$.

Proof. Let $M = M_1 \oplus M_2$. We shall write any element y of $E(M_1) \oplus E(M_2)$ in the form $y = (y_1, y_2)$. Suppose $y \in \tilde{E}(M)$. Then, for any $x_1 \in E(M_1)\backslash\{0\}$, we have $(x_1, 0) \cdot y^{-1}M \neq (0)$ by (8.16). Since $y^{-1}M = y_1^{-1}M_1 \cap y_2^{-1}M_2$, this implies that $(x_1, 0) \cdot y_1^{-1}M_1 \neq (0)$. Hence $x_1 \cdot y_1^{-1}M_1 \neq (0)$ for all $x_1 \in E(M_1)\backslash\{0\}$, and so $y_1 \in \tilde{E}(M_1)$ by (8.16). Similarly, $y_2 \in \tilde{E}(M_2)$, and we have

$$y = (y_1, y_2) \in \tilde{E}(M_1) \oplus \tilde{E}(M_2)\,.$$

□

(8.20) Corollary. *If* M_1, M_2 *are rationally complete, then so is* $M_1 \oplus M_2$.

Proof. By (8.19), we have

$$\tilde{E}(M_1 \oplus M_2) \subseteq \tilde{E}(M_1) \oplus \tilde{E}(M_2) = M_1 \oplus M_2 \subseteq \tilde{E}(M_1 \oplus M_2)\,,$$

so equality must hold throughout. □

In general, although $M_i \subseteq_d \tilde{E}(M_i)$ $(i = 1, 2)$, we need not have

$$M_1 \oplus M_2 \subseteq_d \tilde{E}(M_1) \oplus \tilde{E}(M_2)$$

(see (8.3)(3)). Therefore, we should not expect $\tilde{E}(M_1 \oplus M_2)$ to be given by $\tilde{E}(M_1) \oplus \tilde{E}(M_2)$. Let us make an explicit computation for $\tilde{E}(M_1 \oplus M_2)$ to illustrate this point.

(8.21) Example. Let $R = \mathbb{Z}$, $M = M_1 \oplus M_2$ with

$$M_1 = \mathbb{Z} \subset \tilde{E}(M_1) = E(M_1) = \mathbb{Q}\,,$$
$$M_2 = \mathbb{Z}/p\mathbb{Z} = \tilde{E}(M_2) \subset E(M_2) = C_{p^\infty}\,,$$

where p is a prime. We use the notations in the proof of (8.19). Let

$$y = (y_1, y_2) \in E(M_1) \oplus M_2\,,$$

with $0 \neq y_1 = m/n$ where $(m, n) = 1$ in $\mathbb{Z}$. Then $y^{-1}M = y_1^{-1}M_1 = n\,\mathbb{Z}$. We claim that:

$$y \in \tilde{E}(M) \Longleftrightarrow (n, p) = 1\,. \tag{8.22}$$

Indeed, assume that $y \in \tilde{E}(M)$ but $n = n_0 p$ $(n_0 \in \mathbb{Z})$. Considering the element

$$(y_1 n_0,\ y_2 n_0) \in \tilde{E}(M),$$

we can find $r \in \mathbb{Z}$ such that $y_1 n_0 r \in M_1$ and $(0, \bar{1})r \neq (0,0)$. The former implies that $p|r$, which contradicts the latter. This proves "$\Longrightarrow$" in (8.22). For the converse, assume that $(n, p) = 1$. Consider any

$$x = (x_1, x_2) \neq (0,0) \in E(M).$$

If $x_1 \neq 0$, then $x_1 \cdot y^{-1}M \neq 0$ since $y^{-1}M \neq (0)$. If $x_2 \neq 0$, then

$$x_2 \cdot y^{-1}M = x_2 \cdot n\mathbb{Z} \neq (0)$$

since $(n, p) = 1$ and x_2 is p-primary. Therefore, in any case, $x \cdot y^{-1}M \neq (0)$, and (8.16) yields $y \in \tilde{E}(M)$. Having now proved the claim (8.22), we conclude that

(8.23) $$\tilde{E}(\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})) = \mathbb{Z}_{(p)} \oplus (\mathbb{Z}/p\mathbb{Z}),$$

where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (p). In particular, we see that *$\mathbb{Z}_{(p)} \oplus (\mathbb{Z}/p\mathbb{Z})$ is rationally complete, although its direct summand $\mathbb{Z}_{(p)}$ is not rationally complete.* (We have, of course, $\tilde{E}(\mathbb{Z}_{(p)}) = \mathbb{Q}$.)

In spite of this, one could expect that a rationally complete module M_R has some of the features of an injective module. We finish this subsection with the following result which characterizes a rationally complete module in terms of a property for extending homomorphisms.

(8.24) Theorem. *For any module M_R, the following are equivalent:*

(1) *M is rationally complete.*
(2) *For any right R-modules $A \subseteq B$ such that* $\mathrm{Hom}_R(B/A, E(M)) = (0)$, *any R-homomorphism $f : A \longrightarrow M$ can be extended to B (necessarily uniquely).*

Proof. (1) $\Longrightarrow$ (2). We can certainly extend f to $g : B \to E(M)$. We claim that

(∗) $$M \subseteq_d M + g(B).$$

Once we have proved this, then (1) implies that $g(B) \subseteq M$ and we are done. To prove (∗), it suffices to check

$$\mathrm{Hom}_R((M + g(B))/M, \ \ E(M + g(B))) = (0),$$

by (8.6). This follows from $\mathrm{Hom}_R(B/A, \ E(M)) = (0)$, since $E(M + g(B)) = E(M)$, and g induces a surjection from B/A to $(M + g(B))/M$.

(2)$\Longrightarrow$(1). Suppose $M \subseteq_d M'$. By (8.6), $\mathrm{Hom}_R(M'/M, E(M)) = (0)$, where we have identified $E(M')$ with $E(M)$. By (2), the identity map $M \to M$ extends to some $g : M' \to M$ which is necessarily a split surjection. On the other hand, $(\ker g) \cap M = (0)$ implies that $\ker g = (0)$, since $M \subseteq_e M'$. Therefore, g is an isomorphism, which implies that $M' = M$. □

An easy application of (8.24) leads to the following generalization of (8.20). The details of the proof will be left to the reader.

(8.25) Corollary. *Let* $M = \prod_{i \in I} M_i$. *If each* $(M_i)_R$ *is rationally complete, then so is* M_R.

§8C. Right Kasch Rings

If V is a simple right module over a ring R, it is of interest to know whether V can be embedded in the right regular module R_R. Consideration of this issue leads to the notion of right Kasch rings. Actually, we have already had an earlier encounter with the "right Kasch" property; see (5.74). We shall now reintroduce this property formally, and take a closer look at it in this subsection.

(8.26) Definition. We say that R is a *right Kasch ring* if every simple right R-module V can be embedded in R_R. "Left Kasch ring" is defined similarly. As usual, R is called a *Kasch ring* if it is both right and left Kasch.

We can always think of a simple right R-module as $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal right ideal. The following Proposition gives a characterization for the embedding of $R/\mathfrak{m}$ into R in terms of denseness and double annihilator properties.

(8.27) Proposition. *For any maximal right ideal* $\mathfrak{m} \subset R$, *the following are equivalent:*

(0) $R/\mathfrak{m}$ *embeds into* R_R.
(1) $\mathfrak{m} = \text{ann}_r(x)$ *for some* $x \in R$.
(2) $\text{ann}_\ell(\mathfrak{m}) \neq 0$.
(3) $\mathfrak{m} = \text{ann}_r(\text{ann}_\ell\, \mathfrak{m})$.
(4) $\mathfrak{m}$ *is not dense in* R_R.

Proof. (0) $\Longrightarrow$ (1). There exists an endomorphism of R_R, say, f, with kernel $\mathfrak{m}$. Then f is given by left multiplication by some nonzero element x, and we have $\mathfrak{m} = \text{ann}_r(x)$.

(1) $\Longrightarrow$ (2). From (1), we have clearly $0 \neq x \in \text{ann}_\ell(\mathfrak{m})$.

(2) $\Longrightarrow$ (3). Since $\text{ann}_\ell(\mathfrak{m}) \neq 0$, $\mathfrak{m} \subseteq \text{ann}_r(\text{ann}_\ell\, \mathfrak{m}) \neq R$. The maximality of $\mathfrak{m}$ then implies (3).

(3) $\Longrightarrow$ (4). If $\mathfrak{m}$ is dense, (8.3)(4) implies that $\text{ann}_\ell(y^{-1}\mathfrak{m}) = 0$ for all $y \in R$. In particular, $\text{ann}_\ell(\mathfrak{m}) = 0$, which contradicts (3).

(4) $\Longrightarrow$ (0). Let $x, y \in R$ be such that $x \neq 0$ and $x \cdot (y^{-1}\mathfrak{m}) = 0$. Then, left multiplication by x defines a nonzero R-homomorphism $f : R/y^{-1}\mathfrak{m} \to R$. Also, left multiplication by y gives an R-embedding $g : R/y^{-1}\mathfrak{m} \to R/\mathfrak{m}$. Since $R/y^{-1}\mathfrak{m} \neq 0$, g must be an isomorphism. Thus, $f \circ g^{-1}$ gives an embedding of $R/\mathfrak{m}$ into R_R. □

(8.28) Corollary. *For any ring R, the following are equivalent:*

(0) *R is right Kasch.*
(1) *Any maximal right ideal in R has the form* $\text{ann}_r(x)$ *for some* $x \in R$.
(2) *For any maximal right ideal* $\mathfrak{m}$ *in* R, $\text{ann}_\ell(\mathfrak{m}) \neq 0$.
(3) *For any maximal right ideal* $\mathfrak{m}$ *in* R, $\mathfrak{m} = \text{ann}_r(\text{ann}_\ell\, \mathfrak{m})$.
(4) *The only dense right ideal in R is R itself.*
(5) *For any right ideal* $\mathfrak{A} \subsetneq R$ *in* R, $\text{ann}_\ell(\mathfrak{A}) \neq 0$.

Proof. We deduce the equivalence easily from (8.27), noting that any right ideal $\mathfrak{A} \subsetneq R$ is contained in a maximal right ideal. □

(8.29) Examples.

(0) Any semisimple ring is clearly (right, left) Kasch. (For a more precise statement, see Exercise 15.)

(1) If R is a domain but not a division ring, then R is not (right) Kasch in view of (2) above.

(2) If R is any nonartinian simple ring, then R is not right Kasch. Indeed, if it is, then R will have minimal right ideals, and FC–(3.10) implies that R is artinian.

(3) Let R be any semiprimary ring with $J := \text{rad}\, R$. If either R/J is simple or R is commutative, then R is Kasch by (5.75). In particular, *any local 1-sided artinian ring is Kasch.*

(4) Let $R = \prod_{j\in J} A_j$, where the A_j 's are nonzero rings and the indexing set J is infinite. The ideal $\mathfrak{A} := \bigoplus_{j\in J} A_j \subsetneq R$ clearly has $\text{ann}_\ell(\mathfrak{A}) = \text{ann}_r(\mathfrak{A}) = 0$. Thus, for any maximal right (left) ideal $\mathfrak{m} \supseteq \mathfrak{A}$, the simple right (left) module $R/\mathfrak{m}$ fails to embed in R, so R is neither a right Kasch ring nor a left Kasch ring.

(5) Let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ where k is any field. Consider the maximal right ideals $\mathfrak{m} = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ and $\mathfrak{m}' = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$, from which we can form the two simple right R-modules $V = R/\mathfrak{m}$, $V' = R/\mathfrak{m}'$. A quick calculation shows that $\text{ann}_\ell(\mathfrak{m}) = \mathfrak{m}' \neq 0$, and $\text{ann}_\ell(\mathfrak{m}') = 0$. Therefore, V' cannot be embedded in R_R, but V can. (Of course, V is isomorphic to the minimal right ideal $\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$.) This shows that R is not a right Kasch ring, and a similar argument shows that R is not a left Kasch ring. (Readers with a good memory might recall that this example is just a special case of (5.76)!)

(6) In general, *"right Kasch" and "left Kasch" are independent conditions.* In the following, we shall construct an *artinian* ring that is right Kasch but not left Kasch. Let k be a division ring, and let R be the ring of matrices

$$\gamma = \begin{pmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$$

over k. The set J of $\gamma \in R$ with zero diagonals is a nilpotent ideal with $R/J \cong k \times k$ (the isomorphism being given by $\overline{\gamma} \mapsto (a, e)$), so J is exactly $\text{rad}(R)$. The two simple right R-modules S_1, S_2 can be taken to be k, with γ acting by right multiplication by a for S_1, and by right multiplication by e for S_2. The simple left R-modules S_1', S_2' can be described similarly. Next let us calculate the two socles $\text{soc}(R_R)$ and $\text{soc}({}_RR)$. By *FC*–Exer. (4.20), $\text{soc}(R_R) = \text{ann}_\ell(J)$, which is easily computed to be $\{\gamma \in R : a = 0\}$. The right action of R on this is computed by the following equation:

$$\begin{pmatrix} 0 & 0 & b' & c' \\ 0 & 0 & 0 & d' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e' \end{pmatrix} \begin{pmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{pmatrix} = \begin{pmatrix} 0 & 0 & b'a & c'e \\ 0 & 0 & 0 & d'e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e'e \end{pmatrix},$$

which shows that $\text{soc}(R_R) \cong S_1 \oplus 3\,S_2$. In particular, R *is right Kasch*. On the other hand, $\text{soc}({}_RR) = \text{ann}_r(J)$ is computed to be $\{\gamma \in R : a = e = 0\}$ (which is just J). Since

$$\begin{pmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{pmatrix} \begin{pmatrix} 0 & 0 & b' & c' \\ 0 & 0 & 0 & d' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab' & ac' \\ 0 & 0 & 0 & ad' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we see that $\text{soc}({}_RR) \cong 3\,S_1'$. Thus, S_2' does not embed into ${}_RR$, so R *is not left Kasch.*

This example also provides a good illustration for the result (8.27). In fact, let $\mathfrak{m}_1$ be the ideal $\{\gamma \in R : a = 0\}$, and $\mathfrak{m}_2$ be the ideal $\{\gamma \in R : e = 0\}$. These are both maximal left and maximal right ideals in R, with $(R/\mathfrak{m}_i)_R \cong S_i$ and ${}_R(R/\mathfrak{m}_i) \cong S_i'$. Clearly, the matrix unit E_{13} left and right annihilates $\mathfrak{m}_1$, so (8.27) already predicts that S_1 embeds into R_R and S_1' embeds into ${}_RR$ (and that $\mathfrak{m}_1$ is not dense in R_R or in ${}_RR$). As for $\mathfrak{m}_2$, E_{44} clearly left annihilates it, so S_2 embeds into R_R (and $\mathfrak{m}_2$ is not dense in R_R). On the other hand, a direct calculation shows that $\text{ann}_r(\mathfrak{m}_2) = 0$, so (by the left analogue of (8.27)), S_2' *does not* embed into ${}_RR$ (and ${}_R\mathfrak{m}_2$ is dense in ${}_RR$).

There also exist *local* rings R that are right Kasch but not left Kasch: for an explicit construction, see Exercise 13.

The Kasch property is already of great interest in the category of commutative rings. Let us now conclude this subsection by giving a couple of results to illustrate how Kasch rings arise naturally in the commutative case.

For the first result, we start with a commutative ring S, and construct a commutative Kasch ring $R \supseteq S$, as follows. Let $M = \bigoplus_i V_i$, where $\{V_i\}$ is a complete set of simple S-modules. Viewing M as an (S, S)-bimodule in the obvious way (with identical left, right S-actions), we can form the "trivial extension" R of M by S, as in (2.22)(A).

(8.30) Proposition. *The trivial extension $R = M \oplus S$ (in which M is an ideal of square zero) is a commutative Kasch ring containing S.*

Proof. The defining formula for the multiplication in R shows clearly that R is commutative. Since $M^2 = 0$, R and S have the same simple modules. By construction, each simple S-module V_i is a minimal ideal in $M \subseteq R$, so R is a (commutative) Kasch ring. □

For the second way of constructing commutative Kasch rings, we start with any commutative ring R, and let $Q(R)$ be its total ring of quotients; i.e., the localization of R at the multiplicative set of non 0-divisors in R. It turns out that, under a reasonable finiteness assumption on R, $Q(R)$ will be a (commutative) semilocal Kasch ring.

(8.31) Theorem. *In the above notations, assume that R has* ACC *on annihilator ideals (e.g., R is a subring of a commutative noetherian ring: see* (6.61)*). Then:*

(1) (Small) *An ideal $\mathfrak{A}$ is dense in R iff it contains a non* 0*-divisor;*
(2) (Faith) $K := Q(R)$ *is a (commutative) semilocal Kasch ring.*

Proof. Among the element-annihilators $\{\text{ann}(a) : 0 \neq a \in R\}$, let $\{\text{ann}(a_i) : i \in I\}$ be the maximal members. By Exercise (3.40E)(1), each $\mathfrak{p}_i := \text{ann}(a_i)$ is a prime ideal. *We claim that* $|I| < \infty$. In fact, if we have an infinite set $\mathfrak{p}_{i_1}, \mathfrak{p}_{i_2}, \ldots$, then $\mathfrak{p}_{i_1} \supseteq \mathfrak{p}_{i_1} \cap \mathfrak{p}_{i_2} \supseteq \ldots$ are also annihilator ideals, and so

$$\mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_r} = \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_r} \cap \mathfrak{p}_{i_{r+1}} \subseteq \mathfrak{p}_{i_{r+1}}$$

for some r, by (6.57). Since $\mathfrak{p}_{i_{r+1}}$ is prime, we have $\mathfrak{p}_{i_k} \subseteq \mathfrak{p}_{i_{r+1}}$ for some $k \leq r$, a contradiction to the maximality of $\mathfrak{p}_{i_k}$. Therefore, $|I| < \infty$, and we may relabel I as $\{1, 2, \ldots, n\}$. The given ACC hypothesis also implies that each element-annihilator is contained in a maximal one. Hence

$$T := \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \tag{8.32}$$

gives precisely the set of *all* 0-divisors of R.

(1) If $\mathfrak{A}$ contains a non 0-divisor, then $\mathfrak{A}$ is faithful as an R-module, and so $\mathfrak{A} \subseteq_d R$ by (8.3)(4). On the other hand, if $\mathfrak{A}$ consists of 0-divisors, then $\mathfrak{A} \subseteq T$. By (8.32) and the Lemma of Prime Avoidance (Eisenbud [95: pp. 90-91]), $\mathfrak{A} \subseteq \mathfrak{p}_i$ for some i. But then $a_i \mathfrak{A} = 0$, so (by loc. cit.) $\mathfrak{A}$ is not a dense ideal.

(2) Let $S := R \setminus T$, so that the total ring of quotients of R is $K = R_S$. Consider any maximal ideal M of K. Then M is the localization N_S of some ideal $N \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. By the Lemma of Prime Avoidance again, $N \subseteq \mathfrak{p}_i$ for some i. Thus, $M \subseteq (\mathfrak{p}_i)_S$, so we must have $M = (\mathfrak{p}_i)_S$. This shows that K is semilocal, with maximal ideals $M_i := (\mathfrak{p}_i)_S$ $(1 \leq i \leq n)$. Since $a_i M_i = 0$ for each i, it follows from (8.28) that K is a Kasch ring. □

(8.33) Remark. In the foregoing proof, we worked mostly with element-annihilators. But a moment's reflection shows that the $\text{ann}(a_i)$'s introduced at the beginning of the proof above are also the maximal members in the set of all

(proper) annihilator ideals, under no assumptions other than the commutativity of R.

The main reason we have included a quick exposé on Kasch rings in this subsection is that several classes of Kasch rings prove to be of importance in future chapters. The quasi-Frobenius (QF) rings studied in Chapter 6 and the cogenerator rings studied in Chapter 7 (a generalization of QF rings) will be seen to be both Kasch rings. Small's result in (8.31)(1) above also turns out to be useful in the consideration of maximal rings of quotients of commutative rings; see (13.16).

Exercises for §8

1. Let M' be a submodule of M_R and $N \subseteq_d M$. For any $f \in \text{Hom}_R(M', M)$, show that $f^{-1}(N) \subseteq_d M'$.

2. Let M be an R-module containing the right regular module R_R. Show that $R_R \subseteq_d M$ iff $R_R \subseteq_e M$ and for every $y \in M$, $y^{-1}R \subseteq_d R_R$.

3. Let $C \subsetneqq N \subseteq M$ be R-modules.

   (a) Does $N \subseteq_e M$ imply $N/C \subseteq_e M/C$?
   (b) Does $N \subseteq_d M$ imply $N/C \subseteq_d M/C$?

4. Let M_R be a nonsingular uniform R-module. Show that any nonzero submodule $N \subseteq M$ is dense in M.

5. Let $\mathfrak{A} \neq R$ be an ideal in a commutative ring R, and let $M_R = R/\mathfrak{A}$. Show that any nonzero submodule $N \subseteq M$ is dense in M iff $\mathfrak{A}$ is a prime ideal.

6. Let k be a field and $R = k\langle\{x_i : i \geq 1\}\rangle$ with relations $x_i x_j = 0$ for all unequal i, j. Let $p_i(x_i) \in k[x_i]\backslash\{0\}$. Show that the ideal generated by $\{p_i(x_i) : i \geq 1\}$ is dense in R.

7. Show that a ring R is semisimple iff R is right nonsingular and every right ideal (resp. right R-module) is rationally complete.

8. Let $f : S \to R$ be a surjective ring homomorphism, where S is a commutative ring. If N_R is rationally complete as an R-module, show that N_S is rationally complete as an S-module.

9. Let $N \subseteq_d M$, where M, N are right R-modules. For any $y_1, \ldots, y_n \in M$ and $0 \neq x \in M$, show that there exists $r \in R$ such that $y_1 r, \ldots, y_n r \in N$ and $xr \neq 0$.

10. Let $S \subseteq R$ be rings such that $S_S \subseteq_e R_S$, and let $N \subseteq M$ be right R-modules.

    (1) Assume N_S is nonsingular. Show that
    (a) $N_R \subseteq_e M_R$ iff $N_S \subseteq_e M_S$, and
    (b) N_S is injective $\Longrightarrow$ N_R is injective.

(2) Assume M_S is nonsingular. Show that $N_R \subseteq_d M_R$ iff $N_S \subseteq_d M_S$.
(3) If N_S is nonsingular and rationally complete, show that N_R is also rationally complete.

11. Prove Corollary (8.25).

12. Let R be a local ring with a nilpotent maximal ideal. Show that any module M_R is rationally complete.

13. Give an example of a local ring R that is right Kasch but not left Kasch. (**Hint.** Let S be the ring of power series in two noncommuting variables x, y over a field k, and try $R = S/(yx, y^2)$.)

14. Show that a ring R is right Kasch iff, for any nonzero finitely generated module M_R, $\mathrm{Hom}_R(M, R) \neq 0$.

15. For any ring R, show that the following are equivalent:

    (1) R is semisimple;
    (2) R is von Neumann regular and right Kasch;
    (3) R is Jacobson semisimple (i.e., $\mathrm{rad}(R) = 0$) and right Kasch;
    (4) R is semiprime and right Kasch.

16. Let R be Osofsky's ring of 32 elements in (3.45).

    (1) Show that R is a Kasch ring by finding explicit embeddings of the simple (right, left) R-modules into R.
    (2) Show that the modules R_R and ${}_RR$ are not divisible (and hence not injective).

17. Let A be the direct product $k \times k \times k \times \cdots$ where k is a field. Let e_i $(i \geq 1)$ be the i th "unit vector" and S be the k-subalgebra of A generated by the e_i 's (that is: $S = k \oplus ke_1 \oplus ke_2 \oplus \cdots$).

    (1) Show that the simple S-modules are V_i $(i \geq 0)$ with $V_i \cong ke_i \subset S$ for $i \geq 1$, and $V_0 \cong S/\bigoplus_{i\geq 1} ke_i$ with all e_i 's acting as zero.
    (2) Show that V_0 is the only simple S-module not embeddable into S (so S is not a Kasch ring).

The next four exercises arise from a correspondence between the author and Carl Faith in June, 1998.

18. Show that a commutative Kasch ring R with $|\mathrm{Ass}(R)| < \infty$ (e.g., in case $\mathrm{u.dim}(R_R) < \infty$: see Exercise (6.2)) must be semilocal. Then construct a commutative Kasch ring that is not semilocal.

19. For a prime ideal $\mathfrak{p}$ in a commutative ring R, show that $\mathfrak{p} \in \mathrm{Ass}(R)$ implies that the localization $R_{\mathfrak{p}}$ is a Kasch ring, and conversely if $\mathfrak{p}$ is a f.g. ideal (e.g. if R is noetherian). Give an example to show that the converse need not hold if $\mathfrak{p}$ is *not* a f.g. prime ideal. (**Hint** (for the last part). Use (3.71) and Exercise (3.40G).)

20. Construct a commutative noetherian Kasch ring R with a prime ideal $\mathfrak{p}$ such that the localization $R_{\mathfrak{p}}$ is not a Kasch ring.

21. Let R be a commutative noetherian ring. If $R_{\mathfrak{m}}$ is Kasch for every maximal ideal $\mathfrak{m}$, show that R is Kasch. What if R is not assumed to be a noetherian ring?

22. Compute the left and right singular ideals of the ring R in Example (8.29)(6).

Chapter 4

Rings of Quotients

After developing enough module theory in the three previous chapters, the stage is now set for the study of the theory of rings of quotients. The present chapter is a general introduction to this theory, in the setting of noncommutative rings.

In a preamble section (§9), we discuss the general issues of "inverting" a given multiplicative set S of nonzero elements in a (possibly) noncommutative ring R. If R is a domain and $S = R\backslash\{0\}$, a related issue is that of embedding R into a division ring. Unfortunately, such embeddings need not always exist: in §9, we present Mal'cev's famous example of a domain that *cannot* be embedded in a division ring. Even if such embeddings exist, they may not be unique, as shown by an intriguing example of J. L. Fisher [71].[60]

In the second section (§10) of this chapter, we study Ore's localization theory, developed by Ore in the early 1930s. Here we find the necessary and sufficient conditions for constructing the (Ore) localization RS^{-1} with respect to a given multiplicative set $S \subseteq R$. Letting S be the multiplicative set of all non 0-divisors in R, in particular, we arrive at the notion of *right Ore rings*, which are the rings with a classical (total) right ring of quotients.

Section 11 introduces the important notion of right Goldie rings, and presents the landmark Goldie Theorem(s) proved by A. Goldie (and in part by L. Lesieur and R. Croisot) in the late 1950's. These theorems characterized the "right orders" in semisimple (resp. simple artinian) rings: such right orders are precisely the semiprime (resp. prime) right Goldie rings. On the one hand, these theorems led to various other embedding results of a similar spirit; on the other hand, since right noetherian rings are right Goldie, these theorems provided the first powerful tools for a new in-depth study of noetherian rings. Before the appearance of the Goldie theorems, the theory of noetherian rings was mainly confined to the commutative case (work of Noether, Krull, Zariski, and applications to algebraic geometry).

[60]There are two ring theorists named J. Fisher, and it is not always easy to distinguish their publications. The paper referred to here was written by James L. Fisher. However, in the author index of the AMS collected *Ring Theory Reviews* (1940-1979), this paper was listed under the name of Joe W. Fisher. Later in this Chapter, we shall present some results on nil multiplicative sets; our presentation there follows a paper of J. W. Fisher [70].

In the last three decades, thanks to the pioneering work of Goldie, the theory of noncommutative noetherian rings has also come to fruition.

A natural extension of Goldie's Theorems is the characterization of right orders in right artinian rings. This was successfully carried out by Robson and Small in the 1960s. In §12, we present such a characterization in the convenient form expounded by Warfield. Here we use in an essential way the idea of the "ρ-rank" of a module, which is a variant of Goldie's reduced rank introduced earlier in §7.

Throughout the text, we use the terms "quotients" and "fractions" more or less interchangeably. Most of the time, we speak of "rings of fractions", but "rings of quotients" is in such common use that it can hardly be ignored. As a rule, the reader should feel free to use the term "rings of quotients" as long as there is no possible confusion with "quotient rings" ($R/\mathfrak{A}$ for ideals $\mathfrak{A} \subseteq R$).

§9. Noncommutative Localization

§9A. "The Good"

One of the first things we learn in undergraduate algebra is the fact that, for any commutative domain R, we can formally invert the nonzero elements of R to form a unique quotient field (or field of fractions) for R. Later, in commutative algebra, we are taught the general procedure of localizing any commutative ring R at a multiplicative set S. This procedure yields a commutative ring R_S and a ring homomorphism $\varepsilon : R \to R_S$ such that $\varepsilon(s)$ is a unit in R_S for every $s \in S$, and ε is "universal" with respect to this property. Moreover, we have the following two key facts for ε and R_S:

(9.1a) *Every element in R_S has the form $\varepsilon(r)\varepsilon(s)^{-1}$ where $r \in R$ and $s \in S$.*

(9.1b) $\ker \varepsilon = \{r \in R : rs = 0$ *for some* $s \in S\}$ *(an ideal in R).*

The ring R_S is called the *localization* of R at S. To simplify the notation, we write the elements of R_S as fractions r/s or rs^{-1} (instead of $\varepsilon(r)\varepsilon(s)^{-1}$). We add fractions by taking common denominators, and multiply fractions by multiplying numerators and denominators. The classical case of embedding a commutative domain R into its quotient field corresponds to the localization of R at the multiplicative set $R\backslash\{0\}$.

In commutative algebra, localization provides one of the most powerful tools for proving theorems. Thus, in studying noncommutative rings, it is natural to ask first how much of the localization machinery can be made to work in the *noncommutative* setting. In this initial subsection, we shall give some preliminary answers to this question, mainly by looking at some key examples. The finer issues of localizing a ring R to get "sufficiently decent" rings of quotients will be taken up in more detail in later sections of this Chapter.

In studying the beginning part of the theory of noncommutative localization, we will already encounter new phenomena that are not present in the commutative

setting. Let us now discuss three of these phenomena; we shall refer to them as *The Good, The Bad*, and *The Ugly*. "The Good" is embodied in the fact that, for *any* multiplicative set S in *any* ring R, we can define a "universal S-inverting ring" R_S. At first sight this looks encouraging, but in the general setting we lose both of the properties (9.1a) and (9.1b). This greatly compromises the usefulness of R_S. After this comes "The Bad", a surprising example of a noncommutative domain R found by Mal'cev which cannot be embedded in *any* division ring. This suggests the need to find necessary and sufficient conditions for a domain to be embeddable in a division ring. But even for such embeddable domains, there may not be a *unique* "division ring of fractions". In the subsection dubbed "The Ugly", we'll see that, for the domain $A = \mathbb{Q}\langle u, v\rangle$, there exist embeddings $\varepsilon_n : A \to D_n$ where the D_n's are "minimal" division rings over $\varepsilon_n(A)$ for $n \geq 2$, but there is no isomorphism (or homomorphism) $f : D_m \to D_n$ for $m \neq n$ such that $f \circ \varepsilon_m = \varepsilon_n$. The free algebra $\mathbb{Q}\langle u, v\rangle$, therefore, has *infinitely many* essentially different "division rings of fractions".

We shall begin here with "The Good", since it is quite easy and completely general. By a multiplicative set in a ring R, we shall mean throughout the text a subset $S \subseteq R$ such that S is closed under multiplication, $0 \notin S$, and $1 \in S$. A homomorphism $\alpha : R \to R'$ is said to be *S-inverting* if $\alpha(S) \subseteq \mathrm{U}(R')$ (the group of units of the ring R').

(9.2) Proposition. *Given $S \subseteq R$ as above, there exists an S-inverting homomorphism ε from R to some ring, denoted by R_S, with the following universal property: For any S-inverting homomorphism $\alpha : R \to R'$, there exists a unique ring homomorphism $f : R_S \to R'$ such that $\alpha = f \circ \varepsilon$.*

(As usual, the universal property above guarantees the *uniqueness* of the data $\varepsilon : R \to R_S$. This is why we are justified in using the notation R_S for the receiving ring of the "universal S-inverting homomorphism" ε.)[61]

Proof. Fix a presentation of R by generators and relations. For each $s \in S$, adjoin a new generator s^* and two additional relations $\tilde{s}s^* = 1$, $s^*\tilde{s} = 1$, where $\tilde{s}$ is an element in the free $\mathbb{Z}$-algebra that maps to s in the given presentation. The new set of generators and relations defines a ring R_S, along with a ring homomorphism $\varepsilon : R \to R_S$. For each $s \in S$, the image of s^* in R_S provides an inverse for $\varepsilon(s)$, so $\varepsilon(S) \subseteq \mathrm{U}(R_S)$. The asserted universal property of ε follows quickly from the definition of R_S. □

(9.3) Example. Contrary to the commutative case, the "universal S-inverting ring" R_S may be the zero ring, even though $R \neq (0)$ and $0 \notin S$. For instance, let $R = \mathbb{M}_n(k)$ $(n \geq 2)$, where k is a nonzero ring, and let S be the multiplicative

[61]The Proposition is, in fact, true for *any* subset $S \subseteq R$. The hypotheses we assumed on S are imposed only for convenience, and to avoid trivial situations. (For instance, note that if $0 \in S$, then R_S would be the zero ring.)

set $\{1, E_{11}\}$, where E_{ij} denote the matrix units. Being an ideal in R, the kernel of $\varepsilon : R \to R_S$ has the form $\mathbb{M}_n(\mathfrak{A})$, where $\mathfrak{A}$ is an ideal in k (cf. *FC*–(3.1)). But $E_{11}E_{22} = 0$ implies that $E_{22} \in \ker \varepsilon$, so we have $1 \in \mathfrak{A}$, i.e., $\mathfrak{A} = k$. Therefore, ε is the zero map and $R_S = (0)$! (Here, R is not a domain. But even when R is a domain, R_S may still be equal to (0). For the construction of such an example, see Exercise 5.)

The preceding shows that the nature of the ring R_S may be rather unpredictable. In general, it is difficult to prove things about R_S, since $\varepsilon : R \to R_S$ may no longer have the properties (9.1a) and (9.1b). There is apparently no easy description for the kernel of ε; and, rather than looking like $\varepsilon(r)\varepsilon(s)^{-1}$, elements of R_S are *sums of words in* $\varepsilon(r)$ *and* $\varepsilon(s)^{-1}$, like

(9.4) $$\varepsilon(r)\varepsilon(s)^{-1}\varepsilon(r') + \varepsilon(s')^{-1}\varepsilon(r'')\varepsilon(s'')^{-1},$$

where $r, r', r'' \in R$, and $s, s', s'' \in S$. For instance, if we start with a domain R and take $S = R\backslash\{0\}$, we cannot be sure that $\varepsilon : R \to R_S$ is *injective*, nor can we guarantee that R_S is a *division ring*. (In this case, any nonzero summand in (9.4) is a unit in R_S; but this is a far cry from *the sum* being a unit, if it is nonzero.) In fact, in general, R_S may not even be a domain (cf. Exercise 5), so it is not clear either how we might try to repeat the construction.

§9B. "The Bad"

The above discussion brings us back to the quintessential question: *Can any domain be embedded in a division ring*? Let us now present Mal'cev's negative answer to this question below.

More generally, Mal'cev was concerned with the problem of embedding a cancellative semigroup H into a group G. (A cancellative semigroup is a semigroup in which both cancellation laws hold. *We shall assume that all semigroups considered have an identity element.*[62]) In a famous paper published in 1937, Mal'cev produced an example of a cancellative semigroup H that cannot be embedded into a group. He showed further that the semigroup algebra $\mathbb{Q}H$ is a domain. It then follows that this domain *cannot* be embedded in a division ring D, for if D were to exist, then H would be embeddable in the group of units $\mathrm{U}(D)$ of D!

The main observation that makes the above work possible is the following lemma.

(9.5) Lemma. *Let a, b, c, d, x, y, u, v be elements of a semigroup H. If H is embeddable into a group G, then*

(9.6) $$ax = by,\ \ cx = dy,\ \ au = bv \implies cu = dv \quad \text{in } H.$$

[62]Semigroups with 1 are usually called "monoids". However, this assumption is not really essential for what follows; we make this assumption throughout only for convenience.

Proof. Working in the group G, we have $b^{-1}a = yx^{-1} = d^{-1}c$ from the first two equations, and $b^{-1}a = vu^{-1}$ from the third equation. Therefore, $d^{-1}c = vu^{-1} \in G$, and hence $cu = dv \in H$. (Alternatively, as suggested by D. Moulton, $cu = cx \cdot x^{-1}a^{-1} \cdot au = dy \cdot y^{-1}b^{-1} \cdot bv = dv$ in G.) $\square$

If the equations in (9.6) seem a little mysterious, the following (purely formal) matrix interpretation should help. The three relations on the LHS of (9.6) may be expressed formally by the matrix equation:

$$(9.7) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & u \\ -y & -v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}.$$

By formal multiplication, "$*$" here is "$cu - dv$". Therefore, (9.6) may be expressed by saying that, if (9.7) holds, then the RHS of (9.7) must in fact be the zero matrix. Following Mal'cev, we prove next:

(9.8) Theorem. *There exists a cancellative semigroup H with elements a, b, c, d, x, y, u, v such that $ax = by$, $cx = dy$, $au = bv$, but $cu \neq dv$. In particular, H cannot be embedded in any group G.*

Proof. Let $\hat{H}$ be the free semigroup on the letters A, B, C, D, X, Y, U, V. For two words W and W', let us define $W \sim W'$ if W can be transformed into W' by a finite number of replacements of subwords of length 2 of the following kinds:

$$(9.9) \qquad AX \longleftrightarrow BY, \quad CX \longleftrightarrow DY, \quad AU \longleftrightarrow BV.$$

Clearly, "$\sim$" is an equivalence relation on words. Let H be the set of $\sim$-equivalence classes, and let w denote the class of a word $W \in \hat{H}$. The multiplication in $\hat{H}$ induces a multiplication in H that makes H into a semigroup. The classes $a, b, \cdots \in H$ of $A, B, \ldots$ now satisfy $ax = by$, $cx = dy$ and $au = bv$. But we *don't* have $cu = dv \in H$ since the word CU simply cannot be transformed into DV. The only thing that remains to be verified is the fact that H *does satisfy both of the cancellation laws*!

Let us say that a word $\hat{H}$ is *reduced* if it does not contain a subword AX, CX, or AU. Using the "forward" transformations in (9.9), it is clear that any word $W \in \hat{H}$ is $\sim$-equivalent to a unique reduced word. Equipped with this knowledge, let us now prove the left cancellation law:

$$(9.10) \qquad ww_1 = ww_2 \in H \Longrightarrow w_1 = w_2 \in H.$$

We may assume that w, w_1, w_2 are classes of reduced words W, W_1 and W_2. If WW_1 and WW_2 are both reduced, then we have $WW_1 = WW_2$; hence $W_1 = W_2 \in \hat{H}$ and $w_1 = w_2 \in H$. Now assume, say, WW_1 is *not* reduced. Let us examine a typical case, say,

$$W = \ldots LA, \qquad W_1 = XM_1N_1 \ldots .$$

In this case, the class ww_1 is represented by the reduced word $\ldots LBYM_1N_1 \ldots$. How about the class ww_2? If W_2 did not start with X or U, then WW_2 is already

reduced, and it is *not* $\ldots LBYM_1N_1\ldots$, which contradicts $ww_1 = ww_2$. If W_2 starts with U, then ww_2 is given by a reduced word of the form $\ldots LBV\ldots$, still contradicting $ww_1 = ww_2$. Thus, we must have $W_2 = XM_2N_2\ldots$, so that ww_2 is given by the reduced word $\ldots LBYM_2N_2\ldots$. But then we must have $M_1N_1\cdots = M_2N_2\ldots$, which implies that $W_1 = W_2 \in \hat{H}$ and hence $w_1 = w_2 \in H$. The other cases are similarly dealt with, and we can prove the right cancellation law in the same manner. □

(9.11) Theorem. *Let R be the semigroup algebra kH, where H is as in* (9.8), *and k is any domain. Then R is a domain, and R cannot be embedded into any division ring.*

Proof. In view of our earlier remarks, it suffices to prove that *R is a domain.* Suppose, instead, that there is an equation

$$(*) \qquad \left(\sum_i \alpha_i w_i\right)\left(\sum_j \alpha'_j w'_j\right) = 0 \in R,$$

where $\alpha_i \neq 0 \neq \alpha'_j$, and the w_i 's (resp. w'_j 's) are given by different reduced words W_i 's (resp. W'_j 's). Note that the "length" of an element in H is well-defined, since the transformations allowed in (9.9) are all length-preserving. We may, therefore, assume that all the words W_i (resp. W'_j) have the same length. (If otherwise, we just replace $\Sigma_i\, \alpha_i w_i$ by the subsum given by the terms of the longest length, and do the same for $\Sigma_j\, \alpha_j w'_j$.) In order to "cancel out" the class $w_1 w'_1$, we must have $w_1 w'_1 = w_i w'_j$ for some $i \neq 1$, $j \neq 1$. Since $W_1 \neq W_i$ and they have the same length, the only way for $w_1 w'_1 = w_i w'_j$ to be possible is that we have (say)

$$W_1 = K\ldots LA, \qquad W'_1 = XMN\ldots,$$
$$W_i = K\ldots LB, \qquad W'_j = YMN\ldots.$$

But then on the LHS of $(*)$ above, we have a term $\alpha_1\alpha'_j w_1 w'_j$ corresponding to the reduced word $K\ldots LAYMN\ldots$, which clearly *cannot* be cancelled out by any other term — a contradiction. □

The fact that $R = kH$ is a domain can be strengthened a bit further. As observed independently by Chihata and Vinogradov in 1953, the semigroup H can in fact be *ordered*; that is, there exists a total ordering "$<$" on the elements of H such that

$$(9.12(a)) \qquad \alpha < \beta \text{ in } H \implies \alpha\gamma < \beta\gamma \text{ and } \gamma\alpha < \gamma\beta \quad (\forall \gamma \in H).$$

Upon fixing such an ordering on H, a routine argument involving minimal support terms (cf. the proof of *FC*–(6.29)) shows that R is a domain. To construct an ordering on H is not too difficult. In fact, Chihata has shown that

$$(9.12(b)) \qquad x < u < d < c < y < b < a < v$$

extends *uniquely* to an ordering on H (satisfying (9.12(a))). The details of this construction will be left to the reader: see Exercise 6. Summing up, the ordered (and hence cancellative) semigroup $(H, <)$ cannot be embedded in a group; a fortiori, $\mathbb{Q}H$ cannot be embedded in a division ring. On the other hand, an easy modification of the Mal'cev-Neumann construction in *FC*–§14 shows that $\mathbb{Q}H$ can be embedded in an ordered "formal power series ring".

In another important paper in 1939, Mal'cev also solved the general problem of finding the *necessary and sufficient* conditions for a semigroup to be embeddable in a group. These conditions are in the form of an infinite sequence of implication statements of the kind:

$$A_1, A_2, \ldots, A_n \implies A,$$

where A and A_i 's are equations in the semigroup. Mal'cev called these implication statements "quasi-identities". The simplest quasi-identities required for embeddability are the two cancellation laws. Another quasi-identity required is (9.6), and so forth. However, this study belongs more properly to the domain of Universal Algebra,[63] so we shall not go more deeply into this matter here.

The example of a domain R that is not embeddable in a division ring given in (9.11) is such that the (multiplicative) semigroup $R\backslash\{0\}$ is already not embeddable in a group. This led Mal'cev to ask if there also exist domains R for which $R\backslash\{0\}$ *is* embeddable in a group, and yet R is not embeddable in a division ring. It took nearly thirty years before three mathematicians, L. A. Bokut', A. J. Bowtell, and A. A. Klein, independently and almost simultaneously found such examples.

The point about the existence of such examples is that, besides the semigroup "quasi-identities" such as (9.6), there are also other, more ring-theoretic,[64] necessary conditions for the embeddability of a domain into a division ring. For instance, if a domain R is embeddable in a division ring D, then R must satisfy the following conditions:

(9.13) *R must have* IBN (since D has IBN).

(9.14) *R must be stably finite* (since D is stably finite).

(9.15) (Klein's Nilpotence Condition) *For any nilpotent matrix $A \in \mathbb{M}_n(R)$, we have $A^n = 0$* (since, by linear algebra, the same statement holds for nilpotent matrices over the division ring D).

(9.16) *$\mathbb{M}_n(R)$ must satisfy* ACC *(and* DCC*) on left and right annihilators* (since $\mathbb{M}_n(D)$ does; cf. (6.61)).

These four necessary conditions for the embeddability of R are related as follows. First, (9.14) implies (9.13), by (1.8). Second, each of (9.15), (9.16) implies that

[63] In the terminology of Universal Algebra, the class of semigroups embeddable in groups is a "quasi-variety".

[64] A "ring-theoretic condition" means here a condition expressed not just in terms of multiplication, but in terms of *both* addition and multiplication.

$\mathbb{M}_n(R)$ is Dedekind-finite, by Exercise (1.11) and (6.60), respectively. Since this holds for all n, R is stably finite, by definition. It seems, indeed, that (9.15) is stronger than most of the known conditions for a domain R to be embeddable in a division ring. As for the sufficiency of (9.15) for embedding problems, Klein has shown that (9.15) does imply that the semigroup $R\backslash\{0\}$ embeds into a group. However, Bergman produced an example later to show that (9.15) is still not sufficient for the embeddability of the domain R into a division ring. Bergman's example, incidentally, also answered affirmatively the question raised by Mal'cev mentioned in the second paragraph before (9.13).

The necessary conditions assembled above enable one to come up with other "types" of nonembeddable domains. For instance, Shepherdson's construction (Exercise (1.18)) of a k-domain R over a field k with $\mathbb{M}_2(R)$ not Dedekind-finite gives such an example. While Shepherdson's example has IBN (it has a homomorphism into the field k), there *are* examples of domains that do not have IBN. The construction is again "generic". For a pair of integers $n > m \geq 1$, let $R = R_{m,n}$ be the ring with $2mn$ generators

(9.17) $$\{a_{ij},\ b_{k\ell}\} \qquad (1 \leq i, \ell \leq m;\ \ 1 \leq j, k \leq n),$$

and relations dictated by the matrix equations

(9.17′) $$(a_{ij})(b_{k\ell}) = I_m, \qquad (b_{k\ell})(a_{ij}) = I_n.$$

Over R, we have $(R^m)_R \cong (R^n)_R$. Therefore R does not have IBN, and there is no homomorphism from R to any ring with IBN, let alone a division ring. On the other hand, Cohn [66] has shown that $R_{m,n}$ is a domain whenever $m > 1$.[65] Not surprisingly, Cohn's method is a generalization of the method used in the proofs of (9.8) and (9.11). However, it would take us too far afield to discuss the details here.

§9C. *"The Ugly"*

While not every domain can be embedded in a division ring, various special classes of domains have been proved to be embeddable. For instance, we have the following nice result which will be proved in the next section (see (10.22)):

(9.18) Theorem. *Any right noetherian domain can be embedded in a division ring. In particular, any* PRID *(principal right ideal domain) can be embedded in a division ring.*

We have stated this theorem here with a specific purpose in mind. For a domain A, let us say that a division ring D is a *division hull* of A if there is a given inclusion map $A \hookrightarrow D$ such that D is generated as a division ring by A (i.e. there is no

[65]For $m = 1$, we have $b_{k1}a_{1j} = 0$ for all $k \neq j$, so we do not expect $R_{1,n}$ to be a domain.

division ring D_0 such that $A \subseteq D_0 \subsetneq D$).[66] Clearly, a domain A has a division hull iff A can be embedded in a division ring. Two division hulls of A are regarded as "the same" if they are isomorphic over A. Using Theorem (9.18), we shall carry out the promised construction, due to J. L. Fisher, of an example of a domain A that has *infinitely many* mutually different division hulls. In fact, we shall take A to be $C\langle u, v\rangle$, the free algebra on two generators u, v over any field C.

It is by no means clear that $C\langle u, v\rangle$ can be embedded in a division ring. Since $C\langle u, v\rangle$ is neither left nor right noetherian, Theorem (9.18) does not apply directly. In order to get embeddings of $C\langle u, v\rangle$ into division rings, we shall make crucial use of Hilbert's *skew polynomial rings* (cf. *FC*–(1.7)). Recall that, for any ring k equipped with an endomorphism σ, the skew polynomial ring $k[x; \sigma]$ consists of *left polynomials* of the form $\sum a_i x^i$ ($a_i \in k$) which are multiplied using "Hilbert's twist" $xa = \sigma(a)x$ (for every $a \in k$). The following basic fact will prove to be important for the constructions we have in mind.

(9.19) Lemma. *Let $\sigma : k \to k$ be an injective endomorphism of the ring k, and let $R = k[x; \sigma]$. If $\{t_i : i \in I\} \subseteq k$ are right linearly independent over $\sigma(k)$, then $\{t_i x : i \in I\} \subseteq R_R$ are right linearly independent over R.*

Proof. Suppose $\sum_i (t_i x) f_i = 0$, where $f_i \in R$ are almost all 0. Write $f_i = \sum_j a_{ij} x^j$ ($a_{ij} \in k$). Then

$$0 = \sum_i t_i x \sum_j a_{ij} x^j = \sum_j \Big(\sum_i t_i \sigma(a_{ij}) \Big) x^{j+1}.$$

Therefore, for each j, we have $\sum_i t_i \sigma(a_{ij}) = 0$, and so $\sigma(a_{ij}) = 0$ for all i, j. Since σ is injective, it follows that $f_i = \sum_j a_{ij} x^j = 0$ for all i. □

If k is a division ring, then any endomorphism $\sigma : k \to k$ is automatically injective, from which we can see easily that $k[x; \sigma]$ is a domain. In addition, the usual Euclidean algorithm argument can be used to show that $k[x; \sigma]$ is a PLID (cf. *FC*–(1.25)). Therefore, we have from (9.18):

(9.20) Corollary. *If σ is an endomorphism of a division ring k, then $k[x; \sigma]$ can be embedded in a division ring.*

Therefore, to embed $C\langle u, v\rangle$ into a division ring, we might try to embed it first into $k[x; \sigma]$ where k is a division ring (or even a field). This will be accomplished with the help of the following beautiful observation:

(9.21) Jategaonkar's Lemma. *Suppose a, b are two elements in a ring R that are right linearly independent over R. Let $C \subseteq R$ be any nonzero subring whose*

[66] In the case when A is a commutative domain, this simply means that D is a quotient field of A.

elements commute with a and b. Then the subring of R generated by a, b over C is a free C-ring on a, b.

Proof. If a, b are not free over C, choose a nonconstant polynomial $f(x, y) \in C\langle x, y\rangle$ of the least degree n such that $f(a, b) = 0$. Express f in the form $\alpha + xg(x, y) + yh(x, y)$ ($\alpha \in C$), where, say, $g(x, y) \neq 0$. From

(9.22) $$0 = f(a, b)b = a(g(a, b)b) + b(\alpha + h(a, b)b),$$

we see that $g(a, b)b = 0$. Now write g in the form $\beta + xp(x, y) + yq(x, y)$ ($\beta \in C$). Then we have:

(9.23) $$\deg g \leq n - 1, \quad \deg p \leq n - 2, \quad \deg q \leq n - 2, \quad \text{and}$$

(9.24) $$0 = g(a, b)b = a(p(a, b)b) + b(\beta + q(a, b)b).$$

The latter implies that $p(x, y)y$ and $\beta + q(x, y)y$ are both satisfied by a, b. Using (9.23), we see that $p(x, y) = q(x, y) = 0$ and $\beta = 0$, contradicting $g(x, y) \neq 0$. □

Now let C be a fixed field, and k be the rational function field $C(t)$. Let σ_n be the endomorphism of k such that σ_n is the identity on C, and $\sigma_n(t) = t^n$. *We shall assume that $n > 1$*, so that σ_n is not surjective. Let $R_n = k[x; \sigma_n]$. Then, since $\{1, t\}$ in k are linearly independent over $\sigma_n(k) = C(t^n)$, (9.19) implies that $\{x, tx\} \subseteq R_n$ *are right linearly independent over R_n*. Applying (9.21) with $a = x$ and $b = tx$ in R_n, we see[67] that there is a C-embedding $C\langle u, v\rangle \hookrightarrow R_n$ defined by $u \mapsto x$, $v \mapsto tx$. By (9.20), we can embed R_n into a division hull, say, D_n. Composing the two embeddings, we then obtain a C-embedding

(9.25) $$\varepsilon_n : C\langle u, v\rangle \hookrightarrow D_n, \quad \text{with} \quad \varepsilon_n(u) = x, \ \varepsilon_n(v) = tx.$$

Note that D_n *is also a division hull of* $C\langle u, v\rangle$, for, if E is a division subring of D_n containing $\mathrm{im}(\varepsilon_n)$, then E contains x, tx, and hence t. But then E contains $k = C(t)$ and $k[x; \sigma_n] = R_n$, so $E = D_n$. Next, note that

(9.26) $$(\varepsilon_n(u)^{-1}\varepsilon_n(v))^n = (x^{-1}tx)^n = x^{-1}t^n x = x^{-1}xt = \varepsilon_n(v)\varepsilon_n(u)^{-1} \in D_n.$$

We can now complete the construction of "The Ugly" by proving the following.

(9.27) Theorem. *For $n \neq m$ (both > 1), there does not exist a ring homomorphism $f : D_m \to D_n$ such that $f \circ \varepsilon_m = \varepsilon_n$ (so that D_n and D_m give essentially different division hulls of $C\langle u, v\rangle$).*

Proof. Suppose f exists. Applying it to the equation

(9.28) $$(\varepsilon_m(u)^{-1}\varepsilon_m(v))^m = \varepsilon_m(v)\varepsilon_m(u)^{-1} \in D_m \quad \text{(see (9.26))},$$

[67]Note that, since σ_n is the identity on C, $xc = \sigma_n(c)x = cx$ for every $c \in C$, so C is in the center of R_n.

we get

$$(9.29) \qquad (\varepsilon_n(u)^{-1}\varepsilon_n(v))^m = \varepsilon_n(v)\varepsilon_n(u)^{-1} = (\varepsilon_n(u)^{-1}\varepsilon_n(v))^n \in D_n.$$

Since $\varepsilon_n(u)^{-1}\varepsilon_n(v) = x^{-1}tx$ in D_n, (9.29) gives $t^{n-m} = 1 \in D_n$, a contradiction. □

The embeddability of $C\langle u, v\rangle$ in a division ring is an important fact, even though there is no "uniqueness" in such an embedding. Recalling that any free algebra $C\langle X\rangle$ with X countable can be embedded in $C\langle u, v\rangle$ (cf. *FC*–(1.2)), we see that such $C\langle X\rangle$ can be embedded in a division ring. By a different method, we have in fact shown, in *FC*–(14.25), that $C\langle X\rangle$ can be embedded in a division ring, *for any set X and any division ring C*.

§9D. An Embedding Theorem of A. Robinson

We shall now close §9 by proving a couple of additional embedding results that are related to the notion of strongly (von Neumann) regular rings. A ring R is called *strongly regular* if, for any $a \in R$, there exists $x \in R$ such that $a = a^2x$. In order not to repeat ourselves, we quote the following result from *FC*–Exercises (12.5) and (12.6):

(9.30) Theorem. *A ring R is strongly regular iff it is von Neumann regular and reduced. Such a ring is always a subdirect product of division rings.*

We do not really need the first statement of this theorem below, and, for the purposes of proving the next result, the second statement is needed only to the extent that a nonzero strongly regular ring admits at least one homomorphism into a division ring. Let us now state and prove the following embedding result of A. Robinson. The original proof of this result made use of ultrafilters; the proof presented below avoiding the use of ultrafilters follows Cohn [71].

(9.31) Theorem. *If a domain R can be embedded in a direct product of division rings D_i $(i \in I)$, then R can be embedded in a division ring.*

Proof. Let $P = \prod_i D_i$, and write each element $x \in P$ in the form $(x_i)_{i\in I}$. For such $x \in P$, we define an element $x^* = (x_i^*)_{i\in I} \in P$ by: $x_i^* = 0$ if $x_i = 0$, and $x_i^* = x_i^{-1}$ if $x_i \neq 0$. Also, we define

$$(9.32) \qquad \mathfrak{A}_x = \{(a_i)_{i\in I} : \forall i \in I,\ x_i \neq 0 \Longrightarrow a_i = 0\}.$$

Clearly, $\mathfrak{A}_x$ is an ideal in P (in fact, $\mathfrak{A}_x = \text{ann}_\ell(x) = \text{ann}_r(x)$), and we have

$$(9.33) \qquad 1 - xx^* \in \mathfrak{A}_x,$$

$$(9.34) \qquad (\mathfrak{A}_x + \mathfrak{A}_y + \cdots + \mathfrak{A}_z)xy\cdots z = 0.$$

Viewing R as a subring of P, let $\mathfrak{A} = \sum \mathfrak{A}_x$ where x ranges over $R\backslash\{0\}$. Then $\mathfrak{A} \neq P$, for otherwise $1 \in \mathfrak{A}_x + \mathfrak{A}_y + \cdots + \mathfrak{A}_z$ for suitable $x, y, \ldots, z \in R\backslash\{0\}$,

and (9.34) would give $xy \cdots z = 0$, contradicting the fact that R is a domain. Since each D_i is strongly regular, so are $P = \prod_i D_i$ and $P/\mathfrak{A}$, and hence (by what we said after the statement of (9.30)) there exists a ring homomorphism f from P to a suitable division ring D, with $f(\mathfrak{A}) = 0$. Now, for any $x \in R\backslash\{0\}$, (9.33) implies that

$$0 = f(1 - xx^*) = 1 - f(x)f(x^*), \quad \text{so } f(x) \neq 0 \in D.$$

Therefore $f|R$ gives the desired embedding of R into a division ring. □

Using the full version of the second statement of (9.30), we have the following consequence of (9.31).

(9.35) Corollary. *If a domain R can be embedded in a strongly regular ring R', then R can be embedded in a division ring.*

Proof. By (9.30), R' is a subdirect product of a family of division rings D_i $(i \in I)$. In particular, R' (and hence R) can be embedded into $\prod_{i \in I} D_i$. Since R is a domain, (9.31) implies that R can be embedded in a division ring. □

In view of this Corollary and Mal'cev's result, we see that not every domain can be embedded in a strongly regular ring. By contrast, we shall prove in a future section (see (13.38)′) that every domain R can be embedded in a (simple, right self-injective) von Neumann regular ring $Q^r_{\max}(R)$ — the maximal right ring of quotients of R.

There is also another important application of (9.35). Using it, one can show that the class of domains embeddable in division rings can be defined by an (infinite) system of "ring-theoretic quasi-identities" (see Cohn [77: p. 5]).

Exercises for §9

1. Show that, for any multiplicative set $S \subseteq R$, the universal S-inverting homomorphism $\varepsilon : R \to R_S$ is injective iff R can be embedded into a ring in which all elements of S have inverses.

2. Let S, S' be, respectively, multiplicative sets in the rings R, R', which give rise to the ring homomorphisms $\varepsilon : R \to R_S$ and $\varepsilon' : R' \to R'_{S'}$. For any ring homomorphism $f : R \to R'$ such that $f(S) \subseteq S'$, show that there is a unique ring homomorphism $f_* : R_S \to R'_{S'}$ such that $f_*\varepsilon = \varepsilon' f$.

3. (Cohn) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$ be matrices over a ring T in which b and x are units. If $L := MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$, show that $L = 0$. Does the conclusion hold if one of b, x fails to be a unit in T?

4. *True or False*: The kernel of the universal S-inverting homomorphism $\varepsilon : R \to R_S$ is generated as an ideal by the set
$$A = \{r \in R : s'rs = 0 \text{ for some } s, s' \in S\}.$$

5. Construct a domain B with a multiplicative set S such that $B_S = 0$. (**Hint.** Let $R = kH$ be as in (9.11), and let $B = R/\mathfrak{A}$ where $\mathfrak{A}$ is the ideal of R generated by $cu - dv + 1$. Then, for any multiplicative set $S \subseteq B$ containing b and x, we have $B_S = 0$.)

6. (Chihata, Vinogradov) Show that Mal'cev's semigroup H constructed in (9.8) with generators $\{a, b, c, d, u, v, x, y\}$ can be ordered as follows. First order the eight generators as in (9.12)(b). Then define an order relation "$<$" on H first according to length, and then "lexicographically" for reduced words. In other words, if $\alpha = a_1 \cdots a_m$, $\beta = b_1 \cdots b_n$ are reduced words, where the a_i 's and b_j 's are generators, we define $\alpha < \beta$ if $m < n$, or if $m = n$ and there exists $t \leq m$ such that $a_i = b_i$ for $i < t$, and $a_t < b_t$ in the ordering of (9.12)(b). The heart of the Exercise is to show that the ordering axiom (9.12)(a) is satisfied.

§10. Classical Rings of Quotients

§10A. Ore Localizations

In §10, we continue to write S for a multiplicative set in a ring R, so we have $S \cdot S \subseteq S$, $1 \in S$, and $0 \notin S$. The ring R_S receiving the universal S-inverting homomorphism $\varepsilon : R \to R_S$ is too difficult to work with, since elements of R_S have very complicated forms (cf. (9.4)), and we have little control over $\ker \varepsilon$. What we would like to do in this section is to introduce additional conditions on S so that we can form simpler, "classical" rings of fractions. The following definition sets forth the features of the kind of classical rings of fractions we would like to form (cf. (9.1a) and (9.1b)).

(10.1) Definition. A ring R' is said to be a *right ring of fractions*[68] (with respect to $S \subseteq R$) if there is a given ring homomorphism $\varphi : R \to R'$ such that:

(a) φ is S-inverting
(b) Every element of R' has the form $\varphi(a)\varphi(s)^{-1}$ for some $a \in R$ and $s \in S$.
(c) $\ker \varphi = \{r \in R : rs = 0 \text{ for some } s \in S\}$.

(10.2) Remark. Contrary to the situation with R_S (cf. (9.3)), we have always $R' \neq 0$ here, in view of (c).

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[68] Alternatively "right ring of quotients".

But of course we have no reason to expect such a nice right ring of fractions to exist. In fact, *if R' does exist*, we can quickly deduce two necessary conditions on S, as follows.

(10.3) *For any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$.* (We refer to this property by saying that S is *right permutable*, or that S is a *right Ore set*.) To prove this property, write $\varphi(s)^{-1}\varphi(a)$ in the form $\varphi(r)\varphi(s')^{-1}$, where $r \in R$ and $s' \in S$. Then $\varphi(as') = \varphi(sr)$, so $(as' - sr)s'' = 0$ for some $s'' \in S$ by (10.1)(c). Therefore, we have $as's'' = srs'' \in aS \cap sR$.

(10.4) *For $a \in R$, if $s'a = 0$ for some $s' \in S$, then $as = 0$ for some $s \in S$.* (We refer to this property by saying that S is *right reversible*.) To prove this property, note that $s'a = 0$ implies $\varphi(s')\varphi(a) = 0$. Hence $\varphi(a) = 0$, and (10.1)(c) implies that $as = 0$ for some $s \in S$.

(10.5) Definition. If the multiplicative set $S \subseteq R$ is both right permutable and right reversible, we shall say that S is a *right denominator set*.

We come now to the first major result in this section, which is due to Ore, Asano and others. Ore started the investigation of noncommutative localization in the early 1930s by proving the theorem below for R a domain and $S = R \backslash \{0\}$. Asano and others extended Ore's theory to more general rings. According to P. M. Cohn, Noether was also aware of the key ideas underlying the following theorem, but did not publish them in her writings.[69]

(10.6) Theorem. *The ring R has a right ring of fractions with respect to S iff S is a right denominator set.*

Of course we have already proved the "only if" part (as a motivation for Definition (10.5)). In the following, we shall assume that S is a right denominator set, and construct a right ring of fractions denoted by RS^{-1}. The construction is not very hard; unfortunately, the detailed verifications showing that the construction really gives a ring are very tedious. To save space (and also not to bore our readers to tears), we shall suppress almost all of the details. This will enable us to focus more on the key ideas of the construction instead.

Since elements of RS^{-1} will be right fractions of the form "as^{-1}" ($a \in R$, $s \in S$), we start the construction by working with $R \times S$. We define a relation "$\sim$"

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[69] I made this remark in my second Trjitzinsky Lecture (on ring theory) given at the University of Illinois on April 14, 1998. After the lecture, my friend and collaborator Bruce Reznick came up and showed me a new anagram that he had just composed: it was "Noether" $\rightarrow$ "Then Ore"! I loved this anagram (and found it rather uncanny); however, its composition did not seem to reflect well on my Trjitzinsky Lectures!

on $R \times S$ as follows:

(10.7) $$(a, s) \sim (a', s') \text{ (in } R \times S) \text{ iff there exist } b, b' \in R \text{ such that } sb = s'b' \in S \text{ and } ab = a'b' \in R.$$

Intuitively, the condition means that after we "blow up" s and s' to the common denominator $sb = s'b' \in S$, the numerators ab and $a'b'$ are the same. (This is the time-honored method for checking that two fractions are equal.) Notice that although $sb = s'b' \in S$, b and b' themselves need not belong to S.

We claim that "$\sim$" is an equivalence relation on $R \times S$. Reflexivity and symmetry need no verification, so let us just prove transitivity below. Assume that $(a, s) \sim (a', s')$ as in (10.7), and also that $(a', s') \sim (a'', s'')$, so that we have $c, c' \in R$ with $s'c = s''c' \in S$, and $a'c = a''c' \in R$. From $(s'c)S \cap (s'b')R \neq \emptyset$, there exist $r \in R$ and $t \in S$ such that $s'b'r = s'ct \in S$. Using right reversibility, we have $b'rt' = ctt'$ for some $t' \in S$. Now

$$sbr = s'b'r = s'ct = s''c't \in S \implies s(brt') = s''(c'tt') \in S,$$
$$a(brt') = a'b'rt' = a'ctt' = a''(c'tt'),$$

so we have checked that $(a, s) \sim (a'', s'')$.

In (10.7), if we let $b' = 1$, we see that $(a, s) \sim (ab, sb)$ as long as $sb \in S$. Therefore, we can think of "$\sim$" as the best equivalence relation which "identifies" (a, s) with (ab, sb) ($\forall a \in R,\ s \in S,\ sb \in S$). This remark enables us to work with "$\sim$" very efficiently.

We need a notation for the equivalence class of (a, s). In anticipation of our goal, we write a/s or as^{-1} for this equivalence class. The set of all equivalence classes will be denoted by RS^{-1}; of course as^{-1} is so far only a formal expression in RS^{-1}.

To define addition in RS^{-1}, we observe that any two "fractions" a_1/s_1, a_2/s_2 can be brought to a common denominator. More formally, from $s_1S \cap s_2R \neq \emptyset$, we get elements $r \in R, s \in S$ such that $s_2r = s_1s \in S$, so now $a_1/s_1 = a_1s/s_1s$, and $a_2/s_2 = a_2r/s_2r$. We can then define

(10.8) $$a_1/s_1 + a_2/s_2 = (a_1s + a_2r)/t \quad \text{where} \quad t = s_1s = s_2r.$$

After showing that this is a well-defined binary operation on RS^{-1}, one can go ahead to show that $(RS^{-1}, +)$ is an additive group, with zero element $0/1$. We shall not present the details here, but note quickly that $\varphi(a) = a/1$ gives a group homomorphism $\varphi : R \to RS^{-1}$ with

(10.9) $$\ker \varphi = \{a \in R : (a, 1) \sim (0, 1)\} = \{a \in R : as = 0 \text{ for some } s \in S\},$$

as we had hoped. We also note in passing that, in connection with (10.8), *any finite number of fractions can be brought to a common denominator*, by using the permutability property together with induction.

So far we have used the permutability condition (10.3) only in the case when *both* a and s are in S. We shall need the full version of (10.3) in the next step, when we try to define multiplication on RS^{-1}. In order to multiply a_1/s_1 with a_2/s_2, we

use $s_1 R \cap a_2 S \neq \emptyset$ to find $r \in R$ and $s \in S$ such that $s_1 r = a_2 s$. Then we define

(10.10) $$(a_1/s_1)(a_2/s_2) = (a_1 r)/(s_2 s),$$

keeping in mind that $(a_1 s_1^{-1})(a_2 s_2^{-1})$ should be

$$a_1(s_1^{-1}a_2)s_2^{-1} = a_1(rs^{-1})s_2^{-1} = a_1 r(s_2 s)^{-1}.$$

Again, a substantial amount of work is needed in checking that (10.10) gives a well-defined multiplication on RS^{-1}, and finally, that $(RS^{-1}, +, \times)$ is a ring. After understanding the key ideas in the definition (10.10), however, carrying out all the verification steps is not much more than an exercise in patience. We shall, therefore, suppress all details here. Note that $1/1$ is the multiplicative identity in RS^{-1}, and that the map φ defined just before (10.9) is clearly a ring homomorphism from R to RS^{-1}. Also $1/s$ ($s \in S$) is the inverse of $\varphi(s) = s/1$, so φ is S-inverting. Finally, we see easily that $a/s = \varphi(a)\varphi(s)^{-1}$. Recalling (10.9), we have now shown that RS^{-1} is a right ring of fractions of R with respect to S, completing the proof of (10.6).

(10.11) Corollary. *If S is a right denominator set, then $\varphi : R \to RS^{-1}$ is a universal S-inverting homomorphism. In particular, there is a unique isomorphism $g : R_S \to RS^{-1}$ such that $g \circ \varepsilon = \varphi$, where $\varepsilon : R \to R_S$.*

Proof. It suffices to prove the first statement. Let $\alpha : R \to T$ be any S-inverting homomorphism. We define $f : RS^{-1} \to T$ by

(10.12) $$f(a/s) = \alpha(a)\alpha(s)^{-1} \qquad (a \in R,\ s \in S).$$

If $b \in R$ is such that $sb \in S$, then $\alpha(s)\alpha(b) = \alpha(sb)$ is a unit in T, so $\alpha(b)$ is also a unit in T. But then

$$\alpha(ab)\alpha(sb)^{-1} = \alpha(a)\alpha(b)\alpha(b)^{-1}\alpha(s)^{-1} = \alpha(a)\alpha(s)^{-1}.$$

This shows that $f : RS^{-1} \to T$ is well-defined. From (10.8) and (10.12), we can show easily that f is a ring homomorphism, with $f \circ \varphi = \alpha$. Finally, f as defined in (10.12) is clearly the only homomorphism from RS^{-1} to T satisfying $f \circ \varphi = \alpha$, since $a/s = \varphi(a)\varphi(s)^{-1} \in RS^{-1}$. □

(10.13) Remark. Since the verification for $(RS^{-1}, +, \times)$ to be a ring is not easy, one might wonder if one could bypass this verification by using the ring structure on R_S instead. In view of (10.3), one can show easily (as in the construction of RS^{-1}) that every element of R_S can be expressed in the form $\varepsilon(a)\varepsilon(s)^{-1}$, where $a \in R$ and $s \in S$. *However, we have no direct method for computing* ker ε, so we cannot conclude that R_S is a right ring of fractions, prior to the explicit (and laborious) construction[70] of RS^{-1}.

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[70]There are, indeed, some slick methods available (for instance, using the maximal ring of quotients to be introduced later) by which we can lighten the task of verifying that RS^{-1}

Of course, we also have the notions of "left permutability", "left reversibility", and "left denominator set". If S is a left denominator set, the *left* ring of fractions of R with respect to S is denoted by $S^{-1}R$. From (10.11) and its corresponding left version, we deduce the following result.

(10.14) Corollary. *If both RS^{-1} and $S^{-1}R$ exist, then $RS^{-1} \cong S^{-1}R$ ($\cong R_S$) over R.*

§10B. Right Ore Rings and Domains

To begin this subsection, let us consider some particular choices of the multiplicative set $S \subseteq R$.

(10.15) If S is *central* in R, then S is clearly both a left and a right denominator set, and we can safely identify $S^{-1}R$ with RS^{-1}. We speak of $S^{-1}R = RS^{-1}$ as a "central localization" of R. In this case, we have in fact $RS^{-1} \cong R \otimes_C CS^{-1}$, where C is the center of R.

(10.16) We say that an element $s \in R$ is *regular* if it is neither a left 0-divisor nor a right 0-divisor. If S consists only of regular elements of R, then S is clearly left and right reversible.

(10.17) Let S be the multiplicative set of *all* regular elements. We say that R is a *right Ore ring* iff S is right permutable, iff RS^{-1} exists (by virtue of (10.16) and (10.6)). In this case, we speak of RS^{-1} as the *(total) classical right ring of quotients of* R, and denote it by $Q^r_{cl}(R)$. The left analogues of these notions are defined similarly. If R is both left and right Ore, we shall say that R is an *Ore ring*: in this case, $Q^r_{cl}(R) = Q^\ell_{cl}(R)$ by (10.14). For instance, if $S \subseteq \mathrm{U}(R)$ (R is called a *classical ring* in this case; see (11.4)), then R is clearly an Ore ring, with $Q^r_{cl}(R) = Q^\ell_{cl}(R) = R$. In particular, *any von Neumann regular ring is an Ore ring* (see (11.6)(1) below).

(10.18) *Any commutative ring R is an Ore ring*, according to (10.15).

(10.19) *Let R be a domain and $S = R\backslash\{0\}$*. In this case, the "right permutable" condition (10.3) on S may be re-expressed in the equivalent form:

$$(*) \qquad aR \cap bR \neq (0) \quad \text{for} \quad a, b \in R\backslash\{0\}.$$

This is called the *(right) Ore condition* on R (since it first appeared in the seminal paper [Ore: 31]). Thus, the domain R is right (resp. left) Ore iff R satisfies the right (resp. left) Ore condition. For instance, any division ring is an Ore domain. For other examples, see (10.23), (10.26), (10.28), (10.30), and so on. Some relations

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is a ring. However, these methods do not seem to be in keeping with the classical spirit of the results in this section.

between right Ore domains and the notions of injectivity, flatness, and rank are given in Exercises 20–23 below.

To prepare our way for §11, it is convenient to introduce the following definition.

(10.20) Definition. Let $R \subseteq Q$ be rings. We say that R is a *right order* in Q if (1) every regular element of R is a unit in Q, and (2) every element of Q has the form as^{-1}, where $a \in R$, and s is a regular element of R. Left orders are defined similarly. If R is both a left and right order in Q, we shall simply say R is an order in Q.

Using this terminology, we deduce quickly the following result.

(10.21) Proposition. *The ring R is right Ore iff it is a right order in some ring Q. In this case, $Q \cong Q^r_{cl}(R)$ over R. If, moreover, R is a domain, then Q is a division ring, and up to a unique R-isomorphism, it is the only division hull of R.*

Let us dwell a little more on the case of domains. The formulation of the right Ore Condition $(*)$ (in (10.19)) for domains enables us to make a connection between this section and the section on uniform dimensions.

(10.22) Theorem (Goldie). *For any domain R, the following are equivalent:*

(1) *R is a right Ore domain.*
(2) $\text{u.dim}(R_R) = 1$.
(3) $\text{u.dim}(R_R) < \infty$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) are obvious. We finish by showing that (3) $\Longrightarrow$ (1). Assume that there exist $a, b \in R\backslash\{0\}$ such that $aR \cap bR = (0)$. Following A. Goldie, we show that $\{a^i b : i \geq 0\}$ are right R-linearly independent. Indeed, if $\sum_{i\geq 0} a^i br_i = 0$ where $r_i \in R$ are almost all zero, then

$$br_0 + a(br_1 + abr_2 + \cdots) = 0 \Longrightarrow r_0 = 0 \quad \text{and} \quad br_1 + abr_2 + \cdots = 0.$$

Repeating this argument, we see that all $r_i = 0$. Therefore, R contains $\bigoplus_{i\geq 0} a^i bR$ (a free right module of countably infinite rank), so we have $\text{u.dim}(R_R) = \infty$. □

Note that the equivalence of (2) and (3) is a special feature for domains, and is false in general, even for semisimple rings. For instance, if R is direct product of m division rings, then $\text{u.dim}(R_R) = m$, which can be any positive integer.

(10.23) Corollary. *If R is a right noetherian domain, then R is right Ore. In particular, $Q^r_{cl}(R)$ exists, and it is the unique division hull of R.*

Proof. The noetherian module R_R cannot contain an infinite direct sum of nonzero submodules (cf. (6.7)(1)). □

Needless to say, the converse to the first statement in (10.23) is false: any commutative domain R is Ore, but R need not be noetherian.

Although the notion of a right Ore domain has been known since the early 1930s, the fact that a right noetherian domain is right Ore was apparently not noticed until much later.[71] The following is another result of a similar spirit.

(10.24) Corollary. *A domain R is called a right Bézout domain if every finitely generated right ideal of R is principal. Such a domain is always right Ore.*

Proof. Assume that $aR \cap bR = 0$ where $a, b \in R\backslash\{0\}$. Choose $c \in R$ such that $cR = aR \oplus bR$. Then $c = ar + bs$ and $b = cd$ for suitable $r, s, d \in R$. Right multiplying the former equation by d, we get $b = ard + bsd$, so $rd = 0$. This implies that $r = 0$, so $c = bs$ and hence $cR = bR$, contradicting $a \neq 0$. □

Before we give more explicit examples of right Ore domains, let us note the following remarkable conclusion.

(10.25) Proposition. *If a domain R is not an Ore domain, then R contains a copy of the free algebra $C\langle x_1, x_2, \dots\rangle$, where C is the center of R.*

Proof. Say R is not right Ore. Then there exist a, $b \in R_R$ that are right R-linearly independent. Jategaonkar's Lemma (9.21) then implies that the ring generated by a, b over C is isomorphic to $C\langle x, y\rangle$. It follows from *FC*–(1.2) that the ring generated by a, ab, ab^2, ... over C is isomorphic to $C\langle x_1, x_2, x_3, \dots\rangle$. □

An algebra over a field k is said to be a PI-*algebra* (*polynomial identity algebra*) if there exists a nonzero polynomial

$$f(x_1, \dots, x_n) \in k\langle x_1, \dots, x_n\rangle$$

such that $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in R$. It follows immediately from (10.25) that:

(10.26) Corollary. *If a domain R is a PI-algebra over a field k, then R is an Ore domain.*

We shall offer some examples below. Some of the examples assume Exercise 1 in this section, so we advise our reader to first take a look at Exercise 1 before reading these examples.

(10.27) Examples.

(a) For any domain k, the free k-domain $R = k\langle X\rangle$ is not right (or left) Ore whenever $|X| \geq 2$. For, if a, b are distinct elements in X, we have $aR \cap bR = (0)$.

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[71]Note, however, that the domain assumption is essential here: in general, a right noetherian ring need not be a right Ore ring. For such an example, due to L. Small, see (12.27).

(b) Let R be the ring in (7.6)(4) that is generated by x, y over $\mathbb{Z}$ with the relations $yx = y^2 = 0$. Let $S = \{x^n : n \geq 0\}$. Expressing the elements of R in the form

$$f(x) + g(x)y \quad (f,\ g \in \mathbb{Z}[x])$$

as in (7.6)(4), we see that elements of S are not left 0-divisors, so *S is right reversible*. We claim that *S is also right permutable*. For this, we need to check that $aS \cap sR \neq \emptyset$ for $a = f(x) + g(x)y$ and $s = x^n$ (where we may assume $n \geq 1$). This is clear since $ax^n = f(x)x^n = sf(x)$. Therefore, RS^{-1} exists. Using Exercise 1, it is easy to compute RS^{-1}. The kernel of $\varphi : R \to RS^{-1}$ is $\mathfrak{A} = Ry = \mathbb{Z}[x]y$, so $\bar{R} := R/\mathfrak{A}$ can be identified with $\mathbb{Z}[x]$. According to Exercise 1, RS^{-1} is isomorphic to the localization of the commutative ring $\mathbb{Z}[x]$ at $\{x^n : n \geq 0\}$. On the other hand, *S is not left reversible*, since $yx = 0$ but $x^n y \neq 0$ for all $n \geq 0$. The ring R being left noetherian (cf. *FC*–(1.26)), it follows from Exercise 6 that *S is also not left permutable*. (More directly, $Sy \cap Rx = \emptyset$.) In any case, $S^{-1}R$ does not exist. For more information on the ring R, see Exercise 9.

(c) Let $R = \mathbb{Z}G$ where G is a group, and let $S = \mathbb{Z}\backslash\{0\}$. It is easy to see that $\mathbb{Q}G$ is a right ring of fractions of R with respect to S. Therefore, the central localization RS^{-1} gives a ring naturally isomorphic to $\mathbb{Q}G$. Similarly, if R is the ring of quaternions $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{Z}$, and $S = \mathbb{Z}\backslash\{0\}$, then RS^{-1} is the ring of all rational quaternions. In this case, in fact, $RS^{-1} = Q^r_{cl}(R)$, since it is a division ring.

(d) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. First choose $T = \{n \cdot I : 0 \neq n \in \mathbb{Z}\}$. Using the method in (c), we see easily that the central localization RT^{-1} gives the ring $Q = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. It is easy to see that any regular element of R is a unit in Q. Therefore, R is an order in Q, and $Q = Q^r_{cl}(R) = Q^\ell_{cl}(R)$. In particular, R is an Ore ring. Next, let us consider the multiplicative set

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,\ b,\ c \in \mathbb{Z},\ \ a \neq 0 \right\},$$

whose elements are not necessarily regular. Using the homomorphism $\varphi : R \to \mathbb{Q}$ defined by $\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a$, it is easy to check that $\mathbb{Q}$ is a right ring of fractions of R with respect to S. Therefore, *RS^{-1} exists and is isomorphic* to $\mathbb{Q}$. (The "Ore localization" here kills precisely all matrices of the form $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$.) On the other hand, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is killed by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$ on the right, but for any $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in S$:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq 0.$$

Therefore, S is not left reversible, so *$S^{-1}R$ does not exist.*

(e) For a fixed prime p, let $\mathbb{Z}_p$ denote $\mathbb{Z}/p\mathbb{Z}$, and let $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. Proceeding as in the last example, let T be the central multiplicative set $\{n \cdot I : n \in \mathbb{Z},\ p \nmid n\}$. (Of course, $n \cdot I$ here means $\begin{pmatrix} n & 0 \\ 0 & \bar{n} \end{pmatrix}$.) The central localization RT^{-1} gives the ring

$$Q = \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},$$

where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (p). (Note that $\mathbb{Z}_p = \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is a $\mathbb{Z}_{(p)}$-module.) We can check easily that the multiplicative set of regular elements of R is

$$S = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} : p \nmid x,\ z \neq 0 \in \mathbb{Z}_p \right\},$$

and that these elements are units in Q. Therefore, R is an order in Q, and $Q = Q^r_{cl}(R) = Q^\ell_{cl}(R)$. *In particular, R is an Ore ring.* This fact can be checked directly as follows. To see that S is right permutable, consider any $s = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in S$ and $a = \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \in R$. We can show that $aS \cap sR \neq \emptyset$ by solving the special matrix equation:

$$\begin{pmatrix} u & 0 \\ v & w \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}\begin{pmatrix} u & 0 \\ n & w \end{pmatrix}.$$

This amounts to a single equation $vx = yu + zn$, which has a unique solution $n \in \mathbb{Z}_p$ since z is a unit in $\mathbb{Z}_p$. The fact that S is left permutable can be proved similarly. For later reference, let us note the following three additional properties of R:

(1) For $s \in R$, $\text{ann}_\ell(s) = 0 \Longrightarrow s \in S$.

(2) The element $t = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ has $\text{ann}_r(t) = 0$, but $t \notin S$.

(3) For t as above and $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have $aS \cap tR = \emptyset$.

To see (1), let $s = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \notin S$. If $p \mid x$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} s = 0$. If $p \nmid x$, we must have $z = 0$, in which case $\begin{pmatrix} 0 & 0 \\ -y & \bar{x} \end{pmatrix} s = 0$. For (2), note that $t \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} = \begin{pmatrix} pu & 0 \\ v & w \end{pmatrix}$ is zero only if $u = 0 \in \mathbb{Z}$ and $v = w = 0 \in \mathbb{Z}_p$. For (3), assume there is an equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ v & w \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \in S.$$

This leads to $x = pu$, a contradiction.

(f) Let $R = \begin{pmatrix} k & k[x] \\ 0 & k[x] \end{pmatrix}$, where k is a field. The multiplicative set of regular elements of R is

$$S = \left\{ \begin{pmatrix} c & f(x) \\ 0 & g(x) \end{pmatrix} : c \in k, \ f, g \in k[x], \ c \cdot g \neq 0 \right\}.$$

A quick calculation shows that R is a right order in $Q = \begin{pmatrix} k & k(x) \\ 0 & k(x) \end{pmatrix}$. Therefore, $Q^r_{cl}(R) = RS^{-1} = Q$. On the other hand, $Q^\ell_{cl}(R)$ *does not exist*, as S turns out to be not left permutable: for $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ and $s = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in S$, a direct calculation shows that $Sa \cap Rs = \emptyset$. *Therefore, the ring R is right Ore but not left Ore.* Although every regular element of R becomes invertible in Q, the equation $Sa \cap Rs = \emptyset$ translates into the fact that $as^{-1} = \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix} \in Q$ cannot be written in the form $t^{-1}r$ with $r \in R$ and t a regular element of R. Therefore, Q is *not* a left ring of quotients of R with respect to S.

§10C. Polynomial Rings and Power Series Rings

In ring theory, it is well known that twisted polynomial rings provide a rich source of examples of rings which exhibit different "left" and "right" behavior. We begin this subsection by using twisted polynomials to produce another easy example of a left Ore domain that is not a right Ore domain. The basic facts and notations pertaining to the two types of twisted polynomial rings $R[x;\sigma]$ and $R[x;\delta]$ can be found in *FC*–(1.7) and *FC*–(1.9).

Let σ be an endomorphism of a division ring R, and $S = R[x;\sigma]$. Then S is a PLID; in particular, it is left Ore, by (10.23) or (10.24). If $\sigma(R) \neq R$, say $t \in R \backslash \sigma(R)$, then $\{1, t\} \subseteq R$ are right linearly independent over $\sigma(R)$. By (9.19), $\{x, tx\}$ are right linearly independent over S, so *S is not right Ore* (and hence not right noetherian). On the other hand, if $\sigma(R) = R$, then every left polynomial $\sum a_i x^i \in S$ is also a right polynomial, and we can think of S as a ring of twisted *right* polynomials over R (with the twist rule $ax = x\sigma^{-1}(a)$ for $a \in R$). In this case S is a PRID, and hence right noetherian and right Ore.

More generally, we can start with any domain R, and try to find out when a twisted polynomial ring of the type $S = R[x;\sigma]$ is left Ore. We have the following result.

(10.28) Theorem. *Let σ be an injective endomorphism of a domain R, and let $S = R[x;\sigma]$. If R is left Ore, so is S. The converse holds if σ is an automorphism.*

Proof. We begin by noting that the injectivity of σ guarantees that S is also a domain. Assume S is left Ore, and let $a, b \in R \backslash \{0\}$. Then $fa = gb$ for suitable $f, g \in S \backslash \{0\}$. Considering the leading coefficients of both sides, we obtain

an equation $c\sigma^n(a) = d\sigma^n(b)$ for some $c, d \in R\backslash\{0\}$, and $n \geq 0$. If σ is an automorphism, we can apply σ^{-n} to get $Ra \cap Rb \neq 0$, so R is left Ore. Conversely, assume R is left Ore. Let K be the (unique) division ring of fractions of R (cf. (10.21)). We can extend σ uniquely to an endomorphism of K by defining $\sigma(as^{-1}) = \sigma(a)\sigma(s)^{-1}$. Therefore, we can form

$$K[x;\sigma] \supseteq R[x;\sigma] = S.$$

Since $K[x;\sigma]$ is a PLID, it is left Ore by (10.23) (or (10.24)). Let Q be the division ring $Q_{cl}^{\ell}(K[x;\sigma])$. In view of (10.21), it suffices to show that S *is a left order in* Q. Of course, we have already $S \setminus \{0\} \subseteq \mathrm{U}(Q)$. Next, each element of Q has the form $f^{-1}g$, where

$$0 \neq f = \sum a_i x^i, \quad g = \sum b_i x^i, \quad a_i,\ b_i \in K.$$

Choose a suitable "common denominator" $s \in R\backslash\{0\}$ such that $a_i = s^{-1}c_i$, $b_i = s^{-1}d_i$ $(c_i,\ d_i \in R)$. Then

$$f^{-1}g = \left(s^{-1}\sum c_i x^i\right)^{-1}\left(s^{-1}\sum d_i x^i\right) = \left(\sum c_i x^i\right)^{-1}\left(\sum d_i x^i\right).$$

□

If δ is a *derivation* of the domain R, we can form the domain $R[x;\delta] = \{\sum a_i x^i\}$ using the law $xa = ax + \delta(a)$ for all $a \in R$ (see *FC*–(1.9)). If R is left Ore, with division ring of fractions K, we can again extend δ (uniquely) to a derivation on K by defining:

$$\delta(as^{-1}) = \delta(a)s^{-1} - as^{-1}\delta(s)s^{-1} \qquad (a \in R,\ \ 0 \neq s \in R).$$

The same proof used in (10.28) (with a couple of minor modifications) yields the following analogue.

(10.29) Theorem. *Let δ be a derivation on the domain R. Then the differential polynomial domain $S = R[x;\delta]$ is left Ore iff R is.*

For a useful special case of this, take $R = k[y]$ where k is a left Ore domain, and define δ by $\delta(\sum b_j y^j) = \sum j b_j y^{j-1}$ (formal differentiation with respect to y). Then $R[x;\delta]$ is the first Weyl algebra $A_1(k)$ (cf. *FC*–(1.9)). Applying (10.29) twice, we see that R and hence $A_1(k)$ are left Ore domains. Since the higher Weyl algebras are defined inductively by $A_n(k) = A_1(A_{n-1}(k))$, it follows that:

(10.30) Corollary. *If k is a left Ore domain, so are the Weyl algebras $A_n(k)$. In particular, each $A_n(k)$ has a unique division hull.*

The reason we considered twisted polynomial rings is, in part, to get results such as (10.30) for Weyl algebras. If we are interested only in the usual polynomial ring $S = R[x]$, then another approach is possible. By Shock's Theorem (6.65), u.dim ${}_S S = \text{u.}\dim {}_R R$. It follows immediately from (10.22) that a domain R is left Ore

iff $S = R[x]$ is. Note that this proof does not depend on the existence of a division hull of R. (For yet another such proof, see Exercise 22.)

How about power series rings? Given nonzero power series $f,\ g \in S = R[[x]]$, to show that $fS \cap gS \neq 0$ would involve working with *infinitely many* coefficients in the base domain R. Good intuition would suggest that this is generally impossible. However, good intuition is not always mathematically reliable. What we need is a counterexample in real terms.

The first counterexample in the literature appeared only in 1982: J. Kerr [82] produced a right Ore domain R for which $S = R[[x]]$ is not right Ore. Her example is based on the use of free products of rings. In 1992, G. Bergman produced another, more elementary example, where R is in fact (2-sided) Ore. Moreover, in this example, the power series f (for the inequality $fS \cap gS \neq 0$) is a constant. We are grateful to Bergman who has kindly permitted us to present his example below.

(10.31A) Example. *Let A be any commutative domain with an element z and an automorphism σ with the following properties*[72]*:*

(1) *zA is a prime ideal of A.*
(2) $\bigcap_{i<0} \sigma^i(z)A = 0.$
(3) $\bigcap_{i>0} \sigma^i(z)A \neq 0.$

Then $R = A[t;\sigma]$ is a (2-sided) Ore domain, but the power series domain $S = R[[x]]$ is not right Ore.

To see this, first note that A is Ore, since it is commutative. From (10.28), it follows that R is Ore. To show that S is not right Ore, fix a nonzero element $y \in \bigcap_{i>0} \sigma^i(z)A$. In view of (2), there exists a largest integer $k \leq 0$ such that $y \notin \sigma^k(z)A$. The choices of y and k guarantee that

$$(4) \qquad y \in \sigma^{k+\ell}(z)A \qquad \text{for any } \ell > 0.$$

For the following two (nonzero) power series:

$$(5) \qquad f = y, \qquad g = \sum_{j\geq 0} \sigma^{j-k}(y)t^{2j}x^j \ \text{ in } S,$$

we'll show that $fS \cap gS = 0$. Assume, instead, that $fu = gv$ for certain nonzero power series $u = \sum u_i x^i$, $v = \sum v_i x^i$ in S. After extracting powers of x from the right and cancelling, we may assume that $u_0 \neq 0 \neq v_0$. *For any fixed $i > 0$*, comparing the coefficients of x^i in $fu = gv$ yields

$$(6) \qquad yu_i = \sum_{0\leq j\leq i} \sigma^{j-k}(y)t^{2j}v_{i-j} = \sigma^{i-k}(y)t^{2i}v_0 + \sum_{0\leq j<i} \sigma^{j-k}(y)t^{2j}v_{i-j}\ .$$

For $0 \leq j < i$, we have $y \in \sigma^{k+i-j}(z)A$ by (4), so

$$\sigma^{j-k}(y)t^{2j}v_{i-j} \in \sigma^i(z)R.$$

[72]Explicit constructions for such (A, z, σ) will be given subsequently.

Together with $yu_i \in \sigma^i(z)R$, (6) implies that $\sigma^{i-k}(y)t^{2i}v_0 \in \sigma^i(z)R$. Comparing leading coefficients in the skew polynomial ring $R = A[t; \sigma]$, we get

$$\sigma^{i-k}(y)\sigma^{2i}(a) \in \sigma^i(z)A,$$

where $a \neq 0$ is the leading coefficient of $v_0 \in R$. Applying σ^{-i}, we have $\sigma^{-k}(y)\sigma^i(a) \in zA$. Since $\sigma^{-k}(y) \notin zA$ and zA is a prime ideal, $\sigma^i(a) \in zA$, and hence $a \in \sigma^{-i}(z)A$, for every $i > 0$. This contradicts (2), showing that S is not right Ore. □

To complete our work, we must construct a commutative domain A with the data z, σ as specified. This is easy. Start with the rational function field $K = \mathbb{Q}(z, y)$, and define an automorphism σ on K by $\sigma(z) = z + 1$ and $\sigma(y) = y/(z+1)$. By induction, we have $\sigma^i(y) = y/(z+1)\cdots(z+i)$ for $i \geq 0$. Let

(7) $$A = \mathbb{Q}[z, y, \sigma(y), \sigma^2(y), \ldots] \subseteq K.$$

Since $\sigma(A) \subseteq A$ and $\sigma^{-1}(A) \subseteq A$ (note that $\sigma^{-1}(y) = yz$), σ defines an automorphism on A, which we continue to denote by σ. From $y/(z+1)\cdots(z+i) \in A$, we have $y \in \sigma^i(z)A$ for $i > 0$, so (3) in (10.31) holds. To verify (1), assume $z \mid bc$ in the ring A. This divisibility relation must hold already in a subring

$$A_i = \mathbb{Q}[z,\ y/(z+1)\cdots(z+i)]$$

for a sufficiently large i. Since A_i is a polynomial ring in the two listed generators, we have $z \mid b$ or $z \mid c$ in A_i (and hence in A). Finally, the fact that no nonzero element in A can be divisible by $\sigma^j(z) = z + j$ for all $j < 0$ can be seen by enlarging A to $A' = \mathbb{Q}[z]_T[y]$, where T is the multiplicative set of $\mathbb{Q}[z]$ generated by $\{z + i : i > 0\}$. Since $\mathbb{Q}[z]$ is a UFD, so are $\mathbb{Q}[z]_T$ and A', and $\{z+j : j < 0\}$ remain pairwise nonassociate prime elements of A'. It is therefore clear that no nonzero element of A' is divisible by all $\{z + j : j < 0\}$.

Note that, in the ring A constructed above, $y/z \notin A$, since a polynomial expression in

$$z,\ y,\ y/(z+1),\ y/(z+1)(z+2),\ \ldots$$

cannot involve z in the denominator. Therefore, we could have used this y for the construction of f and g in (5) (with $k = 0$).

Another way to construct (A, z, σ) would be to start with $K = \mathbb{Q}((z_i)_{i\in\mathbb{Z}},\ y)$, with an automorphism σ defined by $\sigma(z_i) = z_{i+1}$ $(i \in \mathbb{Z})$ and $\sigma(y) = y/z_1$. We then take $z = z_0$,

(8) $$A = \mathbb{Q}[(z_i)_{i\in\mathbb{Z}},\ y,\ \sigma(y),\ \sigma^2(y),\ \ldots] \subseteq K,$$

and "σ" $= \sigma|A$. The verifications for (1), (2) and (3) are completely similar to the ones given above for the smaller ring in (7).

Throughout this subsection, we have been studying polynomial rings and power series rings over a base *domain* R. If R is not assumed to be a domain, the situation becomes quite a bit more complicated. In fact, if R is an Ore ring with 0-divisors,

the polynomial ring $R[x]$ may fail to be right or left Ore, as the following remarkable example of Cedó and Herbera [95] shows.

(10.31B) Example. Take a domain A that is neither right nor left Ore which can be embedded in a division ring D. (For instance, $A = \mathbb{Q}\langle x, y\rangle$: see §9C.) View D/A as an (A, A)-bimodule, and form the "trivial extension" $R = A \oplus (D/A)$ (cf. (2.22)(A)) with the multiplication

$$(a, \bar{d})(a', \overline{d'}) = (aa', \overline{ad' + da'}) \qquad (a, a' \in A; \ d, d' \in D),$$

with respect to which D/A becomes an ideal of square zero. Then *any regular element of R is a unit.* In fact, if $r = (a, \bar{d}) \in R$ is regular, we must have $a \neq 0$ in A. For $a' = a^{-1} \in D$, $r(0, \overline{a'}) = 0$ implies $\overline{a'} = 0$, so $a' \in A$. From this we check easily that $rr' = r'r = 1$ for $r' = (a', \ -\overline{a'da'}) \in R$. It follows in particular that R is an Ore ring (see (10.17)). We finish by showing that *$R[x]$ is not right (or left) Ore.*

Let $b, \ c \in A\backslash\{0\}$ be such that $bA \cap cA = 0$, and consider the regular element $1 + (b, \bar{0})x$ in $R[x]$. Assume, for the moment, that $R[x]$ is right Ore. Then there exist

$$p(x) = \sum_{i=0}^{m} \left(p_i, \overline{p_i'}\right) x^i \quad \text{and} \quad q(x) = \sum_{j=0}^{n} \left(q_j, \overline{q_j'}\right) x^j \ \text{ in } R[x],$$

with $q(x)$ regular, such that

$$(*) \qquad (1 + (b, \bar{0})x)\, p(x) = (c, \bar{0})\, q(x) \in R[x].$$

If all $q_j = 0 \in A$, we would have $q(x) \cdot (0, \bar{d}) = 0$ for any $d \in D$. Therefore, we may assume that $q_n \neq 0 \in A$. Projecting the equation $(*)$ from $R[x]$ to $A[x]$ we get

$$(1 + bx) \sum_{i=0}^{m} p_i x^i = c \sum_{j=0}^{n} q_j x^j.$$

A comparison of coefficients shows that $p_i = 0$ for $i \geq n$ and that $bp_{n-1} = cq_n \neq 0$, which contradicts $bA \cap cA = 0$. A similar argument shows that $R[x]$ cannot be left Ore.

Cedó and Herbera have shown further that, as long as A is a subring of a division ring D, then for the ring R defined above, *the power series ring $R[[x]]$ is Ore.*[73] In fact, the Laurent series ring $R((x))$ will be the left and right classical ring of fractions of $R[[x]]$. To see this, it suffices to show that every regular element $\alpha = \sum_{i=0}^{\infty} (a_i, \bar{d}_i)x^i \in R[[x]]$ has an inverse in $R((x))$. First note that the regularity of α implies that the a_i 's are not all 0, so let us write $\alpha = \beta + \gamma x^n$

[73]Together with the preceding material, this shows that $R[[x]]$ being Ore need not imply $R[x]$ being right or left Ore.

where

$$\beta = \sum_{i=0}^{n-1} (0, \bar{d}_i)\, x^i \quad \text{and} \quad \gamma = \sum_{i=0}^{\infty} (a_{n+i}, \bar{d}_{n+i})\, x^i$$

with $a_n \neq 0$ in A. Let

$$\sum_{i=0}^{\infty} b_i x^i = \left(\sum_{i=0}^{\infty} a_{n+i} x^i\right)^{-1} \in D[[x]].$$

An easy calculation shows that

$$\alpha \sum_{i=0}^{\infty} (0, \bar{b}_i)\, x^i = \gamma x^n \sum_{i=0}^{i} (0, \bar{b}_i)\, x^i = (0, \bar{1})\, x^n = 0.$$

Since α is regular in $R[[x]]$, all $\bar{b}_i$ must be 0. In particular, $b_0 \in A$, so $a_n b_0 = 1$ implies that $a_n \in \mathrm{U}(A)$. As before, this yields $(a_n, \bar{d}_n) \in \mathrm{U}(R)$, so γ^{-1} exists in $R[[x]]$. Multiplying $\alpha = \beta + \gamma x^n$ by γ^{-1}, we get $\alpha\gamma^{-1} = \beta_1 + x^n$, where $\beta_1 := \beta\gamma^{-1}$ belongs to $(D/A)[[x]]$. Since $(D/A)^2 = 0$, it follows that $\beta_1^2 = 0$, and so

$$\alpha\gamma^{-1}(x^n - \beta_1) = (x^n + \beta_1)(x^n - \beta_1) = x^{2n}.$$

Thus α has a right inverse in $R((x))$, and a similar argument shows that α has a left inverse in $R((x))$ as well.

For an arbitrary ring R, one can also ask the converse question whether $R[x]$ (or $R[[x]]$) being right Ore would imply R is right Ore. The answer is easily seen to be "yes" if R is a domain. Otherwise, the situation is again more complicated. Cedó and Herbera have shown that, in general, $R[[x]]$ being right Ore need not imply that R is right Ore. But it seems to be an open question whether $R[x]$ being right Ore would imply that R itself is right Ore.[74]

The same kind of questions can be posed for matrix rings. We close this subsection by giving one or two references on the question of whether the right Ore property "goes up" to matrix rings. Again the case of a domain is more special. If R is a right Ore domain, then any matrix ring $\mathbb{M}_n(R)$ is right Ore. This will be deduced a little later from a result on right Goldie rings; see (11.21)(1). In general, however, P. Menal [88] has shown that there exist Ore rings R with $R = Q^r_{cl}(R) = Q^\ell_{cl}(R)$ such that $\mathbb{M}_n(R)$ is neither right nor left Ore for any $n > 1$.

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[74]Although this question is not yet fully answered, Cedó and Herbera have proved the remarkable result that $R[x]$ being right (or left) Ore implies that R is Dedekind-finite. Thus, for any Ore ring R (e.g., a von Neumann regular ring) that is not Dedekind-finite, $R[x]$ fails to be right or left Ore!

§10D. Extensions and Contractions

In this subsection, we return to the study of the right ring of fractions of a ring R with respect to an arbitrary right denominator set $S \subseteq R$. For future reference, it is convenient to collect here some facts on the relationship between the right ideals of R and those of RS^{-1}. To simplify the notation, we shall write $Q = RS^{-1}$, and let φ be the natural homomorphism from R to Q.

For any right ideal $\mathfrak{A}$ of R, we define $\mathfrak{A}^e$ (the *extension* of $\mathfrak{A}$) to be $\varphi(\mathfrak{A}) \cdot Q$, the right ideal generated by $\varphi(\mathfrak{A})$ in Q. For any right ideal $\mathfrak{B}$ of Q, we define $\mathfrak{B}^c$ (the *contraction* of $\mathfrak{B}$) to be $\varphi^{-1}(\mathfrak{B})$. We have then the following properties.

(10.32) Proposition.

(1) $\mathfrak{B}^{ce} = \mathfrak{B}$.

(2) $\mathfrak{A}^e = \{as^{-1} : a \in \mathfrak{A},\ s \in S\}$. *(Here, as before, as^{-1} is used as an informal notation for $\varphi(a)\varphi(s)^{-1} \in Q$.)*

(3) $\mathfrak{A}^{ec} = \{r \in R : \ rs \in \mathfrak{A}$ *for some* $s \in S\}$.

(4) *If we have a direct sum of right ideals $\bigoplus_i \mathfrak{A}_i \subseteq R$, then we have $\bigoplus_i \mathfrak{A}_i^e \subseteq Q$.*

(5) *If we have a direct sum of right ideals $\bigoplus_i \mathfrak{B}_i \subseteq Q$, then we have $\bigoplus_i \mathfrak{B}_i^c \subseteq R$, assuming that $\varphi|\mathfrak{B}_i^c$ is injective for all i.*

(6) *If R is right noetherian (resp. artinian), so is Q.*

Proof. (1) is routine, and it implies (6). For (2), we need only verify the inclusion "$\subseteq$". Consider a sum

$$q = a_1 s_1^{-1} + \cdots + a_n s_n^{-1},$$

where $a_i \in \mathfrak{A}$, $s_i \in S$. We can express $a_i s_i^{-1}$ in the form $a_i' s^{-1}$ $(1 \leq i \leq n)$, where $s \in S$ and $a_i' \in a_i R \subseteq \mathfrak{A}$. Therefore, $q = (a_1' + \cdots + a_n')s^{-1}$. For (3), if $rs \in \mathfrak{A}$ where $s \in S$, then $r \in \mathfrak{A}s^{-1} \subseteq \mathfrak{A}^e$ in Q implies that $r \in \mathfrak{A}^{ec}$. Conversely, if $r \in \mathfrak{A}^{ec}$, then (by (2)) $r = as^{-1}$ in Q where $a \in \mathfrak{A}$ and $s \in S$. This gives $rs = a \in Q$ so we have $r(ss') = as' \in \mathfrak{A}$ for some $s' \in S$. The proofs for (4) and (5) are both routine. □

In general, if $\mathfrak{A}$ is an ideal in R, $\mathfrak{A}^e$ need not be an ideal in Q (see Exercise 11). However, as observed independently by Jategaonkar and Ludgate, a mild assumption on Q will restore this property. This fact and some of its consequences are recorded in the Proposition below.

(10.33) Proposition.

(1) *Assume that Q is right noetherian. If $\mathfrak{A}$ is an ideal in R, then $\mathfrak{A}^e$ is an ideal in Q.*

(2) *Let $\mathfrak{A}$ be an ideal in R such that $\mathfrak{A}^e$ is an ideal in Q. Then for any right ideal $\mathfrak{A}_1 \subseteq R$, $(\mathfrak{A}_1\mathfrak{A})^e = \mathfrak{A}_1^e\mathfrak{A}^e$. If $\mathfrak{A}$ is nilpotent, then $\mathfrak{A}^e$ is also nilpotent.*

(3) *Assume that R is noetherian. Then there is a one-to-one correspondence between* Spec Q *(the set of prime ideals of Q) and* $\text{Spec}_S R$ *(the set of prime ideals of R that are disjoint from S), given by contraction and extension.*

Proof. (1) Consider any $s \in S$. Since $s\mathfrak{A} \subseteq \mathfrak{A}$, we have

$$\mathfrak{A} \subseteq s^{-1}\mathfrak{A} \subseteq s^{-2}\mathfrak{A} \subseteq \cdots \subseteq Q, \quad \text{and hence}$$
$$\mathfrak{A}^e \subseteq s^{-1}\mathfrak{A}^e \subseteq s^{-2}\mathfrak{A}^e \subseteq \cdots \subseteq Q.$$

Since Q is right noetherian, $s^{-n}\mathfrak{A}^e = s^{-(n+1)}\mathfrak{A}^e$ for some $n \geq 0$, and hence $\mathfrak{A}^e = s^{-1}\mathfrak{A}^e$. Left multiplying by any $r \in R$, we get

$$rs^{-1}\mathfrak{A}^e = r\mathfrak{A}^e \subseteq \mathfrak{A}^e,$$

since $\mathfrak{A}$ is a left ideal in R. Therefore, we have $Q\mathfrak{A}^e \subseteq \mathfrak{A}^e$, so $\mathfrak{A}^e$ is an ideal in Q.

(2) The inclusion $(\mathfrak{A}_1\mathfrak{A})^e \subseteq \mathfrak{A}_1^e\mathfrak{A}^e$ is clear. For the reverse inclusion, consider an additive generator $q = (a_1s_1^{-1})(as^{-1})$ of $\mathfrak{A}_1^e\mathfrak{A}^e$, where $a_1 \in \mathfrak{A}_1$, $a \in \mathfrak{A}$, and s, $s_1 \in S$. Since $\mathfrak{A}^e$ is an ideal, $s_1^{-1}as^{-1} \in \mathfrak{A}^e$, so $s_1^{-1}as^{-1} = a's'^{-1}$ for some $a' \in \mathfrak{A}$ and $s' \in S$. Therefore, $q = (a_1a')s'^{-1} \in (\mathfrak{A}_1\mathfrak{A})^e$. This proves $(\mathfrak{A}_1\mathfrak{A})^e = \mathfrak{A}_1^e\mathfrak{A}^e$. In particular, $(\mathfrak{A}^n)^e = (\mathfrak{A}^e)^n$, and $\mathfrak{A}^n = 0$ implies $(\mathfrak{A}^e)^n = 0$.

(3) By (10.32)(6), Q is right noetherian, so (1) above applies. Let $\mathfrak{q} \in \text{Spec}\, Q$; *we claim that* $\mathfrak{p} = \mathfrak{q}^c$ *is a prime*. Indeed, if $\mathfrak{A}$, $\mathfrak{A}'$ are ideals of R such that $\mathfrak{A}\mathfrak{A}' \subseteq \mathfrak{p}$, then, extending to Q and using (2), we get

$$\mathfrak{A}^e\mathfrak{A}'^e = (\mathfrak{A}\mathfrak{A}')^e \subseteq \mathfrak{p}^e = \mathfrak{q}^{ce} = \mathfrak{q}.$$

Therefore, we have (say) $\mathfrak{A}^e \subseteq \mathfrak{q}$, so $\mathfrak{A} \subseteq \mathfrak{q}^c = \mathfrak{p}$. Since $\mathfrak{q} \neq Q$, clearly $\mathfrak{p} \in \text{Spec}_S R$. We have thus a map $\alpha : \text{Spec}\, Q \to \text{Spec}_S R$ defined by contraction. Next, consider any $\mathfrak{p} \in \text{Spec}_S R$, and write $\mathfrak{p}' = \mathfrak{p}^{ec}$. Since R is (left) noetherian, $\mathfrak{p}'$ is finitely generated as a *left* ideal. Using (10.32)(3), we can therefore find an element $s \in S$ such that $\mathfrak{p}'s \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime and $sR \not\subseteq \mathfrak{p}$, this implies that $\mathfrak{p} = \mathfrak{p}' = \mathfrak{p}^{ec}$. In particular, $\mathfrak{p}^e \neq Q$. *We claim that* $\mathfrak{p}^e \in \text{Spec}\, Q$. To see this, suppose $\mathfrak{B}\mathfrak{B}' \subseteq \mathfrak{p}^e$, where $\mathfrak{B}$, $\mathfrak{B}'$ are ideals of Q. Then $\mathfrak{B}^c\mathfrak{B}'^c \subseteq \mathfrak{p}^{ec} = \mathfrak{p}$ implies that (say) $\mathfrak{B}^c \subseteq \mathfrak{p}$. Therefore, by (1), $\mathfrak{B} = \mathfrak{B}^{ec} \subseteq \mathfrak{p}^e$, as desired. It follows that $\mathfrak{p} \mapsto \mathfrak{p}^e$ defines a map $\beta : \text{Spec}_S R \to \text{Spec}\, Q$ which is inverse to the map α above. This completes the proof of (3). □

To get other desirable facts relating R and Q, we shall need to assume that S consists only of regular elements. Recall that $A \subseteq_e B$ is the notation for A being an essential submodule of B. In the following Proposition, we shall use this notation alongside the notation of $\mathfrak{A}^e$ for extensions. This should not cause any confusion since $A \subseteq_e B$ is a subscript notation while $\mathfrak{A}^e$ is a superscript notation.

(10.34) Proposition. *Assume that the right denominator set $S \subseteq R$ consists only of regular elements of R. Let $\mathfrak{A} \subseteq \mathfrak{A}'$ be right ideals of R, and $\mathfrak{B} \subseteq \mathfrak{B}'$ be right ideals in $Q = RS^{-1} \supseteq R$. Then:*

(a) $\mathfrak{A} \subseteq_e \mathfrak{A}'$ *iff* $\mathfrak{A}^e \subseteq_e \mathfrak{A}'^e$.
(b) $\mathfrak{B} \subseteq_e \mathfrak{B}'$ *iff* $\mathfrak{B}^c \subseteq_e \mathfrak{B}'^c$.
(c) *If R is a prime (resp. semiprime) ring,*[75] *so is Q. The converse holds if Q is right noetherian.*

Proof. (a) First assume $\mathfrak{A} \subseteq_e \mathfrak{A}'$, and consider $0 \neq a's^{-1} \in \mathfrak{A}'^e$, where $a' \in \mathfrak{A}'$ and $s \in S$. Fix an element $r \in R$ such that $0 \neq a'r \in \mathfrak{A}$. Then

$$(a's^{-1})(sr) = a'r \in \mathfrak{A}'\backslash\{0\} \subseteq \mathfrak{A}'^e\backslash\{0\}.$$

Next, assume $\mathfrak{A}^e \subseteq_e \mathfrak{A}'^e$, and consider $0 \neq a' \in \mathfrak{A}'$. Then $0 \neq a'(rs^{-1}) = as_1^{-1}$ for some $r \in R$, $a \in \mathfrak{A}$, s, $s_1 \in S$. After expressing rs^{-1} and as_1^{-1} with a common denominator, we may assume that $s = s_1$ (without affecting the condition $a \in \mathfrak{A}$). Therefore, we have $a'r = a \in \mathfrak{A}\backslash\{0\}$.

(b) Assume first $\mathfrak{B}^c \subseteq_e \mathfrak{B}'^c$. Taking extensions and using (a) with (10.32)(1), we get $\mathfrak{B} \subseteq_e \mathfrak{B}'$. Conversely, assume $\mathfrak{B} \subseteq_e \mathfrak{B}'$. Then $(\mathfrak{B}^c)^e \subseteq_e (\mathfrak{B}'^c)^e$ implies $\mathfrak{B}^c \subseteq_e \mathfrak{B}'^c$, again by (a).

(c) Assume first R is prime. Say $as^{-1}Qbt^{-1} = 0$. Then $0 = as^{-1}(sR)b = aRb$, so we have $a = 0$ or $b = 0$. This shows that Q is prime. The semiprime case is similar, upon letting $a = b$. Now let Q be right noetherian. Then we can apply (10.33)(1). Assume Q is prime, and say $aRb = 0$. We are done if we can show that $aQb = 0$ for then $a = 0$ or $b = 0$. Consider an element $rs^{-1} \in Q$, where $r \in R$ and $s \in S$. Since $(RbR)^e$ is an ideal, $s^{-1}b \in s^{-1}(RbR)^e \subseteq (RbR)^e$. Therefore

$$a(rs^{-1})b \in ar(RbR) \cdot Q = 0,$$

as desired. The semiprime case follows similarly, by letting $a = b$. □

(10.35) Corollary. *Under the hypotheses of* (10.34), *we have*

$$\text{u.dim}\, \mathfrak{A}_R = \text{u.dim}\, (\mathfrak{A}^e)_Q = \text{u.dim}\, (\mathfrak{A}^e)_R\,, \quad \textit{and}$$
$$\text{u.dim}\, \mathfrak{B}_Q = \text{u.dim}\, \mathfrak{B}_R = \text{u.dim}\, (\mathfrak{B}^c)_R\,.$$

In particular, $\text{u.dim}\, R_R = \text{u.dim}\, Q_Q = \text{u.dim}\, Q_R$.

Proof. This follows from (10.34), and the easy observation that $\mathfrak{A} \subseteq_e \mathfrak{A}^e$ and $\mathfrak{B}^c \subseteq_e \mathfrak{B}$ as right R-modules. □

Note that, for the last two results (10.34) and (10.35), it is essential to have the assumption that S consist only of *regular* elements (so that we have $R \subseteq Q$). For instance, if R and S are as in Example (10.27)(b), the ideal $\mathfrak{A} = R \cdot y \subseteq R$ is an infinite direct sum $\bigoplus_{i \geq 0} x^i y\mathbb{Z}$ of right ideals. Thus, u.dim $\mathfrak{A}_R = \infty$, but $\mathfrak{A}^e = (0)$ since y maps to zero in Q. Also, we have $\mathfrak{A} \subseteq_e R_R$, but $\mathfrak{A}^e = (0)$ is not essential in Q_Q. Finally, u.dim $R_R = \infty$ here, but u.dim $Q_Q = 1$.

[75]For the notions of prime and semiprime rings, see *FC*–§10.

Exercises for §10

0. Let $S \subseteq R$ be a right permutable multiplicative set. Show that

$$\mathfrak{A} = \{a \in R : \ as = 0 \ \text{ for some } \ s \in S\}$$

is an ideal in R.

1. Let $S \subseteq R$ be a right denominator set, and $\mathfrak{A}$ be as in the above exercise. Let $\bar{R} = R/\mathfrak{A}$ and write "bar" for the natural surjection from R to $\bar{R}$. Show that $\bar{S}$ is a right denominator set in $\bar{R}$ consisting of regular elements and that $RS^{-1} \cong \bar{R}\bar{S}^{-1}$ over R.

2. Let $S \subseteq R$ be a multiplicative set.

   (a) If $s \in S$ has a right inverse, show that $aS \cap sR \neq \emptyset$ holds for any $a \in R$.
   (b) If R is a reduced ring, show that S is right and left reversible.

3. Let $S \subseteq R$ be any commutative multiplicative set, and let

$$A = \{a \in R : \ aS \cap sR \neq \emptyset \ \text{ for every } \ s \in S\}.$$

Show that A is a subring of R containing the centralizer of S in R.

4. Let $x, \ y \in R$ be such that $xy = 1 \neq yx$, and let S be the multiplicative set $\{x^n : \ n \geq 0\}$. Show that

   (1) S is left reversible but not right reversible;
   (2) S is right permutable; and
   (3) if R is generated over a central subring k by x and y, then S is also left permutable.

5. Let V be a right vector space over a field k, with basis $\{e_1, e_2, \cdots\}$. Let $R = \text{End}(V_k)$, and let $x, \ y \in R$ be defined by $y(e_i) = e_{i+1}$ $(i \geq 1)$, and $x(e_1) = 0$, $x(e_i) = e_{i-1}$ $(i \geq 2)$. Show that $S = \{x^n : \ n \geq 0\} \subseteq R$ is a left denominator set, but not a right denominator set.

6. Let R be a ring satisfying ACC on right annihilators of elements. If a multiplicative set $S \subseteq R$ is right permutable, show that it is necessarily right reversible. Conclude from Exercise 4 that the ring R is Dedekind-finite.

7. Prove the last statement of (10.21).

8. Let G be the free group generated by a set X with $|X| \geq 2$. Show that the domain $\mathbb{Z}G$ is not right (or left) Ore.

9. Let R be the ring $\mathbb{Z}\langle x, y\rangle$ defined by the relations $y^2 = yx = 0$. Show that R is left Ore but not right Ore.

10. Let $R = k[x; \sigma]$ where σ is an automorphism of the ring k. Show that $S = \{x^n : \ n \geq 0\}$ is a right and left denominator set of R, and that

RS^{-1} and $S^{-1}R$ are both isomorphic (over R) to the ring of skew Laurent polynomials $k[x, x^{-1}; \sigma]$ (as defined in *FC*–(1.8)).

11. Let k in Exercise 10 be the polynomial ring $\mathbb{Q}[\{t_i : i \in \mathbb{Z}\}]$, with σ defined by $\sigma(t_i) = t_{i+1}$ for all $i \in \mathbb{Z}$, and let R, S be as above. Show that $\mathfrak{A} = t_1R + t_2R + \cdots$ is an ideal in R, but the extension $\mathfrak{A}^e$ is *not* an ideal in $RS^{-1} = k[x, x^{-1}; \sigma]$. (**Hint.** Note that $x^{-1}t_1 = \sigma^{-1}(t_1)x^{-1} = t_0x^{-1} \notin \mathfrak{A}^e$.)

12. Let R be the first Weyl algebra $A_1(k)$ over the field k (see *FC*–(1.3)(c)), identified with $k[y][x; \delta]$ where δ denotes formal differentiation on $k(y)$. Show that $S = k[y]\backslash\{0\}$ is a right and left denominator set of R, and that RS^{-1} and $S^{-1}R$ are both isomorphic (over R) to $k(y)[x; \delta]$.

13. Let R be a right Ore domain with division ring of right fractions K. Show that the center of K is given by

$$\{as^{-1} : a \in R,\ s \in R\backslash\{0\},\ ars = sra \text{ for all } r \in R\}.$$

14. For R, K as in Exercise 13, show that any ring between R and K is a right Ore domain.

15. Let S be a right denominator set in a ring R and let $\varphi : R \to Q$ be the natural map, where $Q = RS^{-1}$. (1) If Q_R is a noetherian R-module, show that $\varphi(R) = Q$. (2) If ${}_RQ$ is a f.g. R-module, show that $\varphi(R) = Q$.

16. Let $f : R \to R'$ be a homomorphism between right Ore rings. Does f induce a ring homomorphism $Q^r_{cl}(R) \to Q^r_{cl}(R')$?

17. Let S be a right denominator set in a ring R. Show that the right ring of fractions $Q = RS^{-1}$ is flat as a left R-module.

18. Let $S \subseteq R$ and Q be as in Exercise 17 and let M be a right R-module.

(1) Generalizing the procedure in the text, show that there exists a "localization" MS^{-1} which is a right Q-module with elements of the form $m/s = ms^{-1}$ $(m \in M,\ s \in S)$.
(2) Show that the kernel of the natural map $M \to MS^{-1}$ is given by

$$M_0 := \{m \in M : ms = 0 \text{ for some } s \in S\}.$$

(3) Show that "localization" is an exact functor from $\mathfrak{M}_R$ (the category of right R-modules) to $\mathfrak{M}_Q$ (the category of right Q-modules).
(4) Show that $MS^{-1} \cong M \otimes_R Q$ in $\mathfrak{M}_Q$. Using this and (3), give another proof for the fact that ${}_RQ$ is flat.
(5) If $M_0 = 0$ in (2), show that

$$\text{u.dim}\, M_R = \text{u.dim}(MS^{-1})_R = \text{u.dim}(MS^{-1})_Q\,.$$

19. For any right R-module M, let

$$t(M) = \{m \in M : ms = 0 \text{ for some regular element } s \in R\}.$$

Show that R is right Ore iff, for any right R-module M, $t(M)$ is an R-submodule of M. In this case, $t(M)$ is called the *torsion submodule* of M; M is called *torsion* if $t(M) = M$, and *torsion-free* if $t(M) = 0$. Show that, in case R is right Ore, $M/t(M)$ is always torsion-free.

20. Let R be a subring of a division ring D. Show that ${}_RD$ is a flat left R-module iff R is a right Ore domain.

21. Show that a *domain* R satisfies the (right) strong rank condition ("for any n, any set of $n+1$ elements in $(R^n)_R$ is linearly dependent": see (1.20)(2)) iff R is right Ore. (This is an interesting conclusion since it means that, to guarantee that we can solve nontrivially m homogeneous linear equations in $n > m$ unknowns over R, it suffices to guarantee that we can always solve *one* such equation in *two* unknowns.)

22. Use Exercise 21 to give another proof for the fact that, if R is a right Ore domain, so is $R[x]$. (**Hint.** Recall Exercise (1.22).)

23. Show that, over a right Ore domain, any f.g. flat right module is projective. (**Hint.** Recall (4.38).)

24. Let $R \subseteq L$ be domains. Show that L is injective as a right R-module iff R is right Ore and L contains the division ring of right fractions K of R.

25. Show that a domain R is right Ore iff it has a nonzero right ideal of finite uniform dimension.

26. Let R be a domain. Show that R is a PRID iff R is right Ore and all right ideals are free, iff R is right noetherian and all right ideals are free.

27. (This exercise leads to another proof of (10.24).) Let $\mathfrak{A} \neq 0$ be a right ideal in a domain R, and $b \in R\backslash\{0\}$. If $\mathfrak{A} \cap bR = 0$, show that $\mathfrak{A} + bR$ is a nonprincipal right ideal.

28. Let $S \subseteq R$ be a right denominator set consisting of regular elements, and let $Q = RS^{-1}$. *True or False*: (1) R is simple iff Q is? (2) R is reduced iff Q is?

29. Let $Q = RS^{-1}$, where S is a right denominator set in R. For any right Q-module N, show that N_Q is injective iff N_R is. (**Hint.** For the "if" part, consider $N_Q \subseteq X_Q$, and show that any direct complement of N_R in X_R is a Q-module (cf. the proof of (3.77)). For the "only if" part, use the flatness of ${}_RQ$ and (3.6A).)

30. In two different graduate algebra texts, the following exercise appeared: "Let S be a multiplicative subset of the commutative ring R. If M is an injective (right) R-module, show that MS^{-1} is an injective (right) RS^{-1}-module." Find a counterexample.

31. Show that the statement in quotes in the last exercise is true under either one of the following assumptions:

(1) R is noetherian (or more generally, R is right noetherian and S is a central multiplicative set in R);
(2) M is S-torsionfree (i.e., for $s \in S$ and $m \in M$, $ms = 0 \Longrightarrow m = 0$).
Your proof should work under the more general assumption that S is a right denominator set in a possibly noncommutative ring R.

§11. Right Goldie Rings and Goldie's Theorems

§11A. Examples of Right Orders

In §11, we shall study a beautiful chapter of noncommutative ring theory inaugurated by Goldie and Lesieur-Croisot in the late 1950s. The class of rings emerging from this study is the class of right Goldie rings (and especially the class of semiprime right Goldie rings). Before we give the motivation for this study, let us first introduce the following

(11.1) Notation. For any ring R, we shall write $\mathcal{C}_R$ for the multiplicative set of regular elements of R, and $\mathrm{U}(R)$ for the group of units of R. (We have, of course, $\mathrm{U}(R) \subseteq \mathcal{C}_R$.)

Recall that, if $R \subseteq Q$ are rings, we say that R is a *right order in* Q if $\mathcal{C}_R \subseteq \mathrm{U}(Q)$ and every element of Q has the form as^{-1}, where $a \in R$ and $s \in \mathcal{C}_R$. We can pose the following two natural questions:

(11.2) *When is a ring R a right order in some other ring Q?*

(11.3) *When does a ring Q have a right order $R \subseteq Q$?*

In the last section, we have fully answered the first question; namely, R is a right order in some ring, say Q, iff R is right Ore, in which case $Q \cong Q^r_{cl}(R)$ over R. The second question can now be answered easily as well.

(11.4) Proposition. *For a ring Q, the following are equivalent:*

(1) *Q has a right order R.*
(2) *Q has a left order R.*
(3) $\mathcal{C}_Q = \mathrm{U}(Q)$.
(4) $Q = Q^\ell_{cl}(Q)$.
(5) $Q = Q^r_{cl}(Q)$.

If Q satisfies these equivalent conditions, we shall say that Q is a classical ring.[76]

[76]Classical rings are also known as "rings of quotients" in the literature.

Proof. By left-right symmetry, it suffices to prove the equivalence of (1), (3), and (5). But clearly, (3)$\Longrightarrow$(5)$\Longrightarrow$(1), so it is enough to prove (1)$\Longrightarrow$(3). Suppose (1) holds, and let us identify Q with $Q^r_{cl}(R)$. Let $q \in \mathcal{C}_Q$. Then $q = as^{-1}$ for some $a \in R$ and $s \in \mathcal{C}_R$. Since $s \in \mathcal{C}_R \subseteq \mathrm{U}(Q) \subseteq \mathcal{C}_Q$, we have

$$a = as^{-1} \cdot s \in \mathcal{C}_Q \cap R \subseteq \mathcal{C}_R. \tag{11.5}$$

Therefore, $q' := sa^{-1}$ is an element of Q, with $qq' = 1 = q'q$, so $q \in \mathrm{U}(Q)$. □

Recall from (10.17) that *any classical ring Q is always Ore*. In the following, we shall give various examples of classical rings.

(11.6) Examples.

(1) *If Q is a von Neumann regular ring, then Q is a classical ring.* Indeed, let $q \in \mathcal{C}_Q$, and write $q = qq'q$ for some $q' \in Q$. Then

$$q(1 - q'q) = 0 = (1 - qq')q$$

gives $q'q = 1 = qq'$, so $q \in \mathrm{U}(Q)$. As a special case, any Boolean ring Q ($q^2 = q\ \forall q \in Q$) is a classical ring. (Here, $\mathcal{C}_Q = \{1\} = \mathrm{U}(Q)$.)

(2) Let Q be a ring in which any chain $qQ \supseteq q^2Q \supseteq \dots$ stabilizes for any $q \in Q$. (Such a Q is called a *strongly π-regular* ring.) Then Q is a classical ring. Indeed, let $q \in Q$ be any element for which $\mathrm{ann}_r(q) = 0$. Then $q^nQ = q^{n+1}Q$ for some $n \geq 1$, so we can write $q^n = q^{n+1}q'$ for some $q' \in Q$. But then $q^n(1 - qq') = 0$ implies $qq' = 1$, and $q(1 - q'q) = q - q = 0$ implies $q'q = 1$, so $q \in \mathrm{U}(Q)$. It follows from *FC*–(24.25) that *all left perfect rings are classical rings*. In particular, all 1-sided artinian rings are classical rings. In the *commutative* category, strongly π-regular rings are precisely the rings of Krull dimension 0 (see Exer. (4.15) in [Lam: 95]), so the latter rings are always classical rings.

(3) Any local ring $(R, \mathfrak{m})$ with a nil maximal ideal $\mathfrak{m}$ is clearly a classical ring, since $\mathcal{C}_R \subseteq R \backslash \mathfrak{m} = \mathrm{U}(R)$.

(4) For any vector space V_k over a division ring k, $Q = \mathrm{End}(V_k)$ (operating on the left of V) is a classical ring. This follows from (1) above since it is well-known that Q is von Neumann regular (see *FC*–(4.27)). More directly, for $q \in Q$, $\mathrm{ann}_r(q) = 0$ means that q is an injective endomorphism, and $\mathrm{ann}_\ell(q) = 0$ means that q is a surjective endomorphism. Therefore, $q \in \mathcal{C}_Q$ clearly implies that $q \in \mathrm{U}(Q)$.

(5) The ring Q in (4) above is a right self-injective ring, by (the right analogue of) (3.74B). More generally, *any right self-injective ring is a classical ring*. The proof of this is left as an exercise (see Exercise 8).

We now turn our attention to right orders.

(11.7) Remarks.

(1) We have noted that any Boolean ring Q is classical. Such a ring has only one left (or right) order, namely, Q itself.

(2) In general, *if $R \subseteq R' \subseteq Q$ are rings, and R is a right order in Q, then so is R'.* (In particular, R' must be right Ore.) First, by the same argument in the proof of (11.4) (applied to R' instead of Q), we see that $\mathcal{C}_{R'} \subseteq \mathrm{U}(Q)$. Next, every element of Q has the form as^{-1}, where $a \in R$ and $s \in \mathcal{C}_R$. But then $s \in \mathrm{U}(Q) \cap R' \subseteq \mathcal{C}_{R'}$, so we have indeed $Q = Q^r_{cl}(R')$.

(3) *Suppose $R \subseteq Q$, where Q is a strongly π-regular ring. To check that R is a right order in Q, it suffices to check that every element of Q has the form as^{-1}, where $a \in R$ and $s \in R \cap \mathrm{U}(Q)$.* The other property, $\mathcal{C}_R \subseteq \mathrm{U}(Q)$, can be deduced from this as follows. Let $q \in \mathcal{C}_R$. Then $\mathrm{ann}^Q_r(q) = 0$, for, if $q \cdot as^{-1} = 0 \in Q$ where $a \in R$ and $s \in R \cap \mathrm{U}(Q)$, then $qa = 0 \in R$ and so $a = 0 \in R$, $as^{-1} = 0 \in Q$. By the proof given in (11.6)(2), we can conclude that $q \in \mathrm{U}(Q)$.

(4) *Let R be a right order in a right artinian ring Q. Then $R_n := \mathbb{M}_n(R)$ is a right order in $Q_n := \mathbb{M}_n(Q)$.* Indeed, if $q \in Q_n$, we can express the entries of the matrix q in the form $a_{ij}s^{-1}$ where $a_{ij} \in R$ and $s \in \mathcal{C}_R$. Therefore, $q = (a_{ij})(sI_n)^{-1}$ where $sI_n \in R_n \cap \mathrm{U}(Q_n)$. It is not difficult to see that Q_n is a right artinian ring, so (3) above implies that R_n is a right order in Q_n.

Some examples of right orders R in Q are given below. Many more examples can be generated from these by using (11.7)(2).

(11.8) Examples.

(1) R is any commutative domain and Q is its quotient field.

(2) More generally, R is any right Ore domain, and Q is its division ring of right fractions. For instance, we can take Q to be the division ring of all rational quaternions, and R to be any subring of Q containing i, j, k (see (11.7)(2)).

(3) $Q = \mathbb{Q} \times \mathbb{Q}$, $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod n\}$ (fixed $n \neq 0$).

(4) $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, $Q = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$; $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$, $Q = \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$; or $R = \begin{pmatrix} k & k[x] \\ 0 & k[x] \end{pmatrix}$, $Q = \begin{pmatrix} k & k(x) \\ 0 & k(x) \end{pmatrix}$, where k is any field. (See (10.27)(d,e,f).)

(5) Let A be any commutative domain with quotient field K, and let Q be any finite-dimensional K-algebra. An A-subalgebra $R \subseteq Q$ is called a (*classical*) *A-order* in Q if R is f.g. as an A-module satisfying $R \cdot K = Q$. (These are the orders studied extensively in the theory of integral representations.) Any such classical A-order R is a right order of Q in our sense. Indeed, consider any $s \in \mathcal{C}_R$. We must have $s \in \mathcal{C}_Q$, for, if $sq = 0$ where $q \in Q$, we can find $a \in A \backslash \{0\}$ such that $r := qa \in R$; but then $sr = 0$ implies $r = 0$, so $q = 0$. Since Q is artinian, (11.6)(2) above implies that $s \in \mathrm{U}(Q)$. Also, any $q \in Q$ has the form ra^{-1} as above where $r \in R$ and $a \in A \backslash \{0\} \subseteq \mathcal{C}_R$, so R is a right order in Q.

(Note that the finite generation of R as an A-module is actually not needed for this conclusion.)

(6) Examples of classical A-orders abound. For instance, in the 2-dimensional $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt{5})$, $R_n = \mathbb{Z} + n\mathbb{Z} \cdot \sqrt{5}$ $(0 \neq n \in \mathbb{Z})$ is a classical $\mathbb{Z}$-order, and $R = \mathbb{Z} + \mathbb{Z} \cdot (\sqrt{5} - 1)/2$ (the ring of algebraic integers in $\mathbb{Q}(\sqrt{5})$) is a maximal classical $\mathbb{Z}$-order. The ring $\begin{pmatrix} \mathbb{Z} & n\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ $(0 \neq n \in \mathbb{Z})$ is a classical $\mathbb{Z}$-order in $\mathbb{M}_2(\mathbb{Q})$, as is $\begin{pmatrix} \mathbb{Z} & n\mathbb{Z} \\ n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. If we write Q for the rational quaternion division algebra, then

$$R = \mathbb{Z} + i\,\mathbb{Z} + j\,\mathbb{Z} + k\,\mathbb{Z} \quad \text{and} \quad S = \mathbb{Z} + 2i\,\mathbb{Z} + 2j\,\mathbb{Z} + 2k\,\mathbb{Z}$$

are both classical $\mathbb{Z}$-orders in Q, and Hurwitz' ring of quaternions

$$H = \tfrac{1}{2}(1 + i + j + k)\mathbb{Z} + i\,\mathbb{Z} + j\,\mathbb{Z} + k\,\mathbb{Z}$$

(cf. *FC*–(1.1)) can be seen to be a *maximal* classical $\mathbb{Z}$-order in Q (containing R and S). The rings

$$\mathbb{M}_2(R), \quad \mathbb{M}_2(H), \quad \begin{pmatrix} R & nR \\ nR & R \end{pmatrix} \quad (0 \neq n \in \mathbb{Z}), \quad \text{and} \quad \begin{pmatrix} R & 2R \\ H & R \end{pmatrix}$$

are all classical $\mathbb{Z}$-orders in the semisimple $\mathbb{Q}$-algebra $\mathbb{M}_2(Q)$. (Note that $2R \cdot H = R \cdot (2H) \subseteq R \cdot R = R$.)

(7) Let $A \subseteq K$ be as in (5), and let G be any *finite* group. Then AG is a classical A-order in KG. In general, it is not a maximal classical A-order. For instance, let K be a number field (i.e., a finite extension of $\mathbb{Q}$), and A be the ring of algebraic integers in K. Consider any (finite) Galois extension L/K, with Galois group G, and let B be the ring of algebraic integers in L. Then L may be viewed as a (say, left) KG-module, and the Galois extension L/K gives rise to a ring

$$R = \{\lambda \in KG : \lambda \cdot B \subseteq B\} \subseteq KG.$$

Since $AG \cdot B \subseteq A \cdot B = B$, we have $AG \subseteq R$. It can be checked that R is a classical A-order of KG (containing AG, usually properly).

§11B. Right Orders in Semisimple Rings

According to Ore's Theorem, right orders in division rings are exactly the right Ore domains. *What about right orders in semisimple rings? In other words, which rings will have classical right rings of fractions $Q^r_{cl}(R)$ that are semisimple?* The answers to these questions are provided by Goldie's Theorem, which we shall now try to formulate.

In §6E, we have investigated various finiteness conditions on rings. By combining two of these finiteness conditions, we arrive at the notion of a right Goldie ring. The justification for this definition will come very shortly.

(11.9) Definition. A ring R is said to be *right Goldie* if it satisfies ACC_{III} and ACC_{IV} in the notation of §6E; that is, if u.dim $R_R < \infty$, *and* R has ACC on right annihilator ideals. (Left Goldie rings are defined similarly.)

The following remarks should shed some light on this definition.

(11.10) *Any right noetherian ring is right Goldie.* The converse, however, is not true, since clearly any commutative domain is also right Goldie.

(11.11) *If Q is right Goldie, so is any right order $R \subseteq Q$.* This follows quickly from (6.61) and (10.35).

(11.12) As a consequence of (11.10) and (11.11), *any right order in a right noetherian ring is right Goldie.*

We are now in a good position to state and prove the celebrated result of Goldie [60]. The main part of the theorem is the equivalence of (1), (2), and (5) below; we have added the other two equivalent conditions (3) and (4) to show more clearly how the ideas in the proof are tied together, and, of course, also to present a more complete form of the theorem. Recall that a ring R is called *right nonsingular* if the right singular ideal $\mathcal{Z}(R_R)$ is zero (see (7.5)), and $\mathcal{C}_R$ denotes the set of regular elements of a ring R.

(11.13) Goldie's Theorem. *For any ring R, the following are equivalent:*

(1) *R is a right order in a semisimple ring, say, Q.*
(2) *R is semiprime right Goldie.*
(3) *R is semiprime,* u. dim $R_R < \infty$, *and R has* ACC *on right annihilators of elements.*
(4) *R is semiprime, right nonsingular, and* u. dim $R_R < \infty$.
(5) *For any right ideal $\mathfrak{A}$ in R, $\mathfrak{A} \subseteq_e R$ iff $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$.*

Note that, by (2)$\Longrightarrow$(1) of this theorem, semiprime right Goldie rings form a subclass of the right Ore rings.

Before we proceed to the proof of Goldie's Theorem, the following cautioning note on terminology is in order. In some books on ring theory (e.g., Renault [75], Passman [91]), a "right Goldie ring" is defined to be a ring R such that u.dim $R_R < \infty$ and R has ACC on right annihilators *of elements*. The equivalence of (2) and (3) in the Theorem above means that, for *semiprime* rings, this definition agrees with ours. But in general, this definition is not equivalent to ours, and it gives a bigger class of rings. We shall, however, follow the majority of practitioners in the field, and define right Goldie rings by using the full right annihilator condition, as in (11.9).

Proof of (11.13). We shall prove

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5) \Longrightarrow (1).$$

(1)$\Longrightarrow$(2). (This basic implication provides the motivation for Def. (11.9).) Suppose (1) holds. Since Q is right noetherian, (11.12) implies that R is right Goldie, and (10.34)(c) implies that R is semiprime. If we don't want to invoke the deeper fact (10.34)(c), we can also deduce the semiprimeness of R as follows. Let $\mathfrak{A} \subseteq R$ be a left ideal with $\mathfrak{A}^2 = 0$. We claim that $\mathfrak{A}' := \text{ann}_\ell\, \mathfrak{A} \subseteq_e R_R$. Indeed, for any $r \notin \mathfrak{A}'$, we have $ra \neq 0$ for some $a \in \mathfrak{A}$. Then $ra\,\mathfrak{A} \subseteq \mathfrak{A}^2 = 0$, so $ra \in \mathfrak{A}'\backslash\{0\}$. This checks that $\mathfrak{A}' \subseteq_e R_R$, and hence $\mathfrak{A}' \cdot Q \subseteq_e Q_Q$ by (10.34)(a). Since Q_Q is semisimple, we must have $\mathfrak{A}' \cdot Q = Q$. In particular, we have $1 = a's^{-1}$ for some $a' \in \mathfrak{A}'$ and $s \in \mathcal{C}_R$ (see (10.32)(2)). But then $0 = a'\,\mathfrak{A} = s\,\mathfrak{A}$ implies that $\mathfrak{A} = 0$.

(5)$\Longrightarrow$(1). *We first check that R is right Ore.* Let $a \in R$ and $s \in \mathcal{C}_R$. By (5), we have $sR \subseteq_e R$. Applying Exercise (3.7) to left multiplication by a, we get

$$\{r \in R : \ ar \in sR\} \subseteq_e R_R.$$

Using (5) again, the LHS contains an element $s' \in \mathcal{C}_R$. Thus, $as' \in sR$, and hence $a\mathcal{C}_R \cap sR \neq \emptyset$. This checks that R is right Ore, so we can form $Q := Q_{cl}^r(R)$. To show that Q is semisimple, consider any right ideal $\mathfrak{B} \subseteq Q$, with contraction $\mathfrak{A} := \mathfrak{B}^c \subseteq R$. Let $\mathfrak{A}'$ be a complement to $\mathfrak{A}$ in R_R (i.e., $\mathfrak{A}'$ is a right ideal in R maximal with respect to $\mathfrak{A} \cap \mathfrak{A}' = (0)$). Clearly, $\mathfrak{A} \oplus \mathfrak{A}' \subseteq_e R_R$, so by (5) again, $\mathfrak{A} \oplus \mathfrak{A}'$ contains a regular element of R. Taking extensions to Q, we get $\mathfrak{A}^e \oplus \mathfrak{A}'^e = Q$ by (10.32)(5), and therefore $\mathfrak{B} = \mathfrak{B}^{ce} = \mathfrak{A}^e$ is a direct summand of Q_Q. This shows that Q is a semisimple ring.

(2)$\Longrightarrow$(3) is a tautology.

(3)$\Longrightarrow$(4) follows from (7.19). (We also have (4)$\Longrightarrow$(3) from (7.51), but we don't need this implication here.)

(4)$\Longrightarrow$(5) is part of the following Proposition:

(11.14) Proposition. *Let R be a ring satisfying* (4) *above.*

(a) *Let $a \in R$ be right regular, in the sense that* $\text{ann}_r(a) = 0$. *Then $aR \subseteq_e R_R$, and in fact $a \in \mathcal{C}_R$.*
(b) *Any right ideal $\mathfrak{A} \subseteq R$ contains an element a such that* $\text{ann}_r(a) \cap \mathfrak{A} = (0)$.
(5) *For any right ideal $\mathfrak{A}$ in R, $\mathfrak{A} \subseteq_e R$ iff $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$.*

Proof. (a) The hypothesis on a implies that $R \cong aR$ as right modules, so

$$\text{u.dim}\,(aR)_R = \text{u.dim}\ R_R < \infty,$$

which clearly implies that $aR \subseteq_e R_R$. (For another argument, see Exercise 2(b).) So far, we have only used the hypothesis u.dim $R_R < \infty$. To see that $a \in \mathcal{C}_R$, we shall also use $\mathcal{Z}(R_R) = 0$. Indeed, if $r \in \text{ann}_\ell(a)$, then $r(aR) = 0$ and $aR \subseteq_e R_R$ imply that $r \in \mathcal{Z}(R_R) = 0$, so a is left regular as well.

(b) *We induct on* $n = \text{u.dim}\, \mathfrak{A}_R < \infty$. The case $n = 0$ being trivial, let us check the crucial case $n = 1$. Here, $\mathfrak{A}_R$ is uniform. Since R is semiprime, $\mathfrak{A}^2 \neq 0$, so there exist $a, a' \in \mathfrak{A}$ with $aa' \neq 0$. We claim that the element a is what we want. Indeed, if $\mathfrak{B} := \text{ann}_r(a) \cap \mathfrak{A} \neq 0$, then $\mathfrak{B} \subseteq_e \mathfrak{A}$ (since $\mathfrak{A}_R$ is uniform). By Exercise (3.7) again,

$$a'^{-1}\mathfrak{B} := \{x \in R : a'x \in \mathfrak{B}\} \subseteq_e R_R.$$

But now $aa' \cdot (a'^{-1}\mathfrak{B}) \subseteq a\mathfrak{B} = 0$ implies that $aa' \in \mathcal{Z}(R_R) = 0$, a contradiction. For the general case, fix a right ideal $\mathfrak{A}_0 \subseteq \mathfrak{A}$ with $\text{u.dim}(\mathfrak{A}_0)_R = n - 1$, and pick $a_0 \in \mathfrak{A}_0$ with $\text{ann}_r(a_0) \cap \mathfrak{A}_0 = 0$. We may assume that $U := \text{ann}_r(a_0) \cap \mathfrak{A} \neq 0$ (for otherwise a_0 already works for $\mathfrak{A}$). Clearly $\text{u.dim}\, U = 1$, so by the work above, there exists $u \in U$ with $\text{ann}_r(u) \cap U = 0$. Then $a := u + a_0 \in \mathfrak{A}$ is what we want since

$$\text{ann}_r(a) \cap \mathfrak{A} = \text{ann}_r(u) \cap \text{ann}_r(a_0) \cap \mathfrak{A} = \text{ann}_r(u) \cap U = 0.$$

(5) First assume $\mathfrak{A} \subseteq_e R$, and pick $a \in \mathfrak{A}$ as in (b). Clearly, $\text{ann}_r(a)$ must be (0), so by (a), $a \in \mathcal{C}_R$. Conversely, if a right ideal $\mathfrak{A}$ contains some $a \in \mathcal{C}_R$, then $aR \subseteq_e R_R$ by (a), so a fortiori $\mathfrak{A} \subseteq_e R_R$. □

Remarks. After proving Goldie's important theorem (11.13), several additional observations on a semiprime right Goldie ring R are worthwhile.

(1) Part (a) of (11.14) shows that, in R, *any right regular element is regular.* However, there are examples of R in which *left* regular elements need not be regular: see Exercise 4 in this section. In particular, such rings R cannot be left Goldie.

(2) According to (7.51)′, all chains of right (resp. left) annihilators in R have bounded length. In particular, *right (resp. left) annihilators in R satisfy both the* ACC *and the* DCC.

(3) In connection to the conclusion (5) in (11.14), more can be said about the relationship between essential right ideals $I \subseteq_e R_R$ and the regular elements in R. For instance, it can be shown that *any coset of I must contain a regular element, and that I is always generated (as a right ideal) by the regular elements it contains.* The proofs of these statements depend on the observation (2) above: see Exercises 24–27. (Some information in the non-Goldie case is given in Exercises 1–2; see also Exercise 7 in §7.)

For any right module M over a ring R, let $\mathcal{Z}(M)$ denote the singular submodule of M (see (7.1)), and let

$$(*) \qquad t(M) = \{m \in M : mr = 0 \text{ for some } r \in \mathcal{C}_R\}.$$

As a consequence of (11.13) and (11.14), we obtain the following simple module-theoretic characterization of semiprime right Goldie rings. The first part of this result is to be compared with Exercise (10.19). The second part gives a natural

interpretation for the reduced rank of a right module over a semiprime right Goldie ring.

(11.15) Proposition. *A ring R is semiprime right Goldie iff $\mathcal{Z}(M) = t(M)$ for all right R-modules M. In this case, if $Q = Q^r_{cl}(R)$, we have in fact*

(A) $\mathcal{Z}(M) = t(M) = \ker\{M \to M \otimes_R Q\}$, *and*
(B) $\text{rank}(M) = \text{length}_Q(M \otimes_R Q)$,

where rank(M) *is the reduced rank of* M_R *defined in* (7.34).

Proof. The "only if" part follows by applying (2)$\Longrightarrow$(5) in Theorem 11.13 to right ideals of the form ann(m) ($m \in M$). Conversely, assuming that $t(M) = \mathcal{Z}(M)$ for all M_R, let us verify (5) in (11.13) for any right ideal $\mathfrak{A} \subseteq R$. First assume $\mathfrak{A} \subseteq_e R_R$. Then $\bar{1} \in R/\mathfrak{A}$ belongs to $\mathcal{Z}(R/\mathfrak{A}) = t(R/\mathfrak{A})$, so $\bar{1} \cdot r = 0$ for some $r \in \mathcal{C}_R$, and this means that $r \in \mathfrak{A}$. Next, assume that $\mathfrak{A}$ contains an element $r \in \mathcal{C}_R$. Then $\bar{1} \in R/\mathfrak{A}$ is killed by r, so $\bar{1} \in t(R/\mathfrak{A}) = \mathcal{Z}(R/\mathfrak{A})$, whence $\mathfrak{A} = \text{ann}(\bar{1}) \subseteq_e R_R$.

Assume now R is a semiprime right Goldie ring. Then the second equality in (A) follows from Exercise (10.18). To compute rank(M), recall that it was defined as u.dim $M/0^{**}$, where 0^{**} is the "closure" of (0) in the sense of (7.31). Since R is a right nonsingular ring, 0^{**} is just $\mathcal{Z}(M)$ by (7.21), so rank M = u.dim $\bar{M}$ where

$$\bar{M} := M/\mathcal{Z}(M) = M/t(M).$$

Applying Exercise (10.18)(5) to this torsion-free module, we have

$$\begin{aligned}\text{rank}(M) &= \text{u.dim}(\bar{M} \otimes_R Q)\\ &= \text{u.dim}(M \otimes_R Q)\\ &= \text{length}_Q(M \otimes_R Q).\end{aligned}$$

Here, we have used the isomorphism $\bar{M} \otimes_R Q \cong M \otimes_R Q$ which results from the right exactness of the tensor product, and the fact that $t(M) \otimes_R Q = 0$. □

Returning now to Goldie's Theorem (11.13), let us state an important special case of it, which is often called "Goldie's First Theorem" ((11.13) being the Second).

(11.16) Corollary (Goldie, Lesieur-Croisot). *A ring R is a right order in a simple artinian ring (say, Q) iff R is a prime right Goldie ring.*

Proof. This follows from (11.13) in view of (10.34)(c) and the fact that simple artinian rings are exactly the prime semisimple rings. Here is a direct argument avoiding the reference to (10.34)(c). Suppose R is prime Goldie, and $Q = Q^r_{cl}(R)$ is its semisimple classical right ring of quotients. If $\mathfrak{B}$, $\mathfrak{B}'$ are right ideals in Q such that $\mathfrak{B}\mathfrak{B}' = 0$, then $\mathfrak{B}^c\mathfrak{B}'^c \subseteq \mathfrak{B}\mathfrak{B}' = 0$ implies (say) $\mathfrak{B}^c = 0$, so $\mathfrak{B} = \mathfrak{B}^{ce} = 0$. Therefore Q is prime and hence simple artinian. Conversely, if R is a right order

in a simple artinian ring Q, then R is right Goldie by (11.13). Suppose $\mathfrak{A}\mathfrak{A}' = 0$, where $\mathfrak{A}$, $\mathfrak{A}'$ are left ideals in R, with $\mathfrak{A} \neq 0$. Then $Q\mathfrak{A}Q = Q$, so there exists an equation $1 = \sum_i x_i a_i y_i$, where $x_i, y_i \in Q$, $a_i \in \mathfrak{A}$. Writing $y_i = r_i s^{-1}$ with a common denominator $s \in \mathcal{C}_R$, we have $s = \sum_i x_i a_i r_i$, so $s\mathfrak{A}' \subseteq Q\mathfrak{A}\mathfrak{A}' = 0$, and hence $\mathfrak{A}' = 0$. This shows that R is a prime ring. □

There is no lack of examples illustrating (11.16). For instance, (11.8)(6) provides many noetherian prime rings that are right orders in simple artinian rings.

As usual, we shall say that R is a *Goldie ring* if it is both left and right Goldie. Let us say that a subring R in a ring Q is a 2-*sided order* (or *order* for short) if R is both a left order and a right order in Q.

(11.17) Corollary. *Let R be a semiprime right Goldie ring, with $Q = Q^r_{cl}(R)$. The following are equivalent:*

(1) *R is a Goldie ring.*
(2) *R is an order in Q.*
(3) $\text{u.dim}({}_R R) < \infty$.

Proof. (1)$\Longrightarrow$(2). Since R is (semiprime) left Goldie, it is also left Ore, so Q coincides with $Q^{\ell}_{cl}(R)$ by (10.14). Therefore, R is a left order in Q as well. (2)$\Longrightarrow$(3) is clear by (the left version of) (10.35). Finally, assume (3). Since Q has DCC on right annihilators, R also has DCC on right annihilators by (6.61) and hence R has ACC on left annihilators by (6.57). Together with (3), this shows that R is left Goldie. □

(11.18) Corollary. *If R is a semiprime (resp. prime) right Goldie ring, then so is* $R_n := \mathbb{M}_n(R)$.

Proof. We first do the semiprime case. Since R is a right order in $Q = Q^r_{cl}(R)$ and Q is right artinian, (11.7)(4) implies that R_n is a right order in $Q_n := \mathbb{M}_n(Q)$. But Q_n is also semisimple, so (11.13) implies that R_n is semiprime right Goldie. The prime case is entirely similar. □

Remark. The proof of (11.18) using Goldie's Theorem (11.13) may seem a little circuitous. However, *if R is only a right (or even 2-sided) Goldie ring,* $\mathbb{M}_n(R)$ *need not be right Goldie*! J. Kerr [79] has found an example of a commutative Goldie ring R such that $A = \mathbb{M}_2(R)$ does not have ACC on right annihilators. In particular, A is not right Goldie, although

$$\text{u.dim}\, A_A = 2(\text{u.dim}\, R_R) < \infty$$

by (6.62). In light of this, the *semiprime* assumption is essential for the truth of (11.18), and Goldie's Theorem does play a key role in its proof.

(11.19) Corollary (Small). *Let $S = R[x]$. Then R is semiprime (resp. prime) right Goldie iff S is.*

Proof. We shall focus on the semiprime case, as the prime case is similar. This time we cannot use the main equivalence (1)$\Longleftrightarrow$(2) in Goldie's Theorem (11.13). However, we can use (2)$\Longleftrightarrow$(4) there. By (6.65), R is right finite-dimensional iff S is. By Exercise (7.35), R is right nonsingular iff S is. By *FC*–(10.18), R is semiprime iff S is. The desired conclusion follows from these.[77] □

Remarks.

(1) We could have applied the same method above to the proof of (11.18), if we only know that R is right nonsingular iff $\mathbb{M}_n(R)$ is. This is, in fact, true, but we have not yet proved it. An indication of the proof will be given later in the context of the Morita Theory in Chapter 7: see Exercise (18.3).

(2) As in the Remark following (11.18), if R is just right Goldie (without the semiprime condition), then the polynomial ring $S = R[x]$ need not be right Goldie. Indeed, J. Kerr [90] has constructed a commutative Goldie ring R for which $S = R[x]$ does not have ACC on annihilator ideals; in particular, S fails to be Goldie. Kerr's ring R is an algebra over the field of two elements. On the other hand, V. Camillo and R. Guralnick [86] have shown that, if R is an algebra over an *uncountable* field, then R being right Goldie implies that $R[X]$ is right Goldie for any set of (commuting) variables X.

(11.20) Corollary. *A domain R is right Goldie iff it is right Ore.*

Proof. The "only if" part follows from the fact (shown in the course of proving (11.13)) that semiprime right Goldie rings are right Ore. Conversely, if R is a right Ore domain, then u.dim $R_R = 1$ and the only right annihilator ideals are $\{0\}$ and R, so R is clearly right Goldie. (In this case, R is a right order in the division ring $Q^r_{cl}(R)$.) □

(11.21) Examples.

(1) If R is any right Ore domain with $Q = Q^r_{cl}(R)$, then any ring between $\mathbb{M}_n(R)$ and $\mathbb{M}_n(Q)$ is a prime right Goldie (and hence right Ore) ring. (This follows from (11.20), (11.18), and (11.7)(2).)

(2) Let G be any finite group. By Maschke's Theorem (*FC*–(6.1)), $\mathbb{Q}\,G$ is semisimple. It follows as in (1) that *any ring between $\mathbb{Z}\,G$ and $\mathbb{Q}\,G$ is a semiprime Goldie ring.*

(3) Let R be the ring in Example (7.6)(4). Since R is left noetherian, it is left Goldie. However, $\sum x^i y\mathbb{Z}$ is an infinite *direct* sum of right ideals, so R is not right Goldie. Of course, this R is not semiprime. It is just as easy to find semiprime examples. For instance, if A is a left Ore domain that is not right Ore, as in the second

[77] Since Shock's results used here are also valid for $R[X]$ where X is any set of commuting indeterminates (see Exercises (6.26), (7.35)), the conclusion of (11.19) holds for $S = R[X]$ as well.

paragraph before (10.28), then, by (11.20), A is (semiprime) left Goldie, but not right Goldie. We have here u.dim ${}_A A = 1$, but u.dim $A_A = \infty$.

(4) There is a very remarkable example, due to G. Bergman, of a prime (in fact primitive) ring R which has left and right uniform dimension 1 but is neither left nor right Goldie. Such a ring R is necessarily not a domain, and is *neither left nor right nonsingular*. In fact, for any left zero-divisor $a \in R$, we have $\text{ann}_r(a) \neq 0$; hence $\text{ann}_r(a) \subseteq_e R_R$, and so $\mathcal{Z}(R_R) \neq 0$ (and similarly $\mathcal{Z}({}_R R) \neq 0$). (Of course, the fact that R fails to be left or right nonsingular is also predicted by the characterization (11.13)(4) for semiprime 1-sided Goldie rings.) Bergman's example is not easy, but the details for the construction are well covered in pp. 27-30 of the book by Chatters and Hajarnavis [80], so let us just refer the reader to that source.

At this point, let us record one more useful result related to (11.13); this concerns the structure of minimal prime ideals. Again, we work in the setting of a semiprime right Goldie ring; somewhat more general results on minimal primes in *any* semiprime ring will be presented later in §11D.

(11.22) Proposition. *Let R be a semiprime right Goldie ring, and let $Q_1 \times \cdots \times Q_t$ be the Wedderburn decomposition of the semisimple ring $Q := Q^r_{cl}(R)$. Let $\mathfrak{q}_i = \sum_{j \neq i} Q_j$ and $\mathfrak{p}_i = \mathfrak{q}_i^c = \mathfrak{q}_i \cap R$. Then $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ are all the minimal prime ideals of R. Each $R/\mathfrak{p}_i$ is a prime right Goldie ring with $Q^r_{cl}(R/\mathfrak{p}_i) \cong Q_i$ $(1 \leq i \leq t)$.*

Proof. Each $\mathfrak{p}_i$ is prime by the first argument used in the proof of (10.33)(3), and by (10.32)(1), $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for $i \neq j$. Also, $\bigcap_i \mathfrak{p}_i \subseteq \bigcap_i \mathfrak{q}_i = (0)$. If $\mathfrak{p} \subseteq R$ is any prime ideal,

$$\mathfrak{p}_1 \cdots \mathfrak{p}_t \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = (0) \subseteq \mathfrak{p} \tag{11.23}$$

implies that $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. From this, we see easily that $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are all the minimal prime ideals of R. Identifying $Q/\mathfrak{q}_i$ with Q_i, we have $R/\mathfrak{p}_i \subseteq Q/\mathfrak{q}_i = Q_i$ and

$$R \hookrightarrow R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_t \subseteq Q_1 \times \cdots \times Q_t = Q. \tag{11.24}$$

By (11.7)(2), $R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_t$ is a right order in Q, so by Exercise 0 below, each $R/\mathfrak{p}_i$ is a right order in Q_i. Therefore, by (11.16), $R/\mathfrak{p}_i$ is a prime right Goldie ring, with $Q^r_{cl}(R/\mathfrak{p}_i) = Q_i$. □

To illustrate this Proposition, consider, as in (11.21)(2) above, the case $R = \mathbb{Z}G$ where G is a finite group. Let $Q_1 \times \cdots \times Q_t$ be the Wedderburn decomposition of the semisimple ring $\mathbb{Q}G$, and let π_i be the projection map from $\mathbb{Z}G$ to Q_i with respect to this decomposition. Then the minimal primes of $\mathbb{Z}G$ are precisely the kernels of the π_i 's. In the case when G is abelian, the situation is especially transparent. Here, the Q_i 's are cyclotomic fields $\mathbb{Q}[\zeta_i]$ where the ζ_i 's are various roots of unity, and the π_i 's are ring homomorphisms from $\mathbb{Z}G$ onto

$\mathbb{Z}[\zeta_i] \subseteq \mathbb{Q}[\zeta_i]$. The kernels of these homomorphisms give all minimal primes of the abelian group ring $\mathbb{Z}G$.

§11C. Some Applications of Goldie's Theorems

Using the Goldie Theorems, one can often prove results about right noetherian rings which may not be easy to prove otherwise. To see how Goldie's Theorems can be applied, let us sample a few such applications below. The first application concerns the structure of the injective hull $E(R/\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal in a right noetherian ring R. This leads, in particular, to a criterion for $\mathfrak{p}$ to be right meet-irreducible (in the sense of the definition given at the beginning of §3F).

(11.25) Theorem. *Let $\mathfrak{p}$ be a prime ideal in a right noetherian ring R. Let $A = R/\mathfrak{p}$ and $n = \text{u.dim } A_R$. Then the injective hull $E(A_R)$ is a direct sum of n mutually isomorphic indecomposable injective R-modules. In particular, $\mathfrak{p}$ is right meet-irreducible iff $A = R/\mathfrak{p}$ is a domain.*

Proof. Since the prime ring A is right noetherian and hence right Goldie, $Q := Q^r_{cl}(A)$ exists and is simple artinian, by (11.16). Therefore, $Q \cong \mathbb{M}_m(D)$ where $m \geq 1$ and D is a division ring. In particular, Q has a unique simple right module B_Q such that $Q_Q \cong mB$. By (6.7)(2) and (10.35):

(11.26) $$m(\text{u.dim } B_A) = \text{u.dim } Q_A = \text{u.dim } Q_Q = m,$$

(11.27) $$m = \text{u.dim } Q_Q = \text{u.dim } A_A = \text{u.dim } A_R = n.$$

Therefore, u.dim B_R = u.dim $B_A = 1$, so $E(B_R)$ is an *indecomposable* injective R-module, and, since $A_R \subseteq_e Q_R$,

(11.28) $$E(A_R) \cong E(Q_R) \cong m \cdot E(B_R) = n \cdot E(B_R),$$

as desired. In this notation, $\mathfrak{p}$ is right meet-irreducible iff $n = 1$, iff Q is a division ring, iff $A = R/\mathfrak{p}$ is a domain. □

(11.29) Remark. In the last conclusion of (11.25), neither implication is true without the right noetherian assumption on the ring R. For instance, if R is a domain that is not right Ore, then the prime $\mathfrak{p} = (0)$ is not right meet-irreducible. On the other hand, in the example of Bergman referenced in (11.21)(4), we have a prime ring R with u.dim(R_R) = u.dim$({}_RR) = 1$ that is neither left Goldie nor right Goldie. Thus, $\mathfrak{p} = (0)$ in R is left and right meet-irreducible, but R fails to have ACC on left as well as on right annihilators. In particular, R is *not* a domain.

Note that Theorem (11.25) has a very natural interpretation in terms of the classification of the indecomposable injective right modules over R. Recall from (3.60) that we have a natural "fibration" $\alpha : \mathcal{I}(R) \to \text{Spec } R$ where $\mathcal{I}(R)$ denotes the set of isomorphism classes of the indecomposable injective right R-modules. If we define $\beta : \text{Spec } R \to \mathcal{I}(R)$ by taking $\beta(\mathfrak{p})$ to be any indecomposable direct

summand of $E((R/\mathfrak{p})_R)$, then β gives a natural splitting for α (over any right noetherian ring R).

Let us now give another nice application of Goldie's Theorems to bimodules, due to T. H. Lenagan. The term "length" below refers, as usual, to the length of a composition series.

(11.30) Theorem. *Let I be an (S, R)-bimodule, where R and S are arbitrary rings. Assume that* $\text{length}({}_SI) < \infty$. *Then I_R is noetherian iff* $\text{length}(I_R) < \infty$.

Lenagan proved this result in the case when $S = R$ and I is an ideal in R, but his arguments carry over verbatim to (S, R)-bimodules. Before presenting the proof of this beautiful result, let us first record a couple of its consequences.

(11.31) Corollary. *Let I be an ideal in a noetherian ring R. Then* $\text{length}({}_RI) < \infty$ *iff* $\text{length}(I_R) < \infty$.

This follows from (11.30) and left-right symmetry. On the other hand, if we apply (11.30) to the (R, R)-bimodule $I = R$, we get the following:

(11.32) Corollary. *If a ring R is left artinian, then it is right noetherian iff it is right artinian.*

For readers familiar with the results in *FC*, this should come as no surprise. In fact, according to the Hopkins-Levitzki Theorem, *FC*–(4.15), the conclusion in (11.32) is true more generally for any semiprimary ring; that is, any ring R such that rad R is nilpotent and $R/\text{rad}\, R$ is semisimple. To place (11.30) in perspective, we should think of it as a generalization of the classical result (11.32). Let us now present:

Proof of (11.30). ("Only if" part.) Assume that I_R is noetherian. By induction on $\text{length}({}_SI)$, we are reduced to the case when I is a *simple* (S, R)-bimodule. Let $\mathfrak{p} = \text{ann}(I_R) \neq R$. This is a prime ideal in R, for, if $\mathfrak{A}$, $\mathfrak{B} \subseteq R$ are ideals not contained in $\mathfrak{p}$, then $I\mathfrak{A} = I = I\mathfrak{B}$, so $I\mathfrak{A}\mathfrak{B} = I$ and hence $\mathfrak{A}\mathfrak{B} \not\subseteq \mathfrak{p}$. After replacing R by $R/\mathfrak{p}$, we may therefore assume that R is a prime ring acting faithfully on I. Since ${}_SI$ is artinian, there exist $a_1, \ldots, a_n \in I$ such that $I = \sum S \cdot a_i$. The map $R \to I^n$ sending $r \in R$ to $(a_1r, \ldots, a_nr)$ is a monomorphism of right R-modules, so R_R is noetherian; that is, *R is now a prime right noetherian (in particular right Ore) ring.* Consider

(11.33) $$t(I) := \{a \in I : ar = 0 \text{ for some } r \in \mathcal{C}_R\}.$$

By Exercise (10.19), $t(I)$ is an R-submodule of I_R, and from its definition, $t(I)$ is also an S-submodule of ${}_SI$. Since ${}_SI_R$ is a simple bimodule, we have either $t(I) = I$ or $t(I) = 0$. If $t(I) = I$, then $a_ir_i = 0$ for suitable $r_i \in \mathcal{C}_R$. Taking a common right multiple $r \in \mathcal{C}_R$ for $\{r_1, \ldots, r_n\}$, we get $Ir = \sum S \cdot a_ir = 0$, contradicting the faithfulness of I_R. Therefore, we must have $t(I) = 0$. This

means that each element $s \in \mathcal{C}_R$ acts as an *injective* endomorphism of ${}_SI$. Since $\text{length}({}_SI) < \infty$, s must act as an automorphism of ${}_SI$. Therefore, I_R can now be made into a Q-module I_Q, where $Q := Q^r_{cl}(R)$ is a simple artinian ring, by (11.16). Fix a simple submodule $J_Q \subseteq I_Q$. Then

$$Q_Q \cong k \cdot J_Q \subseteq k \cdot I_Q$$

for some $k > 0$, and in particular, $Q_R \hookrightarrow k \cdot I_R$. Since I_R is noetherian, so is Q_R, and by Exercise (10.15)(1), this implies that $R = Q$. Therefore, R is (right) artinian, and hence $\text{length}(I_R) < \infty$ as desired. □

The above remarkable argument (in the case when $R = S$ and I is an ideal in R) appeared in Lenagan [75] in the "Short Notes" section of the *Proceedings* of the AMS. Lenagan's paper seemed to fit perfectly the explicitly stated policy of the "Short Notes" department, which is, "to publish very short papers of an unusually elegant and polished character, for which there is no other outlet." Indeed, although the argument above is quite short, it seemed to have used the full force of Goldie's First Theorem (11.16), and it is not clear at all how one could have proved (11.30) otherwise.

It is of interest to mention an application of (11.30) to the so-called artinian radicals of a ring. For any ring R, let $A^r(R)$ be the sum of all right ideals $\mathfrak{A} \subseteq R$ which are artinian as right R-modules. Since $r\,\mathfrak{A}$ is a homomorphic image of $\mathfrak{A}$ for any $r \in R$, $A^r(R)$ is an ideal of R. This is called the *right artinian radical* of R; the *left artinian radical* $A^\ell(R)$ is defined similarly. Clearly, $A^r(R) \supseteq \text{soc}(R_R)$, and $A^\ell(R) \supseteq \text{soc}({}_RR)$.

(11.34) Remark. $A^r(R) = R$ *iff* R *is right artinian.* The "if" part is clear. For the "only if" part, assume $A^r(R) = R$. Then $1 \in \mathfrak{A}_1 + \cdots + \mathfrak{A}_n$ for suitable right ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ which are artinian as right R-modules. But then R_R is a homomorphic image of $\mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$, so it is artinian. Using this remark, we see easily that $A^\ell(R) \neq A^r(R)$ in general. For, if R is a ring which is left artinian but not right artinian, then $A^\ell(R) = R \neq A^r(R)$. For another, somewhat different example, see Exercise 12.

Now consider a ring R which is *right noetherian.* Then there exists a maximal artinian right ideal $\mathfrak{A}_0$. Since the sum of any two artinian right ideals is artinian, $\mathfrak{A}_0$ contains all artinian right ideals. This implies that $\mathfrak{A}_0 = A^r(R)$, and this is the largest artinian right ideal, and hence the largest right ideal of finite length. Similarly, if R is left noetherian, $A^\ell(R)$ is the largest left ideal of finite length. We close with the following interesting application of (11.30).

(11.35) Corollary. *For any noetherian ring R, $A^\ell(R) = A^r(R)$.*

Proof. Since the ideal $A^\ell(R)$ has finite length as a left R-module, it also has finite length as a right R-module, by (11.30). Therefore, $A^\ell(R) \subseteq A^r(R)$, and similarly, $A^r(R) \subseteq A^\ell(R)$. □

§11D. Semiprime Rings

In this subsection, we collect some special results on semiprime rings that are relevant to Goldie's Theorem (11.13). In particular, at the end of this subsection, we shall give a reworked list of characterizations for *commutative* semiprime Goldie rings.

We begin with the following basic observation on semiprime rings.

(11.36) Lemma. *Let $\mathfrak{A}$ be an ideal in a semiprime ring. Then* $\operatorname{ann}_\ell \mathfrak{A} = \operatorname{ann}_r \mathfrak{A}$. *In particular, this is an ideal (which will henceforth be denoted by* $\operatorname{ann} \mathfrak{A}$*).*

Proof. If $x\mathfrak{A} = 0$, then $(\mathfrak{A}x)^2 = \mathfrak{A}x\mathfrak{A}x = 0$ implies $\mathfrak{A}x = 0$. Similarly, $\mathfrak{A}x = 0$ implies $x\mathfrak{A} = 0$. □

(11.37) Definition. In a semiprime ring R, an ideal $\mathfrak{B}$ is said to be an *annihilator* (ideal) if $\mathfrak{B} = \operatorname{ann} \mathfrak{A}$ for some ideal $\mathfrak{A}$. As is easily seen, $\mathfrak{B}$ is an annihilator iff $\mathfrak{B} = \operatorname{ann}(\operatorname{ann} \mathfrak{B})$. (See also Exercise 16(a).)

For any ring R and (R, R)-bimodules, ${}_RM_R$, we can define essential submodules, uniform submodules, uniform dimensions, complements, and the like, by interpreting "submodules" to mean (R, R)-submodules of M. The general results developed in §6 can all be carried over to this setting. To see this, we can simply remark that an (R, R)-bimodule is the same as a right module over the ring $R \otimes_{\mathbb{Z}} R^{\text{op}}$. Therefore, we need only apply the results of §6 to the ring $R \otimes_{\mathbb{Z}} R^{\text{op}}$ to get the desired results for bimodules. In particular, we can talk about the essentiality of $\mathfrak{A}$, u.dim $\mathfrak{A}$, complements to $\mathfrak{A}$, and so on, for ideals in $\mathfrak{A}$ in a ring R, by viewing $\mathfrak{A}$ as ${}_R\mathfrak{A}_R$.

(11.38) Lemma. *Let $\mathfrak{A}$ be an ideal in a semiprime ring R. Then $\mathfrak{A}$ has a unique complement* $\operatorname{ann} \mathfrak{A}$, *and we have* $\mathfrak{A} \oplus \operatorname{ann} \mathfrak{A} \subseteq_e {}_RR_R$ *(essential in the two-sided sense).*

Proof. Let $\mathfrak{B}$ be *any* complement to $\mathfrak{A}$; that is, $\mathfrak{B}$ is an ideal maximal with respect to the property of having zero intersection with $\mathfrak{A}$. (Recall that such a complement always exists, by Zorn's Lemma.) Note that $\mathfrak{A}\mathfrak{B} \subseteq \mathfrak{A} \cap \mathfrak{B} = 0$, so $\mathfrak{B} \subseteq \operatorname{ann} \mathfrak{A}$. Since R is semiprime, $\mathfrak{A} \cap \operatorname{ann} \mathfrak{A} = 0$, so we must have $\mathfrak{B} = \operatorname{ann} \mathfrak{A}$. This shows that $\operatorname{ann} \mathfrak{A}$ is the unique complement to $\mathfrak{A}$ (in R), and by (6.19), $\mathfrak{A} \oplus \operatorname{ann} \mathfrak{A} \subseteq_e {}_RR_R$. □

An immediate consequence of the lemma is the following characterization of annihilator ideals in a semiprime ring.

(11.39) Corollary. *Let $\mathfrak{B}$ be an ideal in a semiprime ring R. Then $\mathfrak{B}$ is an annihilator iff it is a complement.*

Remark. In a semiprime ring R, the partially ordered set of all annihilator ideals has a natural structure of a "complete Boolean algebra". The "meet" of an arbitrary family of annihilators $\{\mathfrak{B}_i\}$ is their intersection $\bigcap_i \mathfrak{B}_i$ (which is clearly an annihilator), and the "join" of the same family is the intersection of all annihilators containing all of the $\mathfrak{B}_i$'s. The complement operation in the Boolean algebra is given by taking annihilators.[78] For more details of this construction, we refer the reader to Lambek [66: p. 111]. To make our exposition self-contained, however, we shall carry out all of our proofs in this subsection *without* explicitly assuming the existence of such a Boolean structure. Nevertheless, it will be convenient to speak of this Boolean structure at least in some of the examples given below.

(11.40) Lemma. *Let $\mathfrak{A}$ be an ideal in a semiprime ring R. Let S be the set of minimal prime ideals of R which do not contain $\mathfrak{A}$. Then* $\text{ann}\,\mathfrak{A} = \bigcap\{\mathfrak{p} : \mathfrak{p} \in S\}$.

Proof. Let $\mathfrak{B} = \bigcap\{\mathfrak{p} : \mathfrak{p} \in S\}$. Any element in $\mathfrak{A} \cap \mathfrak{B}$ is in the intersection of all minimal primes of R, so $\mathfrak{A} \cap \mathfrak{B} = 0$ (*FC*–Exercise (10.14)). In particular, $\mathfrak{A}\mathfrak{B} = 0$, so $\mathfrak{B} \subseteq \text{ann}\,\mathfrak{A}$. On the other hand, for any $\mathfrak{p} \in S$, $\mathfrak{A} \cdot \text{ann}\,\mathfrak{A} = (0) \subseteq \mathfrak{p}$ implies that $\text{ann}\,\mathfrak{A} \subseteq \mathfrak{p}$ (since $\mathfrak{A} \nsubseteq \mathfrak{p}$). Therefore, $\text{ann}\,\mathfrak{A} \subseteq \mathfrak{B}$. □

In a (semiprime) ring R, an annihilator $\mathfrak{B}$ is said to be a *maximal annihilator* if $\mathfrak{B} \neq R$, and there is no annihilator strictly between $\mathfrak{B}$ and R.

(11.41) Theorem. *For any ideal $\mathfrak{A}$ in a semiprime ring R, the following are equivalent:*

(1) *$\mathfrak{A}$ is a maximal annihilator.*
(2) *$\mathfrak{A}$ is a minimal prime and an annihilator.*
(3) *$\mathfrak{A}$ is a prime and an annihilator.*
(4) $\mathfrak{A} = \text{ann}\,U$ *for some uniform ideal* $U \subseteq R$.

If R has only finitely many minimal primes, then these are also equivalent to:

(5) *$\mathfrak{A}$ is a minimal prime.*

Proof. (1)$\Longrightarrow$(2). Say $\mathfrak{A} = \text{ann}\,\mathfrak{A}'$ ($\neq R$). Suppose $\mathfrak{B}\mathfrak{C} \subseteq \mathfrak{A}$ but $\mathfrak{C} \nsubseteq \mathfrak{A}$, where $\mathfrak{B}$, $\mathfrak{C}$ are ideals. We have $0 \neq \mathfrak{C}\mathfrak{A}' \subseteq \mathfrak{A}'$, so

$$R \neq \text{ann}(\mathfrak{C}\mathfrak{A}') \supseteq \text{ann}\,\mathfrak{A}',$$

hence $\text{ann}(\mathfrak{C}\mathfrak{A}') = \text{ann}\,\mathfrak{A}'$. Since $\mathfrak{B}(\mathfrak{C}\mathfrak{A}') = 0$, we must have $\mathfrak{B}\mathfrak{A}' = 0$; i.e., $\mathfrak{B} \subseteq \text{ann}\,\mathfrak{A}' = \mathfrak{A}$. This shows that $\mathfrak{A}$ is prime. If $\mathfrak{p}$ is a prime $\subsetneq \mathfrak{A}$, then $\mathfrak{A}'\mathfrak{A} = (0) \subseteq \mathfrak{p}$ implies $\mathfrak{A}' \subseteq \mathfrak{p} \subseteq \mathfrak{A}$; hence $(\mathfrak{A}')^2 = 0$ (while $\mathfrak{A}' \neq 0$), a contradiction to R being semiprime.

[78]Note that the Boolean algebra obtained here always contains the Boolean algebra $B'(R)$ defined in Exercise (7.30).

(2)$\Longrightarrow$(3) is a tautology.

(3)$\Longrightarrow$(4). Let $U := \text{ann}\ \mathfrak{A}$. Since $\mathfrak{A}$ is an annihilator, $\mathfrak{A} = \text{ann}\ U$. If U is not uniform, there would exist nonzero ideals U_1, U_2 with $U_1 \oplus U_2 \subseteq U$. Since all minimal primes intersect at (0), $U_1 \not\subseteq \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Then $U_1 \cdot \text{ann}\ U_1 = (0) \subseteq \mathfrak{p}$ implies $\mathfrak{p} \supseteq \text{ann}\ U_1$. But

$$U_1 U_2 = (0) \neq U U_2 \implies \text{ann}\ U_1 \supsetneq \text{ann}\ U = \mathfrak{A},$$

and hence $\mathfrak{p} \supsetneq \mathfrak{A}$, a contradiction.

(4)$\Longrightarrow$(1). For $\mathfrak{A}$ as in (4), suppose $\mathfrak{A} \subsetneq \mathfrak{B}$, where $\mathfrak{B}$ is an annihilator. Since $\mathfrak{A} = \text{ann}\ U$ is a complement to U, we have $\mathfrak{B} \cap U \neq (0)$, and so $\mathfrak{B} \cap U \subseteq_e U$ (as ideals). Therefore, by (11.38),

$$(\mathfrak{B} \cap U) \oplus \text{ann}\ U \subseteq_e U \oplus \text{ann}\ U \subseteq_e {}_R R_R\ .$$

Since the LHS is contained in $\mathfrak{B}$, we have $\mathfrak{B} \subseteq_e {}_R R_R$. This (together with $\mathfrak{B} \cap \text{ann}\ \mathfrak{B} = 0$) implies that $\text{ann}\ \mathfrak{B} = 0$ and so $\mathfrak{B} = \text{ann}(\text{ann}\ \mathfrak{B}) = R$, proving (1).

Finally, assume R has only finitely many minimal primes, say, $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Using $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = 0$, we see easily from (11.40) that each $\mathfrak{p}_i$ is the annihilator of the intersection of the other $\mathfrak{p}_j$'s. Thus, in this case (5)$\Longleftrightarrow$(2). □

(11.42) Corollary. *Let R be a semiprime ring with finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Then an element $a \in R$ is regular iff $a + \mathfrak{p}_i$ is regular in $R/\mathfrak{p}_i$ for every i.*

Proof. For the "if" part, suppose $ab = 0$. Working in $R/\mathfrak{p}_i$, we see that $b \in \mathfrak{p}_i$ and hence $b \in \bigcap_i \mathfrak{p}_i = 0$. Similarly, $ba = 0 \Rightarrow b = 0$. Conversely, assume a is regular in R, and fix any index i. As observed above, $\mathfrak{p}_i = \text{ann}(\mathfrak{A})$ where $\mathfrak{A}$ is the intersection of the $\mathfrak{p}_j$'s for $j \neq i$. Suppose $ab \in \mathfrak{p}_i$. Then $ab\,\mathfrak{A} = 0$ and so $b\,\mathfrak{A} = 0$ since a is regular. Therefore $b \in \text{ann}\ \mathfrak{A} = \mathfrak{p}_i$. Similarly $ba \in \mathfrak{p}_i \implies b \in \mathfrak{p}_i$, so $a + \mathfrak{p}_i$ is regular in $R/\mathfrak{p}_i$. □

The following theorem offers various criteria for a semiprime ring to have only finitely many minimal primes.

(11.43) Theorem. *For any semiprime ring R, the following are equivalent:*

(1) $n := \text{u.dim}\ {}_R R_R < \infty$.
(2) *The number t of minimal primes in R is finite.*
(3) *The number m of annihilators in R is finite.*
(4) *R has* ACC *on annihilators.*
(4′) *R has* DCC *on annihilators.*
(5) *R has* ACC *on complements.*
(5′) *R has* DCC *on complements.*

If these conditions hold, then $n = t$ and $m = 2^t$. Finally, $n = t = 1$ iff R is a prime ring.

Proof. (1)$\Longrightarrow$(2). Let U_i $(1 \le i \le n)$ be uniform ideals of R such that

$$U_1 \oplus \cdots \oplus U_n \subseteq_e {}_R R_R.$$

By (11.41), $\mathfrak{p}_i := \operatorname{ann} U_i$ are minimal primes, and we have

$$(*) \qquad \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t \subseteq \operatorname{ann}(U_1 \oplus \cdots \oplus U_n) = 0,$$

since $\operatorname{ann}(U_1 \oplus \cdots \oplus U_n)$ has zero intersection with $U_1 \oplus \cdots \oplus U_n$. If $\mathfrak{p}$ is any minimal prime, $(*)$ implies that $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some i and hence $\mathfrak{p} = \mathfrak{p}_i$. This proves (2), and we see that $t = n$.

(2)$\Longrightarrow$(3). Given $t < \infty$, we see right away from (11.40) that $m \le 2^t < \infty$. Using the last part of (11.41) (and the definition of a prime ideal), we see further that $m \ge 2^t$, so we have $m = 2^t$.

(3)$\Longrightarrow$(4) is a tautology.

(4)$\Longleftrightarrow$ (5) and (4$'$) $\Longleftrightarrow$ (5$'$) follow from (11.39).

(4)$\Longleftrightarrow$ (4$'$) follows by taking annihilators.

(5)$\Longrightarrow$(1) follows by applying (6.30)$'$ to the bimodule ${}_R R_R$.

The last statement in the theorem is now immediate. □

For semiprime rings, (4) is generally weaker than ACC on *right* annihilators, and (5) is weaker than ACC on *right* complements (in R_R). The equivalence of (4) and (5) is clear from (11.39), but ACC on right annihilators and ACC on right complements are independent in general! Semiprime rings satisfying the conditions in (11.43) comprise a class that is considerably larger than the class of semiprime one-sided Goldie rings. (Recall that for semiprime right Goldie rings, we have given another construction of the minimal primes of R in (11.22).)

(11.44) Corollary. *Let R be a semiprime ring with finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Then R is right Goldie iff each $R/\mathfrak{p}_i$ $(1 \le i \le t)$ is right Goldie, in which case $Q^r_{cl}(R) \cong \prod_i Q^r_{cl}(R/\mathfrak{p}_i)$.*

Proof. Assume that each $R/\mathfrak{p}_i$ is right Goldie, and consider the embedding $R \to \prod_i R/\mathfrak{p}_i$. Since $\prod_i R/\mathfrak{p}_i$ satisfies ACC on right annihilators, so does R by (6.61). Also,

$$\text{u.dim } R_R \le \sum_i \text{u.dim}(R/\mathfrak{p}_i)_R = \sum_i \text{u.dim}(R/\mathfrak{p}_i)_{R/\mathfrak{p}_i} < \infty,$$

so R is right Goldie. The rest follows from (11.22). □

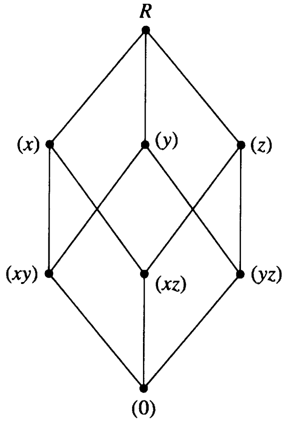
We pause to offer some examples for (11.43).

(11.45) Examples.

(1) Let $R = R_1 \times \cdots \times R_t$, where the R_i 's are prime rings. Then the minimal primes of R are $\mathfrak{p}_i = \prod_{j \ne i} R_j$ $(1 \le i \le t)$. These are clearly maximal annihilators, and each $\mathfrak{p}_i$ is the annihilator of the uniform ideal R_i in R, as predicted

by (11.41). The 2^t annihilator ideals in R are given by all subproducts of the R_i 's. Thus, the Boolean algebra of annihilators in this example is the same as the Boolean algebra $B'(R)$ in Exercise (7.30).

(2) Let k be any field, and R be the (commutative) semiprime ring $k[x, y, z]$ with the defining relation $xyz = 0$. Since the principal ideals (x), (y), (z) are prime and their intersection is (0), they are all the minimal primes in R. (Note also that $(x) = \text{ann}(yz)$, $(y) = \text{ann}(xz)$, and $(z) = \text{ann}(xy)$.) In view of (11.40), the Boolean algebra of all annihilators in R is given by:



This is, of course, isomorphic to the Boolean algebra of all subsets of a 3-element set. Note that this is bigger than the Boolean algebra $B'(R)$ which has only two elements since R has no nontrivial idempotents. This example generalizes readily to the case when $R = k[x_1, \ldots, x_t]$ with the relation $x_1 \cdots x_t = 0$.

In the rest of this subsection, let us focus on *commutative* rings. In this more specialized setting, we shall offer a reworked list of characterizations for semiprime Goldie rings. The redundancies in some of the conditions in (11.13) in the commutative case are eliminated in this version. Also, we shall give a direct proof for this new version using only results in this subsection, mainly independently of (11.13).

(11.46) Theorem. *For a reduced (i.e., semiprime) commutative ring R, the seven conditions* (1) *through* (5′) *in* (11.43) *are equivalent to each of the following:*

(6) *R is Goldie.*
(7) *Every dense ideal $\mathfrak{A}$ of R contains a regular element.*[79]
(8) *Every essential ideal $\mathfrak{A}$ of R contains a regular element.*
(9) *R is an order in a finite direct product of fields.*

[79]It is helpful to recall here that, in a commutative ring R, $\mathfrak{A} \subseteq_d R$ simply means that $\text{ann}(\mathfrak{A}) = 0$; that is, $\mathfrak{A}$ is *faithful* as an R-module (see (8.3)(4)).

Proof. (9)$\Longrightarrow$(6) follows from (11.12), and (6)$\Longrightarrow$(4) (in (11.43)) is a tautology. (4)$\Longrightarrow$(7) is true by (8.31)(1) (even without the assumption that R is semiprime). (7)$\Longrightarrow$(8). By (7.8), R is a nonsingular ring, so by (8.9), essential ideals are dense in R.

(8)$\Longrightarrow$(9). Let $S := \mathcal{C}_R$ be the multiplicative set of regular elements of R and let $Q = RS^{-1} = R_S$. Repeating the last part of the argument for (5)$\Longrightarrow$(1) in (11.13), we see that Q is semisimple. Since Q is commutative, it is a finite direct product of fields. □

It is also of interest to see directly, by using the standard tools of commutative algebra, that Condition (2) (R is reduced and has only finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$) implies Condition (9) on the structure of the classical ring of quotients $Q = R_S$. In fact, from (11.42), we obtain the equation[80]

(11.47) $$S = \mathcal{C}_R = R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t).$$

Thus, by the Lemma of Prime Avoidance and standard facts about localizations, the only primes in Q are $(\mathfrak{p}_1)_S, \ldots, (\mathfrak{p}_t)_S$, and these are all maximal. Since their intersection is (0), the Chinese Remainder Theorem gives

$$Q = R_S \cong \prod_i D_i\,,$$

where $D_i := R_S/(\mathfrak{p}_i)_S$ ($1 \leq i \leq t$) are fields.

§11E. Nil Multiplicatively Closed Sets

In the ring theory literature, there are many theorems about the nilpotency of nil subrings and nil one-sided ideals in rings. Clearly, when we talk about "nil subrings" S in a ring R, the word "subring" means that S is a subgroup of R that is closed under multiplication, but which *may not contain the identity* 1 *of* R. This more general notion of "subring" will be used throughout this subsection. In particular, any one-sided ideal in a ring R is a subring of R in this sense.

The earliest results on "nil $\Longrightarrow$ nilpotent" were due to Levitzki and Hopkins, who proved that any nil subring in a right artinian ring is necessarily nilpotent. Later, Levitzki proved that any nil one-sided ideal in a right noetherian ring is nilpotent (cf. *FC*–(10.30)). As a further improvement of this result, Goldie showed that any nil subring of a right noetherian ring is nilpotent. In this subsection, we shall present a theorem of Herstein, Small, and Lanski which includes all of the above results. Our proof of this theorem follows that of J. W. Fisher [70].

Instead of working with subrings, we shall work more generally with *multiplicatively closed* (m.c.) *subsets* below. These are simply nonempty subsets $I \subseteq R$ that are closed under multiplication. By I^n, we shall mean the set of products $x_1 \cdots x_n$, where $x_i \in I$. Clearly, I^n is also m.c., with $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$. We say that I is

[80] See also Exercise 15.

nil if, for any $x \in I$, $x^n = 0$ for some n (depending on x), and we say that I is *nilpotent* if $I^n = (0)$ for some $n \geq 1$. In general, of course I being nilpotent is a much stronger property than I being nil. For some classes of rings, however, these two properties for an m.c. set turn out to be equivalent, as we shall see below.

In *FC*–(23.13), we have defined a set $A \subseteq R$ to be *right T-nilpotent* if, for any sequence $\{x_i : i \geq 1\}$ in A, there exists an integer $m \geq 1$ such that $x_m \cdots x_2 x_1 = 0$. This notion is a useful tool for proving the nilpotency of m.c. sets, since we have the following convenient result.

(11.48) Lemma. *Let R be a ring with* ACC *on right annihilator ideals. Then an m.c. set $I \subseteq R$ is nilpotent iff it is right T-nilpotent.*

Proof. We need only prove the "if" part. Assume that the m.c. set I is not nilpotent. Since $I \supseteq I^2 \supseteq \ldots$, we have $\text{ann}_r(I) \subseteq \text{ann}_r(I^2) \subseteq \ldots$, so there exists an integer n such that $\text{ann}_r(I^n) = \text{ann}_r(I^{n+1})$. Since $I^{n+1} \neq (0)$, there exists $x_1 \in I$ such that $I^n x_1 \neq 0$, and so $I^{n+1} x_1 \neq 0$. This means that $I^n x_2 x_1 \neq 0$ for some $x_2 \in I$, and so $I^{n+1} x_2 x_1 \neq 0$. Continuing in this manner, we get a sequence $\{x_i : i \geq 1\} \subseteq I$ with $x_m \cdots x_2 x_1 \neq 0$ for all $m \geq 1$, so I is not right T-nilpotent. □

We shall now state the main theorem in this subsection. We stress again that it applies to subrings and one-sided ideals, since these are all m.c. sets.

(11.49) Theorem. *Let R be a ring with* ACC *on right annihilators, and assume, furthermore, that one of the following two conditions holds:*

(1) *R has* ACC *on left annihilators.*
(2) $\text{u.dim}\, R_R < \infty$.

Then any nil m.c. set $I \subseteq R$ is nilpotent.

(In case (1), this theorem is due to Herstein and Small. In case (2), it is due to Lanski. Note that in case (2), the Theorem simply says that *any nil m.c. set in a right Goldie ring is nilpotent.* In particular, this subsumes the theorems of Levitzki, Hopkins, and Goldie mentioned earlier in this subsection.)

Proof of (11.49). In view of (11.48), it suffices to show that I is right T-nilpotent. *Assume this is not true.* Let us say that an element $y_1 \in I$ is "bad" if there exist $y_2, y_3, \cdots \in I$ such that $y_n \cdots y_2 y_1 \neq 0$ for all n. Among elements in

(11.50) $$S_1 := \{y \in I : y \text{ is bad}\} \neq \emptyset,$$

choose $x_1 \in S_1$ with $\text{ann}_r(x_1)$ maximal. We then define $S_n \subseteq I$ and $x_n \in I$ $(n \geq 1)$ inductively as follows. If $x_1, \ldots, x_{n-1}$ are already defined, we take:

(11.51) $$S_n := \{y \in I : y x_{n-1} \cdots x_2 x_1 \text{ is bad}\}, \quad \text{and}$$

(11.52) $$x_n \in S_n \text{ such that } \text{ann}_r(x_n) \text{ is maximal.}$$

(Note that, by choice $x_{n-1}(x_{n-2}\cdots x_2x_1)$ is bad, so there exists at least one $y\in I$ such that $yx_{n-1}\cdots x_2x_1$ is bad; this guarantees that S_n is nonempty.) For the elements

$$a_n := x_n\cdots x_2x_1 \in I\backslash\{0\} \quad (n\geq 1),$$

we make the following two claims:

(A) $\text{ann}_r(x_i) = \text{ann}_r(x_{i+j}\cdots x_{i+1}x_i)$, *for any* $i, j\geq 1$;
(B) $x_ia_n = 0$ *for any* $n\geq i\geq 1$.

For (A), note that since $x_{i+j}\cdots x_i\cdots x_2x_1$ is bad, we have $x_{i+j}\cdots x_{i+1}x_i\in S_i$ and so, by the choice of x_i (cf. (11.52)):

$$\text{ann}_r(x_i)\subseteq \text{ann}_r(x_{i+j}\cdots x_{i+1}x_i) \text{ is an equality.}$$

For (B), assume, for the moment, that $x_ia_n\neq 0$, for some $n\geq i\geq 1$. Then, by (A),

$$x_{i+j}\cdots x_ia_n\neq 0 \text{ for every } j\geq 1.$$

This implies that

$$x_ia_n = (x_ix_n\cdots x_i)(x_{i-1}\cdots x_1) \text{ is bad.}$$

Therefore, $x_ix_n\cdots x_i\in S_i$, and hence (again by (11.52)):

(11.53) $$\text{ann}_r(x_i)\subseteq \text{ann}_r(x_ix_n\cdots x_i) \text{ is an equality.}$$

Fix an integer $k\geq 1$ such that $(x_n\cdots x_i)^k = 0$. Then, by (11.53):

$$x_i(x_n\cdots x_i)(x_n\cdots x_i)^{k-1} = 0 \implies x_i(x_n\cdots x_i)^{k-1} = 0.$$

Repeating this argument, we get $x_i(x_n\cdots x_i) = 0$ and hence $x_ia_n = 0$, a contradiction.

Having established (A) and (B), we now make the following two new claims:

(C) $\sum_{i=1}^{\infty} a_iR$ *is a direct sum of right ideals.*
(D) *If* $A_i = \{a_j : j\geq i\}$, *then* $\text{ann}_\ell(A_1)\subsetneq \text{ann}_\ell(A_2)\subsetneq\cdots$.

Once we have established (C) and (D), the theorem will clearly follow. To prove (D), note that (B) implies that $x_{i+1}a_j = 0$ whenever $j\geq i+1$, while $x_{i+1}a_i = a_{i+1}\neq 0$. Therefore, we have

$$x_{i+1}\in \text{ann}_\ell(A_{i+1})\backslash \text{ann}_\ell(A_i).$$

To prove (C), consider any relation:

$$a_ir_i + \cdots + a_{i+p}r_{i+p} = 0\,, \quad \text{where } r_j\in R,\ i\geq 1,\ p\geq 2.$$

Left multiplying by x_{i+1}, we have

$$x_{i+1}a_ir_i = -x_{i+1}(a_{i+1}r_{i+1} + \cdots + a_{i+p}r_{i+p}) = 0.$$

This means that $x_{i-1} \cdots x_1 r_i \in \text{ann}_r(x_{i+1}x_i)$. But, by (A), $\text{ann}_r(x_{i+1}x_i) = \text{ann}_r(x_i)$ so we have

$$0 = x_i(x_{i-1} \cdots x_1 r_i) = a_i r_i.$$

Repeating this argument, we get $a_j r_j = 0$ for $i \le j \le i + p$, thus proving (C). □

Exercises for §11

0. Let $R_i \subseteq Q_i$ ($i \in I$) be rings. Show that $\prod_i R_i$ is a right order in $\prod_i Q_i$ iff each R_i is a right order in Q_i.

1. Let $a \in R$ be right regular (i.e., $\text{ann}_r(a) = 0$), and let $I \subseteq R$ be a right ideal such that $aR \cap I = 0$.

   (a) Show that the sum $\sum_{i \ge 0} a^i I$ is direct.
   (b) From (a), deduce that if u.dim $R_R < \infty$, we must have $aR \subseteq_e R_R$.
   (c) Give an example to show that $aR \subseteq_e R_R$ need not hold if u.dim $R_R = \infty$.

2. Let R be a right Ore ring.

   (a) Show that any right ideal $\mathfrak{A}$ containing a regular element is essential (in R_R).
   (b) Show that R is semiprime right Goldie iff any essential right ideal of R contains a regular element.
   (c) Give an example of a commutative (hence Ore) ring with an essential ideal $\mathfrak{A} \subseteq R$ not containing any regular elements.

3. (Goldie) For any element a in a right Goldie ring R, show that there exists an integer $n \ge 1$ such that $\mathfrak{A} := a^n R + \text{ann}_r(a^n)$ is a direct sum and $\mathfrak{A} \subseteq_e R_R$.

4. Let R be a right Ore domain that is not left Ore, say, $Ra \cap Rb = 0$, where $a, b \in R \backslash \{0\}$. The ring $A = \mathbb{M}_2(R)$ is prime right Goldie by (11.18), so right regular elements of A are regular by (11.14)(a). Show, however, that $\sigma = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in A$ is left regular but not regular.

5. Let R be a semiprime right Goldie ring with

$$Q = Q^r_{cl}(R) \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_t}(D_t),$$

   where the D_i 's are division rings. Show that u.dim $R_R = n_1 + \cdots + n_t$, and that a right ideal $\mathfrak{A} \subseteq R$ is uniform iff $\mathfrak{A}^e$ is a minimal right ideal in Q. If, in addition, R is prime, show that u.dim R_R is the largest index of nilpotency of the nilpotent elements in R.

6. Let $x, y \in R$ where R is a semiprime right Goldie ring. If $xy \in \mathcal{C}_R$, show that $x, y \in \mathcal{C}_R$.

7. Let Q be an algebraic algebra over a field k (see FC–(4.19)). Show that any $q \in \mathcal{C}_Q$ is a unit by considering the minimal polynomial of q over k. (Thus, Q is a classical ring. This is a special case of (11.6)(2), since an algebraic algebra over a field is always strongly π-regular: see Exer. (23.6) in [Lam: 95].)

8. Show that any right self-injective ring is a classical ring.

9. Show that a f.g. projective right module M over a commutative classical ring Q is "cohopfian", in the sense that any injective Q-endomorphism $\varphi : M \to M$ is an automorphism. Is this still true if Q is not commutative? (**Hint.** Reduce to the case $M = Q^n$, and show that $\det(\varphi)$ is regular in Q.)

10. Let $\bar{R} = R/\mathfrak{A}$ where $\mathfrak{A}$ is an ideal of R.
(a) If u.dim $R_R < \infty$, is u.dim $\bar{R}_{\bar{R}} < \infty$?
(b) Exhibit a right Goldie ring R with a quotient ring $\bar{R}$ that is not right Goldie.

11. *True or False*: Every right Goldie ring is stably finite?

12. The ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right noetherian but not left noetherian, by FC–(1.22). Show that the right artinian radical $A^r(R) = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, but the left artinian radical $A^\ell(R) = 0$. (**Hint.** For the latter, use the fact that the only artinian $\mathbb{Z}$-submodule of $\mathbb{Z} \oplus \mathbb{Q}$ is (0).)

13. Find $A^r(R)$ and $A^\ell(R)$ for $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q}(t) \\ 0 & \mathbb{Q}(t) \end{pmatrix}$.

14. Use (11.43) to show that, if R has ACC on ideals, then R has only finitely many minimal primes. (For a different approach to the same problem, see FC–Exer. (10.15).)

15. Prove the equation (11.47) by a direct localization argument, without using (11.42). (Note that there is no noetherian assumption on the ring R here!)

16. Let R be a semiprime ring. (a) Show that an ideal $\mathfrak{B} \subseteq R$ is an annihilator (in the sense of (11.37)) iff $\mathfrak{B}$ is a right annihilator. (b) If R is also right Goldie and $Q = Q^r_{cl}(R)$ has t Wedderburn (simple) components, show that the annihilator ideals of R are exactly the contractions to R of the 2^t ideals of Q.

17. Let R be a semiprime ring with only finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Let $\mathfrak{B}_i = \prod_{j \neq i} \mathfrak{p}_j$, and let $\bar{\mathfrak{B}}_i = (\mathfrak{B}_i + \mathfrak{p}_i)/\mathfrak{p}_i$ in $\bar{R}_i = R/\mathfrak{p}_i$. After identifying R with a subring of $S := \prod_i \bar{R}_i$, show that $\mathfrak{B} := \prod_i \bar{\mathfrak{B}}_i$ is an ideal of S lying in R, and that $\text{ann}^S_\ell(\mathfrak{B}) = 0$.

18. Keep the notations in Exercise 17, and assume that each $\bar{R}_i = R/\mathfrak{p}_i$ is right Goldie, with $Q_i := Q^r_{cl}(\bar{R}_i)$. Independently of (11.44), show that R is also right Goldie, with $Q^r_{cl}(R) \cong Q := \prod_i Q_i$.

19. (See Exercise (8.6).) Let R be the (commutative) ring $\mathbb{Q}\langle\{x_i : i \geq 1\}\rangle$ with relations $x_i x_j = 0$ for all unequal i, j.

(1) Show that R is semiprime (i.e., reduced).
(2) R does not satisfy ACC on annihilators.
(3) u.dim $R_R = \infty$.
(4) Let $p_i(x_i)$ $(i \geq 1)$ be nonzero polynomials without constant terms. Then the ideal generated by $\{p_i(x_i) : i \geq 1\}$ is dense, but contains no regular elements of R.
(5) Show that the minimal primes of R are given precisely by $\mathfrak{p}_i = \sum_{j \neq i} Rx_j$ for all $i \geq 1$.

20. Show that R is a reduced right Goldie ring iff R is a right order in a finite direct product of division rings.

The following four exercises (with hints) are taken from Procesi-Small [65], where they used these results to give an alternative proof for the main equivalence (1)⇔(2) in Goldie's Theorem (11.13). We assume, in these exercises, that *R is a semiprime ring satisfying* ACC *on right annihilators*.

21. Let $B \subseteq A$ be right ideals in R such that $\text{ann}_\ell(A) \subsetneq \text{ann}_\ell(B)$. Show that there exists $x \in A$ such that $xA \neq 0$ and $xA \cap B = 0$. In particular, B cannot be essential in A. (**Hint.** Pick a left annihilator U minimal w.r.t. $\text{ann}_\ell(A) \subsetneq U \subseteq \text{ann}_\ell(B)$.)

22. Deduce from Exercise 21 that any chain of right annihilators in R has length $\leq$ u. dim R_R. (Note that this conclusion is already available in (7.51)′ in view of (11.13). Exercise 21 just provided an alternative route to the same result.)

23. Let $x,\ y \in R$. If $xR \subseteq_e R$ and $yR \subseteq_e R$, show that $xyR \subseteq_e R$. (**Hint.** For any right ideal $C \neq 0$, consider

$$A = \{s \in R :\ xs \in C\} \supseteq B = \text{ann}_r(x),$$

and use Exercise 21 to show that $C \cap xyR \neq 0$.)

24. Let $a \in R$. If $aR \subseteq_e R$, show that a is regular in R. (**Hint.** Apply Exercise 21 with $B = aR$ and $A = R$ to show that $\text{ann}_\ell(a) = 0$. Then use Exercise 23 to show that $\text{ann}_r(a) = 0$.)

25. Let R be a prime right Goldie ring, and I be an essential right ideal in R. Show that any coset $c + I$ $(c \in R)$ contains a regular element of R. (**Hint.** Choose $a \in c + I$ with $\text{ann}_r(a)$ minimal and check that a is regular.)

26. Prove the result in Exercise 25 for any *semiprime* right Goldie ring R (by using a reduction to the prime case).

27. (Robson) Let R be any semiprime right Goldie ring, and J be an essential right ideal in R. Show that J is generated as a right ideal by the regular elements in J.

28. (Small) If a ring R is right perfect and right Rickart, show that it is semiprimary. (**Hint.** By Bass' Theorem (FC–(24.25)), R satisfies DCC on principal left ideals, so R has no infinite orthogonal set of nonzero idempotents. Then use (7.55) and (11.49).)

29. For any idempotent e in a semiprime ring R, show that the following are equivalent: (a) e is central in R; (b) $(1-e)Re = 0$; (c) eR is an ideal in R.

30. Let R be a semiprime ring with $\text{u.dim}\, R_R < \infty$.

    (1) Show that $\text{soc}(R_R) = eR$ for a central idempotent $e \in R$.
    (2) There exists a direct product decomposition $R = S \times T$ where S is a semisimple ring, and T is a semiprime ring with $\text{soc}(T_T) = 0$.
    (3) If $\text{soc}(R_R) \subseteq_e R_R$, show that R is a semisimple ring.

    Deduce from the above that a prime ring R is simple artinian if and only if $\text{u.dim}\, R_R < \infty$ and R has a minimal right ideal.

31. (Attarchi) Let R be a ring such that $dR \subseteq_e R_R$ whenever $\text{ann}_r(d) = 0$, and $Rd \subseteq_e {}_RR$ whenever $\text{ann}_\ell(d) = 0$. Let S be the ring of 2×2 upper triangular matrices over R. Show that $\alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is regular in S iff a and c are both regular in R.

32. Let R and S be as in the last exercise. If R is right Ore, with $Q = Q^r_{cl}(R)$, show that S is also right Ore, with $Q^r_{cl}(S)$ given by the ring T of 2×2 upper triangular matrices over Q.

§12. Artinian Rings of Quotients

§12A. Goldie's ρ-Rank

In the last section, we have characterized the class of rings that are right orders in semisimple rings: these are precisely the semiprime right Goldie rings. The next natural step to take will be to characterize, more generally, the class of rings that are right orders in *right artinian* rings. In other words, what rings R have a classical right ring of quotients $Q^r_{cl}(R)$ that is right artinian? This question was considered, and satisfactorily answered, by J. C. Robson, L. W. Small, and T. D. Talintyre in the mid 1960s. In this section, we shall give a characterization of the rings in question, following a 1979 paper of R. Warfield. Warfield's characterization involves the notion of the "ρ-rank" of a module, which is a variant of the reduced rank defined in §7C. The idea of using such ranks comes from the seminal work of A. Goldie in 1964.

To begin with, we introduce a notation which seems to have become rather standard in the ring theory literature. For any ideal N in a ring R, we shall write $\mathcal{C}(N)$ (or, if necessary, $\mathcal{C}_R(N)$) for the set of elements $r \in R$ such that $r + N$ is a regular element in R/N. In particular, $\mathcal{C}(0) = \mathcal{C}_R(0)$ is the set of regular elements

of R. (In §11, we have written $\mathcal{C}_R$ for this set.) It is easy to see that $\mathcal{C}(N)$ is a multiplicatively closed set, for any ideal $N \subseteq R$. In general, we do not have an inclusion relationship between $\mathcal{C}(N)$ and $\mathcal{C}(0)$. But for right artinian rings, we have the following result, which turns out to play a rather special role in this section.

(12.1) Lemma. *Let Q be a right artinian ring. Then $\mathcal{C}_Q(0) = \mathcal{C}_Q(J)$ for any nil ideal $J \subseteq Q$.*

Proof. Recall that an element of a right artinian ring is regular iff it is a unit. (This follows by an easy length argument: see *FC*–Exercise (4.16).) On the other hand, since J is nil, an element $q \in Q$ is a unit in Q iff $q + J$ is a unit in Q/J. Taken together, these two observations clearly imply the Lemma. □

Next we shall introduce the notion of the "ρ-rank" of a module. *First consider the case of a semiprime right Goldie ring S.* Let $T = Q_{cl}^r(S)$, which is a semisimple ring. For any right S-module M, we have defined in (7.34) its *reduced rank*, $\text{rank}_S(M)$ which is a nonnegative integer or the symbol ∞. In (11.15), we have further obtained a simple interpretation of this rank, namely,

(12.2) $$\text{rank}_S(M) = \text{length}_T(M \otimes_S T).$$

Using this interpretation, we can, for instance, give a "more conceptual" proof for the fact that "rank" is additive over short exact sequences. In fact, this follows quickly now from the exactness of the tensor product with T (Exercise 10.18), and the additivity of the length function over short exact sequences of T-modules.

Let us now consider more general rings. Specifically, take any ring R that satisfies the following two properties:

(12.3) *The prime radical (a.k.a. lower nilradical) $N = \text{Nil}_* R$ is nilpotent.*

(12.4) *$S := R/N$ is (semiprime) right Goldie.*

The idea is to define a new "ρ-rank" for right R-modules by using the reduced rank for right S-modules. For any right R-module M, a filtration

(12.5) $$0 = M_n \subseteq \cdots \subseteq M_0 = M$$

is said to be a *Loewy series* for M if $M_i N \subseteq M_{i+1}$ for all i. (Such a series always exists; e.g., fixing an integer n with $N^n = 0$, we can take $M_i = MN^i$.) Noting that each M_i/M_{i+1} in (12.5) is a right S-module, we define

(12.6) $$\rho_R(M) = \sum \text{rank}_S(M_i/M_{i+1})$$

for *any* Loewy series (12.5) in M.

(12.7) Proposition. *The ρ-rank $\rho(M)$ is independent of the choice of the Loewy series, and ρ is additive over short exact sequences of right R-modules.*

Proof. Since (by Schreier's Theorem) any two Loewy series for M have a "common refinement," it suffices to show that the sum in (12.6) is unchanged if we refine a given Loewy series (12.5). But each M_i/M_{i+1} is a right S-module, so this follows from the additivity of "rank_S" over short exact sequences of right S-modules. The last part of (12.7) now follows easily by "joining" Loewy series. □

Having successfully defined the ρ-rank for modules over rings satisfying (12.3), (12.4), let us note below the most important special case which perhaps provided the main impetus for all of this work.

(12.8) Corollary. *Let R be any right noetherian ring. Then ρ_R is defined, and $\rho_R(M) < \infty$ for any f.g. right R-module M.*

Proof. By Levitzki's Theorem (*FC*–(10.30)), $N = \text{Nil}_* R$ is nilpotent, and since $S = R/N$ remains right noetherian, S is semiprime right Goldie. Therefore, ρ_R is defined. To prove the last part of the Corollary, it suffices to consider the case when $MN = 0$. In this case M is a f.g. S-module, so for $T = Q^r_{cl}(S)$, $M \otimes_S T$ is a f.g. T-module. Therefore, by (12.2):

$$\rho_R(M) = \text{rank}_S M = \text{length}_T(M \otimes_S T) < \infty.$$

□

Next, we shall characterize R-modules of zero ρ-rank. This characterization will be crucial for the work in the rest of this section.

(12.9) Proposition. *Let M be a right R-module, where R satisfies* (12.3) *and* (12.4). *Then, $\rho_R(M) = 0$ iff, for any $m \in M$, there exists $r \in \mathcal{C}(N)$ such that $mr = 0$.*

Proof. Fix a Loewy series $0 = M_n \subseteq \cdots \subseteq M_0 = M$. First assume $\rho_R(M) = 0$. Then, for $S = R/N$, each $\text{rank}_S(M_i/M_{i+1}) = 0$, and this means that each M_i/M_{i+1} is a torsion S-module. (For the notion of S-torsion modules, see Exercise (10.19).) Thus, for any $m \in M$, we can find $r_1, \ldots, r_n \in \mathcal{C}(N)$ such that $mr_1r_2\cdots r_n \in M_n = 0$, so we are done by choosing $r = r_1 \cdots r_n \in \mathcal{C}(N)$. Conversely, if every $m \in M$ is killed by some element of $\mathcal{C}(N)$, each M_i/M_{i+1} is clearly a torsion S-module, so $\rho_R(M) = \sum \text{rank}_S(M_i/M_{i+1}) = 0$. □

§12B. Right Orders in Right Artinian Rings

We are now ready to state and prove the following characterization theorem for rings that are right orders in right artinian rings. The characterization is in terms of the prime radical $N = \text{Nil}_* R$, the multiplicative sets $\mathcal{C}(0)$, $\mathcal{C}(N)$, and Goldie's ρ-rank ρ_R.

(12.10) Theorem. *A ring R is a right order in a right artinian ring iff the following conditions are satisfied:*

(A) $N = \text{Nil}_* R$ *is nilpotent.*
(B) $S = R/N$ *is (semiprime) right Goldie.*
(C) $\rho_R(R_R) < \infty$.
(D) $\mathcal{C}(N) \subseteq \mathcal{C}(0)$.

For any such ring R, we have $\mathcal{C}(N) = \mathcal{C}(0)$ and

$$\rho_R(R_R) = \text{length}_Q(Q_Q) \geq \text{u.dim}\, R_R,$$

where $Q = Q^r_{cl}(R)$.

Proof. First assume (A), (B), and (C) are satisfied. We claim that:

$$(12.11) \qquad \forall\, s \in \mathcal{C}(0),\ a \in R,\ \exists\, t \in \mathcal{C}(N),\ b \in R \text{ such that } at = sb.$$

In fact, since $sR \cong R$ as right R-modules, $\rho(R) = \rho(sR)$. From

$$(12.12) \qquad \rho(R) = \rho(sR) + \rho(R/sR) < \infty,$$

it follows that $\rho(R/sR) = 0$. Therefore, (12.11) follows from (12.9). In particular, taking $a = 1$, we see that for any $s \in \mathcal{C}(0)$, there exist $t \in \mathcal{C}(N)$ and $b \in R$ such that $t = sb$. In the semiprime right Goldie ring S, $\bar{s}\bar{b} = \bar{t}$ is regular, so $\bar{s}$ is also regular by Exercise (11.6). Therefore, $s \in \mathcal{C}(N)$, so we have proved that $\mathcal{C}(0) \subseteq \mathcal{C}(N)$. If (D) is also satisfied, we'll get $\mathcal{C}(0) = \mathcal{C}(N)$. With this, (12.11) says that $\mathcal{C}(0)$ is right permutable, so $Q := Q^r_{cl}(R)$ exists. To prove that Q is right artinian, consider first a pair of right ideals $\mathfrak{A} \subsetneq \mathfrak{B}$ in Q. Let $A = \mathfrak{A} \cap R$ and $B = \mathfrak{B} \cap R$. Since $\mathfrak{A} = AQ$ and $\mathfrak{B} = BQ$, we have $A \subsetneq B$. If $\rho_R(B/A) = 0$, then (by (12.9)) for every $b \in B$, there exists $s \in \mathcal{C}(N) = \mathcal{C}(0)$ such that $bs \in A$; that is,

$$b \in R \cap As^{-1} \subseteq R \cap \mathfrak{A} = A,$$

a contradiction. Therefore, we must have $\rho_R(B/A) > 0$. Thus, for any chain of right ideals

$$0 = \mathfrak{A}_n \subsetneq \cdots \subsetneq \mathfrak{A}_0 = Q,$$

we have a bound $n \leq \rho_R(R_R) < \infty$. In particular, the ring Q is right artinian.

For the converse, assume that $Q := Q^r_{cl}(R)$ exists and is right artinian. Let $J = \text{Nil}_* Q$, which is, of course, just the Jacobson radical of Q (since the latter is nilpotent). Let $\bar{Q}$ be the semisimple ring Q/J, and let $\bar{R} = R/J \cap R \subseteq \bar{Q}$. For any $s \in \mathcal{C}(0)$, we have $s \in \text{U}(Q)$; hence $\bar{s} \in \text{U}(\bar{Q})$ and so $\bar{s}$ is regular in $\bar{R}$. Since every element in Q has the form as^{-1} $(a \in R,\ s \in \mathcal{C}(0))$, every element in $\bar{Q}$ has the form $\bar{a}\bar{s}^{-1}$, with $\bar{s}$ regular in $\bar{R}$. It follows from (11.7)(3) that $\bar{Q} = Q^r_{cl}(\bar{R})$, so by Goldie's Theorem (11.13), $\bar{R}$ is semiprime right Goldie. In particular, $J \cap R$ is a semiprime ideal in R, so $J \cap R \supseteq N$. On the other hand, $J \cap R$ is nilpotent, so $J \cap R = N$, and $\bar{R}$ is just $S = R/N$. We have now proved the properties (A) and (B) in the Theorem. By (12.1), we also have

$$\mathcal{C}(N) \subseteq \mathcal{C}_Q(J) \cap R = \mathcal{C}_Q(0) \cap R \subseteq \mathcal{C}(0),$$

which gives (D). To prove (C), first note that $J = (J \cap R)Q = NQ$, from which we see easily that $J^i = N^i Q$ for any $i \geq 0$. We leave it to the reader to show that

$$(12.13) \qquad (N^i/N^{i+1}) \otimes_{\bar{R}} \bar{Q} \cong N^i Q/N^{i+1} Q = J^i/J^{i+1}.$$

Using this isomorphism, we have:

$$\begin{aligned}\rho_R(R_R) &= \sum \operatorname{rank}_{\bar{R}}(N^i/N^{i+1}) \\ &= \sum \operatorname{length}_{\bar{Q}}(J^i/J^{i+1}) \\ &= \sum \operatorname{length}_{Q}(J^i/J^{i+1}) \\ &= \operatorname{length}_Q(Q_Q) < \infty.\end{aligned}$$

Finally, by (6.7)(2) and Exercise (10.18)(5), $\operatorname{length}_Q(Q_Q) \geq \text{u.dim}\ Q_Q = \text{u.dim}\ R_R$. $\square$

(12.14) Corollary. *Let $\mathcal{C}'(0)$ denote the set of right regular elements of R; i.e., those elements whose right annihilators are zero. If R satisfies* (A), (B), (C) *in* (12.10), *then $\mathcal{C}'(0) \subseteq \mathcal{C}(N)$. If R also satisfies* (D), *then $\mathcal{C}'(0) = \mathcal{C}(0) = \mathcal{C}(N)$, and every nil 1-sided ideal of R is nilpotent. (In particular, the upper and lower nilradicals of R coincide.)*

Proof. In retrospect, the proof for (12.11) is valid under the assumptions (A), (B), (C), as long as $s \in \mathcal{C}'(0)$. Therefore, our earlier work shows that $\mathcal{C}'(0) \subseteq \mathcal{C}(N)$. Now assume (D) also holds. Then $\mathcal{C}'(0) \subseteq \mathcal{C}(N) \subseteq \mathcal{C}(0)$, so we must have equalities. Also, since R embeds into a right artinian ring, every nil multiplicatively closed set of R is nilpotent by the theorem of Hopkins and Levitzki (cf. §11E). This gives the last part of the Corollary about nil 1-sided ideals in R. $\square$

If R is a right noetherian ring, we have observed already (cf. (12.8)) that (A), (B), and (C) all hold for R. Therefore, for such a ring, we have the following greatly simplified form of (12.10) (and (12.14)), due to L. Small (and in part to T. Talintyre).

(12.15) Theorem. *Let R be any right noetherian ring, with $N = \text{Nil}_* R$. Then $\mathcal{C}(0) \subseteq \mathcal{C}'(0) \subseteq \mathcal{C}(N)$; R is a right order in a right artinian ring iff $\mathcal{C}(N) \subseteq \mathcal{C}(0)$.*

(12.16) Corollary. *Let R be any noetherian ring with $\mathcal{C}(N) \subseteq \mathcal{C}(0)$. Then R is a (2-sided) order in an artinian ring.*

Proof. Let Q be the right artinian ring $Q^r_{cl}(R)$, in which R is a right order. Since the condition $\mathcal{C}(N) \subseteq \mathcal{C}(0)$ is left-right symmetric, we know that $Q^\ell_{cl}(R)$ also exists and is left artinian. By (10.14), we can identify $Q^\ell_{cl}(R)$ with $Q^r_{cl}(Q) = Q$. Therefore, Q is artinian, and R is an order in Q. $\square$

The following are two nice applications of (12.15), due to L. Small.

(12.17) Corollary. *Let R be a right noetherian ring that is right Rickart. Then R is a right order in a right artinian ring.*

Proof. It suffices to check $\mathcal{C}(N) \subseteq \mathcal{C}(0)$. Let $s \in \mathcal{C}(N)$. Then $\text{ann}_r(s) \subseteq N$. The ring R being right Rickart, $\text{ann}_r(s) = eR$ for some idempotent $e \in \text{ann}_r(s) \subseteq N$. Since N is nil, we have $e = 0$, so $\text{ann}_r(s) = 0$. *It now remains to show that* $\text{ann}_\ell(s) = 0$. For any $a \in \text{ann}_\ell(s)$, $\text{ann}_r(a)$ contains sR which, by Exercise (11.1), is essential in R_R. Therefore, by definition, $a \in \mathcal{Z}(R_R)$ (the right singular ideal of R). But by (7.6)(7), R is right nonsingular, so $a = 0$ as desired. □

(12.18) Corollary. *A right hereditary ring R is right noetherian iff it is a right order in a right artinian ring.*

Proof. The "only if" part follows from (12.17). The "if" part follows from (7.58) and the last part of (10.35). □

(12.19) Examples.

(1) Consider the noetherian ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, with prime radical $N = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Clearly, $R/N \cong \mathbb{Z} \times \mathbb{Z}$. If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{C}(N)$, we must have $a, c \neq 0$, so $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{C}(0)$. Therefore, $\mathcal{C}(N) \subseteq \mathcal{C}(0)$, and (12.16) implies that R is an order in an artinian ring. In fact, we have seen in (10.27)(d) that R is an order in the artinian ring $\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$.

(2) For a fixed prime p, let $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and R be the noetherian ring $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. The prime radical of R is $N = \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_p & 0 \end{pmatrix}$, with $R/N \cong \mathbb{Z} \times \mathbb{Z}_p$. Thus,

$$\mathcal{C}(N) = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} : x \neq 0,\ z \neq 0 \right\},$$

and from this, we see easily that $\mathcal{C}(N) = \mathcal{C}'(0)$. On the other hand, we have seen in (10.27)(e) that

$$'\mathcal{C}(0) = \mathcal{C}(0) = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} : p \nmid x,\ z \neq 0 \right\},$$

where $'\mathcal{C}(0)$ denotes the set of left regular elements of R. Therefore, we have

$$\mathcal{C}(0) = {}'\mathcal{C}(0) \subsetneq \mathcal{C}'(0) = \mathcal{C}(N),$$

and (12.15) implies that R is *not* a right order in a right artinian ring. In fact, we have seen in (10.27)(e) that R is an Ore ring, with

$$Q = Q^r_{cl}(R) = Q^\ell_{cl}(R) = \begin{pmatrix} \mathbb{Z}_{(p)} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

This is not an artinian ring since $Q/\text{Nil}_* Q \cong \mathbb{Z}_{(p)} \times \mathbb{Z}_p$ is not. (For a generalization of this example, see Exercise 3.)

(3) Let A be a commutative noetherian domain of characteristic $p > 0$ and K be its quotient field. Let G be any finite group whose order is divisible by p. Then the group ring AG is an order in the group algebra KG. Since KG is not semisimple (FC–(6.1)), AG gives an example of a non-semiprime noetherian ring with an artinian (left and right) ring of quotients.

§12C. *The Commutative Case*

In this subsection, we point out that the result (12.15) of Small and Talintyre is already of interest for *commutative* noetherian rings. Let us therefore take a careful look at (12.15) in the context of commutative rings, in order to get more insight into this important result. We start with the following easy observation.

(12.20) Lemma. *For any commutative ring R, with nilradical $N = \text{Nil}_* R$, $\mathcal{C}(0) \subseteq \mathcal{C}(N)$.*

Proof. Let $s \in \mathcal{C}(0)$. If $t \in R$ is such that $st \in N$, then, for some $k \geq 1$, $0 = (st)^k = s^k t^k$. Therefore $t^k = 0$ and so $t \in N$. This shows that $s \in \mathcal{C}(N)$. $\square$

The condition $\mathcal{C}(N) \subseteq \mathcal{C}(0)$ is already rather subtle for commutative rings. The following result, (12.21), gives an interpretation of this condition for commutative noetherian rings. The proof of this result assumes familiarity with some basic facts about associated primes in the commutative (and mainly noetherian) case, as given explicitly in Exercises (3.40B) and (3.40E). In fact, in the proofs of all remaining results in this subsection, we shall be using these facts freely *without giving further references*. Thus, the reader should first carefully review the two exercises (3.40B) and (3.40E) before proceeding with the material below.

(12.21) Proposition. *Let R be a commutative noetherian ring, and let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the set of associated primes of R_R. Let $N = \text{Nil}_* R$. Then $\mathcal{C}(N) \subseteq \mathcal{C}(0)$ iff each $\mathfrak{p}_i$ is a minimal prime of R.*

Proof. Say $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are minimal, and the others are not.[81] *First assume that $t = n$.* Consider any element $a \notin \mathcal{C}(0)$. Then $a \in \mathfrak{p}_i$ for some i. By assumption,

[81]The primes $\mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_n$ are called "embedded primes" in the theory of primary decomposition in commutative algebra.

$\mathfrak{p}_i$ is a minimal prime over N, so $\mathfrak{p}_i$ is an associated prime of $(R/N)_R$. This means $\mathfrak{p}_i$ has the form $\{x \in R : xb \in N\}$ for some element $b \notin N$. But then $ab \in N$ implies that $a \notin \mathcal{C}(N)$. This shows that $\mathcal{C}(N) \subseteq \mathcal{C}(0)$. *Next, assume that* $t < n$. Then, by the Lemma of Prime Avoidance (Eisenbud [95: pp. 90-91]), there exists an element

$$(\dagger) \qquad c \in \mathfrak{p}_n \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t).$$

Since $\mathfrak{p}_n$ consists of 0-divisors of R, $c \notin \mathcal{C}(0)$. *We claim that* $c \in \mathcal{C}(N)$. For, if $cd \in N$ $(d \in R)$, then, for any $i \leq t$, we have $cd \in \mathfrak{p}_i \Longrightarrow d \in \mathfrak{p}_i$ by ($\dagger$). This shows that

$$d \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = N.$$

Therefore, $\mathcal{C}(N) \not\subseteq \mathcal{C}(0)$. □

In view of the above, (12.15) can be restated as follows in the commutative case.

(12.22) Proposition. *A commutative noetherian ring R is an order in a (commutative) artinian ring iff all the associated primes of R_R are minimal primes.*

Of course, this result could have been checked directly using commutative algebra techniques alone, without invoking the general result (12.15). We continue to use the notation in the proof of (12.21), and recall the key fact (already used above) that $S := \mathcal{C}(0)$ is the complement of $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. The classical ring of quotients of R is the localization R_S, which is a noetherian ring. If $\mathfrak{p}_i \subsetneq \mathfrak{p}_j$ for some i, j, then $(\mathfrak{p}_i)_S \subsetneq (\mathfrak{p}_j)_S$ in R_S, so R_S cannot be artinian. If, on the other hand, there is no inclusion relation among $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then there is also no inclusion relation among $(\mathfrak{p}_1)_S, \ldots, (\mathfrak{p}_n)_S$, and, by the Lemma of Prime Avoidance (along with standard facts about localizations), these give *all* prime ideals of R_S. Hence R_S has Krull dimension 0, and is artinian by *FC*–(23.12).

In general, if R is a commutative noetherian ring (or just a commutative ring with ACC on its annihilator ideals), then by (8.31)(2) the classical ring of quotients of R is a (commutative) semilocal Kasch ring. For a related result, see Exercise 10.

(12.23) Examples.

(a) For any field k, let R be the (noetherian) commutative k-algebra generated by x, y, with relations $yx = y^2 = 0$. (This is a "commutative version" of the ring studied in Example (7.6)(4).) We see easily that $N = \text{Nil}_* R = (y)$, and that the associated primes of R_R are

$$\mathfrak{p}_1 = \text{ann}(x) = (y) \quad \text{and} \quad \mathfrak{p}_2 = \text{ann}(y) = (x, y) \supsetneq \mathfrak{p}_1.$$

Here $\mathfrak{p}_1$ is a minimal prime of R, but $\mathfrak{p}_2$ is not. Hence R is not an order in an artinian ring. In the spirit of (12.15), we have

$$\mathcal{C}(0) = R \setminus \mathfrak{p}_2 \quad \text{and} \quad \mathcal{C}(N) = R \setminus \mathfrak{p}_1 \quad (\text{since } R/N \cong k[x]);$$

in particular, $\mathcal{C}(0) \subsetneq \mathcal{C}(N)$. It is easy to check that

$$(*) \qquad (0) = (y) \cap (x + ay) = \mathfrak{p}_1 \cap (x + ay) \qquad (\forall a \in k),$$

and that each $(x + ay)$ is a primary ideal with radical $\mathfrak{p}_2$. $(*)$ is often cited in commutative algebra to show that the zero ideal may have more than one primary decomposition: any $(x + ay)$ may be taken as a primary component of (0) within the "embedded prime" $\mathfrak{p}_2$. (See the Hint for Exercise (3.40B).)

(b) To make an even nicer example, we can take, instead, $R' = k[[X, Y]]/(YX, Y^2)$, which is a commutative noetherian *local* ring. For $x = \bar{X}$ and $y = \bar{Y}$, we have exactly the same information as before. But now $\mathcal{C}_{R'}(0)$ is exactly the complement of the unique maximal ideal (x, y) of R'. Hence $\mathcal{C}_{R'}(0) = \mathrm{U}(R')$, which means that *$R'$ is its own classical ring of quotients.* Here, R' is local, nonartinian, with Krull dimension 1. (A noncommutative version of the ring R' has been used before as an example of a right Kasch ring that is not left Kasch: see Exercise 13 in §8.)

While we are on the subject of commutative noetherian rings, the following two consequences of (12.22) are worth recording.

(12.24) Proposition. *Let R be a commutative noetherian ring in which* (0) *is a primary ideal. Then $Q_{cl}(R)$ is a local artinian ring.*

Proof. As usual, let $N = \mathrm{Nil}_* R$. Since (0) is primary, N is prime, and is in fact the unique associated prime of R. Thus, (12.22) applies, and $Q_{cl}(R)$ is a (commutative) local artinian ring. □

(**Note.** Here, the condition $\mathcal{C}(N) \subseteq \mathcal{C}(0)$ is also very easy to check directly. Indeed, if $a \in \mathcal{C}(N)$ and $ab = 0$, then $a \notin N = \sqrt{(0)}$ implies $b = 0$ since (0) is primary. This shows that $a \in \mathcal{C}(0)$. We note further that, in the case when (0) is meet-irreducible in (12.24), $Q_{cl}(R)$ is in fact a quasi-Frobenius ring in the sense of §15; see Exercise (15.19).)

(12.25) Corollary. *Every commutative noetherian ring R can be embedded in a commutative artinian ring.*

Proof. By the Primary Decomposition Theorem, $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are suitable primary ideals. By (12.24), each $Q_{cl}(R/\mathfrak{q}_i)$ is artinian. Now $R \subseteq \prod_i R/\mathfrak{q}_i$ embeds into the commutative artinian ring $\prod_i Q_{cl}(R/\mathfrak{q}_i)$. □

(12.26) Remarks.

(1) In contrast to (12.25), not every commutative ring can be embedded in a 1-sided artinian or noetherian ring. This can be easily seen from (6.61).

(2) In contrast to (12.25), there also exist right noetherian rings R that cannot be embedded in a 1-sided artinian ring or a left noetherian ring. (See Exercise 8.)

(3) The result (12.22) characterizing commutative noetherian rings R with $Q_{cl}(R)$ artinian in terms of the associated primes and minimal primes of R does admit an analogue for *noncommutative* noetherian rings, due to J. T. Stafford. (See e.g., Goodearl-Warfield [89: p. 173].)

In conclusion, let us make one more observation about Goldie rings in the commutative setting. Since commutative noetherian rings are always Goldie, one may ask whether some kind of "strengthened" Goldie condition might imply noetherianness. Such a result has indeed been obtained by V. Camillo [75], who proved that *a commutative ring R is noetherian iff all homomorphic images of R are Goldie.* The proof of this is not easy, depending on a result of R. C. Shock. There seems to be no known analogue of this result for noncommutative rings.

§12D. Noetherian Rings Need Not Be Ore

Consider any right noetherian ring R with prime radical N. If $\mathcal{C}(N) \subseteq \mathcal{C}(0)$, then by (12.15), $Q_{cl}^r(R)$ exists and is right artinian; in particular, R is right Ore. Various other classes of right noetherian rings have also been proved to be right Ore. This raises the following question: *Is every right noetherian ring right Ore?* In other words, *does every right noetherian ring possess a classical right ring of fractions?* This question has been answered in the negative by L. Small. We shall close this section by presenting below the example from [Small: 66].

(12.27) Example. *There exists a noetherian ring that is neither left Ore nor right Ore.*

For the construction, we start, curiously enough, with an Ore ring. In (10.27)(e), we have seen that $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ is an Ore ring with the following properties:

(A) $\mathcal{C}_R(0) = {}'\mathcal{C}_R(0)$ (the set of left regular elements of R).
(B) There exist $a \in R$ and $t \in \mathcal{C}'_R(0)$ such that $aS \cap tR = \emptyset$, where $S = \mathcal{C}_R(0)$.
(C) There exists a subring $T \subseteq R$ with $t \in T$ and $\text{ann}_\ell^T(t) = 0$.

For (C), we simply take $T = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z}_p \end{pmatrix}$. For the element $t = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in T$ in (10.27)(e), it is clear that $\text{ann}_\ell^T(t) = 0$. The idea now is that, starting from *any* ring R, with the properties (A), (B), and (C) above, we can construct a non-Ore ring (even when R itself is Ore).

(12.28) Lemma. *Let R be any ring satisfying the properties* (A), (B), *and* (C). *Then the triangular ring $R' = \begin{pmatrix} R & 0 \\ R & T \end{pmatrix}$ is not right Ore. (In defining this triangular ring, the off-diagonal entry R above is to be viewed as a (T, R)-bimodule, by means of the ring inclusion $T \subseteq R$.)*

Proof. Let $S' = \mathcal{C}_{R'}(0)$. Using the elements $t,\ a \in R$, we construct:

$$t' = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \in R \quad \text{and} \quad a' = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in R'.$$

We finish by proving that: (1) $t' \in S'$, and (2) $a'S' \cap t'R' = \emptyset$. To prove (1), suppose $t'\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} = 0$. Then $u = 0$ and $tv = tw = 0$. Since $t \in \mathcal{C}'_R(0)$, we have $v = w = 0$. Next, suppose $\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} t' = 0$ instead. Then $u = v = 0$ and $wt = 0$. Since $w \in T$ and $\text{ann}_\ell^T(t) = 0$ by (C), we have also $w = 0$. To prove (2), assume that there exists an equation (in R'):

$$(12.29) \qquad a'\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} = t'\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, \qquad \text{where } \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \in S'.$$

We claim that $u \in {}'\mathcal{C}_R(0)$. In fact, if $cu = 0$ in R, then $\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} = 0$ in R', and hence $c = 0$ since $\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \in S'$. In view of (A), we have therefore $u \in \mathcal{C}_R(0) = S$. Reading the (2,1)-entries in the equation (12.29), we now obtain $au = ty$, which contradicts (B). Therefore, (12.29) cannot exist. □

If we let R above be the ring $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ in (10.27)(e), then the ring R' just obtained is noetherian, since R' is clearly a f.g. abelian group. By (12.28), this noetherian ring is not right Ore. It does turn out that R' is left Ore, although we shall not digress to give the proof here. Suffice it to note that, if we form the direct product $A = R' \times R''$, where R'' is the opposite ring of R', then A remains a noetherian ring, while an easy argument shows that A is neither left Ore nor right Ore, as required in (12.27).

Exercises for §12

In all of the following Exercises, $N = \text{Nil}_* R$.

1. Name a ring that is not right noetherian, but satisfies the conditions (A), (B), (C), and (D) in (12.10).

2. Show that a ring R is right artinian iff it is right noetherian and $\mathcal{C}(N) \subseteq \text{U}(R)$.

3. Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, where $n \geq 2$. Compute $\mathcal{C}(0)$ and $\mathcal{C}(N)$ for the ring $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_n & \mathbb{Z}_n \end{pmatrix}$, and show that R is a 2-sided order in a noetherian, nonartinian ring.

4. Let k be a field.

(1) Compute $\mathcal{C}(0)$ and $\mathcal{C}(N)$ for the ring $R = \begin{pmatrix} k & k[x] \\ 0 & k[x] \end{pmatrix}$.
(2) Show that R is right noetherian and is a right order in the right artinian ring $Q = \begin{pmatrix} k & k(x) \\ 0 & k(x) \end{pmatrix}$.
(3) Show that every right regular element of R is (right) regular in Q, but a left regular element of R need not be left regular in Q.

5. Prove or disprove: every left regular element in a right artinian ring is a unit.

6. Name a ring R for which N is nilpotent, R/N is noetherian, R_R has finite ρ-rank, but ${}_RR$ has infinite ρ-rank. (**Hint.** Use the ring in Exercise 4, and recall that it is not left Ore by (10.27)(f).)

7. Let $R \neq (0)$ be a ring with a faithful, singular right module M, and let A be the triangular ring $\begin{pmatrix} \mathbb{Z} & M \\ 0 & R \end{pmatrix}$, where M is viewed as a $(\mathbb{Z}, R)$-bimodule.
(1) Show that, for any finite set $N \subseteq M$, $\mathrm{ann}_r^A \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{A} \end{pmatrix}$ for some essential right ideal $\mathfrak{A} \subseteq R$.
(2) Show that $\mathrm{ann}_r^A \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \neq \mathrm{ann}_r^A \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ for any finite subset $N \subseteq M$. Deduce from Exercise (6.21) that A does not satisfy DCC on right annihilators.
(3) Using (2), show that A cannot be embedded in a right artinian or a left noetherian ring.

8. Construct a right noetherian ring A that cannot be embedded in a right artinian or a left noetherian ring.

9. For an ideal $\mathfrak{A} \subseteq R$, let $\mathcal{C}(\mathfrak{A})$ (resp. $\mathcal{C}'(\mathfrak{A})$) be the set of elements $r \in R$ such that $r + \mathfrak{A}$ is regular (resp. right regular) in $R/\mathfrak{A}$. If R is right noetherian and $\mathfrak{A}$ is a semiprime ideal, show that $\mathcal{C}(\mathfrak{A}) = \mathcal{C}'(\mathfrak{A})$. Exhibit an example to show that the hypothesis that $\mathfrak{A}$ be semiprime cannot be removed. (**Hint.** Apply (11.14)(a) to the semiprime right Goldie ring $R/\mathfrak{A}$.)

10. (E. Davis) For any commutative ring R, show that $Q_{cl}(R)$ is a semilocal ring iff the set of 0-divisors of R is a finite union of prime ideals.

Chapter 5

More Rings of Quotients

In this Chapter, we study rings of quotients of a different sort, breaking away from the "classical" rings of quotients studied in Chapter 4. Injective modules will play a major role here.

We begin the Chapter with some basic theorems on the endomorphism ring of an injective (or more generally, quasi-injective) module. By applying these results to the regular module R_R in case it is injective, we get a first glimpse into the structure of a right self-injective ring. If R_R is not injective, we can nevertheless apply these results to the endomorphism ring of $E(R_R)$, the injective hull of R_R. This leads us to the Findlay-Lambek-Utumi theory of maximal rings of quotients, which occupies the main part of §13.

Contrary to the situation with $Q^r_{cl}(R)$ (the classical right ring of quotients of R), the maximal right ring of quotients $Q^r_{\max}(R)$, always exists. In the case when R is a right Ore ring so that $Q^r_{cl}(R)$ does exist, $Q^r_{cl}(R)$ can be identified naturally with a subring of $Q^r_{\max}(R)$. In this sense, one can say that the study of $Q^r_{\max}(R)$ subsumes the study of $Q^r_{cl}(R)$. We shall also see that, if we extend the notion of "classical rings of quotients" to "general rings of quotients", then $Q^r_{\max}(R)$ is indeed the *biggest* general (right) ring of quotients one can associate with R.

Our exposition of the theory of maximal rings of quotients is based on the work of Utumi, Findlay, and Lambek, which was inspired by the pioneering work of R. E. Johnson on nonsingular rings. After developing the basic theory of $Q^r_{\max}(R)$, we shall specialize to the case when R is a right nonsingular ring. In this case, we shall retrieve Johnson's theory of (von Neumann) regular rings of quotients for right nonsingular rings.

Throughout §13, we shall use freely the theory of dense submodules and rational extensions developed in §8. In particular, the notations $N \subseteq_d M$ (for N being a dense submodule of M) and $\tilde{E}(M)$ (for the rational completion of a module M) will be in force. The reader will be well advised to first review the material of §8 before reading this section.

The Chapter concludes with §14 which is an introduction to Martindale's rings of quotients. This theory is included in our exposition since it has played an increasingly important role in the recent study of prime and semiprime rings. Although we stop short of venturing into the full Galois theory of such rings, a

quick exposition is given for important topics such as normalizing elements in ring extensions and X-inner automorphisms of semiprime rings. Limitation of space, however, precludes us from presenting the many applications of this theory to the study of group rings and rings with polynomial identities.

§13. Maximal Rings of Quotients

§13A. Endomorphism Ring of a Quasi-Injective Module

The theme of this subsection is that the endomorphism ring of an injective module is a very nice kind of ring. For instance, over the ring $R = \mathbb{Z}$, the injective module $\mathbb{Q}_{\mathbb{Z}}$ has endomorphism ring $\cong \mathbb{Q}$, and the injective module $\mathbb{Z}_{p^\infty} := \varinjlim \mathbb{Z}/p^n\mathbb{Z}$ (for a fixed prime p) has endomorphism ring $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$, the ring of p-adic integers.

As it turns out, in analyzing the structure of the endomorphism ring of an injective module I, what we need is not so much the injective property of I, but the *quasi-injective* property. Recall that a module I_R is quasi-injective (or QI for short) if, for any submodule $L \subseteq I$, any R-homomorphism from L to I extends to an R-endomorphism of I. As long as we are dealing with *endomorphisms* of I (rather than homomorphisms from arbitrary modules into I), it is reasonable to expect that the QI property on I will be strong enough to give us good information. Thus, in this subsection, we shall analyze the structure of $\text{End}(I_R)$ assuming only that I is a QI module.

Let us first set up some general notations *which will be fixed throughout this subsection.* For any right R-module I, let $H = \text{End}(I_R)$ be its endomorphism ring, operating on the left of I. It is convenient to think of I as an (H, R)-bimodule. We shall be working heavily with the following subset of endomorphisms:

$$N := \{f \in H : \ker(f) \subseteq_e I\}.$$

We start with the following general result which is valid for any module I_R.

(13.0) Lemma. *With the notations above:*

(1) *N is an ideal of H.*
(2) *If $f = f^2 \in H$, then for any R-submodule $A \subseteq_e I$, we have $fA \subseteq_e fI$.*
(3) *Let f_j $(j \in J)$ be idempotents in H whose images $\overline{f}_j$ in $\overline{H} := H/N$ are mutually orthogonal. Then $\sum f_j I$ is a direct sum in I.*

Proof. (1) Let $f, g \in N$. Then $\ker(f)$, $\ker(g)$ are both essential in I; hence so is $\ker(f) \cap \ker(g)$. Since this intersection is contained in $\ker(f - g)$, it follows that $\ker(f - g) \subseteq_e I$, so $f - g \in N$. Now let $h \in H$. Then

$$\ker(hf) \supseteq \ker(f), \quad \text{and} \quad \ker(fh) \supseteq h^{-1}(\ker(f)).$$

Thus, $\ker(f) \subseteq_e I$ implies that $\ker(hf)$ and $\ker(fh)$ are both essential in I (for the latter, see Exercise (3.7)), and hence $hf,\ fh \in N$. This checks that N is an *ideal* in H.

(2) Consider any nonzero element $fx \in fI$ (where $x \in I$). Since $A \subseteq_e I$, there exists $r \in R$ such that $0 \neq fxr \in A$. Noting that $(fx)r = f(fxr) \in fA$, we have checked that $fA \subseteq_e fI$.

(3) It suffices to verify (3) when $|J| < \infty$, say, $J = \{1, 2, \dots, n\}$. Since $\overline{f_i f_j} = \delta_{ij}$ in $\overline{H}$, we have $\ker(f_i f_j) \subseteq_e I$ whenever $i \neq j$. Then, by Exercise (3.6)(A):

$$A := \bigcap_{i \neq j} \ker(f_i f_j) \subseteq_e I,$$

and we have $f_i f_j A = 0$ whenever $i \neq j$. It follows that $\sum f_j A$ is a direct sum, for, if we have an equation $f_1 a_1 + \cdots + f_n a_n = 0$ where $a_i \in A$, an application of f_j gives $f_j a_j = 0$. Now from (2), $f_j A \subseteq_e f_j I$ for each j, so by Exercise (3.8), we conclude that $\sum f_j I$ is also a direct sum in I. □

We shall now specialize to QI modules. As we proceed formally with our results on the endomorphism rings of such modules, the reader should keep in mind the basic examples mentioned in the opening paragraph of this subsection. It would also be helpful if the reader is familiar with Exercise (6.32), the gist of which is that the endomorphism ring of an *indecomposable* (equivalently, *uniform*) QI module M is *local*, with Jacobson radical consisting of all the endomorphisms with nonzero kernels. The following theorem is a generalization of this fact to the case of an *arbitrary* QI module, due to Utumi, Faith, Johnson, Wong, Osofsky, Renault, and others.

(13.1) Theorem. *Let I_R be a QI right module over any ring R, and let $\overline{H} = H/N$, where H and N are as above. Then*

(1) $N = \operatorname{rad} H$ *(the Jacobson radical of H; see FC–§4).*
(2) $\overline{H}$ *is a von Neumann regular ring.*
(3) $\overline{H}$ *is a right self-injective ring.*
(4) *Idempotents of $\overline{H}$ can be lifted to (idempotents of) H.*
(5) *If I_R is nonsingular or semisimple, then $N = 0$ (so by (2) and (3) above, H itself is a von Neumann regular right self-injective ring).*

Proof. (1) Let $f \in \operatorname{rad} H$. Suppose $M \subseteq I$ is any submodule with $M \cap \ker f = 0$. Then $f|M : M \to I$ is a monomorphism. The quasi-injectivity of I implies that there exists $g \in H$ such that $g(f(m)) = m$ for every $m \in M$. Since $1 - gf$ is a unit in H (*FC*–(4.3)), $M = (0)$. Therefore, $\ker f \subseteq_e I$ and we have $f \in N$. For the converse, we use the fact that N is an ideal of H. To see that $N \subseteq \operatorname{rad} H$, it suffices to show (according to *FC*–(4.5)) that, for any $f \in N$, $1 - f$ is left invertible in H. Now, from

$$\ker f \cap \ker(1 - f) = (0), \quad \text{and} \quad \ker f \subseteq_e I,$$

we have $\ker(1 - f) = (0)$. Therefore, by the quasi-injectivity of I, there exists $g \in H$ whose restriction to $(1 - f)M$ is the inverse of the isomorphism $1 - f : M \to (1 - f)M$. We have thus $g(1 - f) = 1 \in H$, as desired.

(2) Let $f \in H$ and let M be a complement to $K := \ker f$, so $M \oplus K \subseteq_e I$. Again, there exists $h \in H$ whose restriction to $f(M)$ is the inverse of $f : M \to f(M)$. Then $fhf = f$ on M and, of course, fhf and f are both zero on K. Therefore, $fhf - f$ is zero on $M \oplus K \subseteq_e I$, and hence $fhf - f \in N$. We have then $\bar{f} = \bar{f}\bar{h}\bar{f} \in \overline{H}$, so $\overline{H}$ is a von Neumann regular ring.

(5) For any $f \in N$, consider $K := \ker f \subseteq_e I$. If I is semisimple, the latter implies that $K = I$. If I is nonsingular instead, consider the injection $f' : I/K \to I$ induced by f. Since I/K is singular (by (7.6)(3)) and I is nonsingular, f' is the zero map, so we have again $K = I$. Therefore, in either case, $f = 0$, and we have shown that $N = 0$, as desired.

(4) Let $f \in H$ be such that $\bar{f} \in \overline{H}$ is an idempotent. Then, by (1),

$$L := \ker(f^2 - f) = \{x \in I : f^2(x) = f(x)\} \subseteq_e I.$$

For $x \in L$, we have

$$(*) \qquad x = f(x) + (x - f(x)) \in \ker(1 - f) \oplus \ker f.$$

Therefore, $\ker(1 - f) \oplus \ker f \subseteq_e I$. Taking injective hulls in $E(I)$, we get

$$E(I) = E(\ker(1 - f)) \oplus E(\ker f).$$

Let π be the projection of $E(I)$ onto $E(\ker(1 - f))$ with respect to this decomposition, and let $\pi_0 \in H$ be its restriction to I (see (6.74)). For $x \in L$, the decomposition $(*)$ above shows that $\pi_0(x) = f(x)$. Therefore, $L \subseteq \ker(\pi_0 - f)$. This implies that $\overline{\pi_0 - f} = 0 \in \overline{H}$, so the idempotent $\pi_0 \in H$ lifts $\bar{f}$.

We now come to the most difficult part of the Theorem, which is the proof of the right self-injectivity of $\overline{H}$. We first prove the following improvement of (13.0) in the QI case.

(13.0)′ Lemma. *Assume I_R is* QI, *and let e_j $(j \in J)$ be idempotents in H such that $\sum_j \overline{e_j H}$ is a direct sum in $\overline{H}$. Then $\sum_j e_j I$ is a direct sum in I.*

Proof. We may assume, as before, that $J = \{1, \ldots, n\}$. Since $\overline{e_1 H} \oplus \cdots \oplus \overline{e_n H}$ is a f.g. right ideal in the von Neumann regular ring $\overline{H}$, we can write

$$\overline{H} = \overline{e_1 H} \oplus \cdots \oplus \overline{e_n H} \oplus X$$

for some right ideal $X \subseteq \overline{H}$ (see *FC*–(4.23)). Let $\bar{1} = \overline{f_1} + \cdots + \overline{f_n} + \overline{f}$ be the corresponding decomposition of $\bar{1}$. Then $\{\overline{f_1}, \ldots, \overline{f_n}, \overline{f}\}$ are *mutually orthogonal* idempotents in $\overline{H}$, with $\overline{f_j H} = \overline{e_j H}$ for all j and $\overline{fH} = X$ (see *FC*–Exercise (1.7)). By (4), we may assume that $\overline{f_j}$ is the image of an idempotent $f_j \in H$. Since $\overline{f_j e_j} = \overline{e_j}$, there exist (by (1)) R-submodules $K_j \subseteq_e I$ such that $(e_j - f_j e_j)K_j = 0$. From this, we have $e_j K_j \subseteq f_j I$. Now by Lemma

(13.0), $\sum f_j I$ is a direct sum; hence so is $\sum e_j K_j$. But $K_j \subseteq_e I$ implies that $e_j K_j \subseteq_e e_j I$ by (13.0)(2), so it follows (again from Exercise (3.8)) that $\sum e_j I$ is a direct sum in I. □

With the aid of the above lemma, we are now ready to give the proof of (13.1)(3).

(3) We'll check Baer's Criterion for the injectivity of $\overline{H}_{\overline{H}}$. Let $\varphi : C \to \overline{H}$ be a right $\overline{H}$-homomorphism, where C is any right ideal of $\overline{H}$. By Zorn's Lemma, there exists a maximal family of nonzero principal right ideals C_j $(j \in J)$ with $\bigoplus_j C_j \subseteq C$. Then we must have $\bigoplus_j C_j \subseteq_e C$. By (2) and (4) in (13.1), there exist idempotents $e_j \in H$ such that $C_j = \overline{e_j}\,\overline{H}$, for all j. Say $\varphi(\overline{e_j}) = \overline{t_j}$, where $t_j \in H$. By the lemma above, the sum $\sum e_j I$ is *direct*, and so $\bigoplus t_j$ defines an R-homomorphism from $\bigoplus e_j I$ to I, which can then be extended to an endomorphism h of I (by the quasi-injectivity of I). Since h agrees with t_j on $e_j I$, we have $he_j = t_j e_j \in H$, and so

$$\bar{h}\,\overline{e_j} = \varphi(\overline{e_j})\,\overline{e_j} = \varphi(\overline{e_j}\,\overline{e_j}) = \varphi(\overline{e_j}).$$

Hence φ agrees with left multiplication by $\bar{h}$ on $\bigoplus \overline{e_j}\,\overline{H}$. At this point, we repeat an argument used in the proof of (13.1)(5). Since $\overline{H}$ is von Neumann regular, it is right nonsingular (by (7.7)), and $\bigoplus_j C_j \subseteq_e C$ implies that $C/\left(\bigoplus_j C_j\right)$ is singular (by (7.6)(3)). From these, it follows that φ must agree with left multiplication by $\bar{h}$ on C, as desired. □

Remarks.

(A) It is worth pointing out that all of the results obtained above on $\text{End}(I_R)$ remain valid (with slight modifications of the proofs) for a larger class of modules I called *continuous* modules, which we briefly introduced in Exercise (6.36). (Of course, the conclusion (3) in this case is to be modified into: $\overline{H}_{\overline{H}}$ is continuous.) Since we do not intend to make use of continuous modules in the text, it seemed best to restrict the formulation of (13.1) to quasi-injective modules here. For a full treatment of continuous (quasi-continuous, and discrete) modules, including detailed information on the endomorphism rings thereof, see the book of Mohamed and Müller [90].

(B) If no assumption is imposed on I_R, the equation $N = \text{rad}\ H$ certainly need not hold. For instance, let R be a domain with $\text{rad}\ R \neq 0$, and let $I = R_R$. We may identify H with R (acting on the left of R). Obviously, we have here $N = (0) \subsetneq \text{rad}\ H$. And, even if $N = \text{rad}\ H$ holds, the quotient ring $\overline{H}$ need not be either right self-injective or von Neumann regular: just consider $I = \mathbb{Z}$ over the ring $\mathbb{Z}$.

(C) In case I is a semisimple module, I is automatically quasi-injective (by (6.71)(1)), so (13.1)(5) applies. The fact that $\text{End}(I_R)$ is von Neumann regular is well known in this case (see FC–(4.27)), and (13.1)(5) yields the additional information that H is a right self-injective ring. This latter conclusion can also be deduced directly as follows. First, by decomposing I into a direct sum of its isotypic components and using (3.11B), we can reduce our considerations to the

case when $I = \sum_J S$, where S is a simple right R-module (and J is some indexing set). By Schur's Lemma, $k := \text{End}(S_R)$ is a division ring, and, exploiting the fact that S is a f.g. (indeed cyclic) R-module, we can identify $H = \text{End}_R(I)$ with the ring of "column-finite" matrices over k whose rows and columns are indexed by J. The fact that such a ring is right self-injective follows from (3.74B) with a change of side: see the last paragraph of that example.

If the module I_R is QI and "essentially semisimple" (in the sense that it has an essential socle), the situation is still somewhat close to that in the case (C) above. To make this explicit, we state the following result which actually gives a good illustration of (13.1) in a special case.

(13.1)′ Theorem. *Let I_R be a* QI *module such that $S := \text{soc}(I) \subseteq_e I$, and as before let $H = \text{End}(I_R)$. Then*

(1) rad $H = \{f \in H : f(S) = 0\}$, *and*
(2) $H/\text{rad } H \cong \text{End}(S_R)$. *(By the remarks made in* (C) *above, it can be checked directly that the latter is a right self-injective von Neumann regular ring.)*

Proof. (1) Let $f \in \text{rad } H$. Then $\ker(f) \subseteq_e I$, so $\ker(f) \supseteq S$; that is, $f(S) = 0$. Conversely, if $f \in H$ is such that $f(S) = 0$, then $f \in \text{rad } H$ since $S \subseteq_e I$.

(2) Clearly, any $f \in H$ maps S to S. Therefore, we have a ring homomorphism $\theta : H \to \text{End}(S_R)$ defined by $\theta(f) = f|S$. By (1), $\ker(\theta) = \text{rad } H$. Since I_R is QI, any endomorphism of S extends to an endomorphism of I, so θ is onto. Therefore, θ induces a ring isomorphism $H/\text{rad } H \cong \text{End}(S_R)$. □

Theorems (13.1) and (13.1)′ have some very nice consequences on the structure of right self-injective rings, which we record below.

(13.2) Corollary. *Let R be any right self-injective ring. Then:*

(1) rad $R = \mathcal{Z}(R_R)$ *(right singular ideal of R).*
(2) $R/\text{rad } R$ *is a von Neumann regular ring.*
(3) $R/\text{rad } R$ *is a right self-injective ring.*
(4) *Idempotents of* $R/\text{rad } R$ *can be lifted to R.*
(5) *If R is right nonsingular, then it is a von Neumann regular ring.*

In case $S := \text{soc}(R_R) \subseteq_e R_R$, we have rad $R = \text{ann}_\ell(S)$, *and* $R/\text{rad } R \cong \text{End}(S_R)$.

Proof. Apply Theorem (13.1) to $I = R_R$, for which $H = \text{End}(R_R) \cong R$. This gives (2)–(5) right away.[82] The ideal N in (13.1) is given here by

$$N = \{a \in R : \text{ann}_r(a) \subseteq_e R_R\},$$

which is exactly $\mathcal{Z}(R_R)$. This gives (1). (For a more general version of (1), see Exercise 25.) Finally. in the case when $\text{soc}(R_R) \subseteq_e R_R$, the additional information on rad R and $R/\text{rad } R$ follows from Theorem (13.1)′. □

For instance, by (3.15E), the conclusions (1)–(4) above apply to any group ring kG where G is any finite group, and k is any field. It follows, in particular, that kG is (right) nonsingular iff it is semisimple. (Of course, by Maschke's Theorem *FC*–(6.1), kG is semisimple iff char k does not divide $|G|$.)

Continuing our study of the endomorphism ring of an injective module I_R, we shall now specialize to the case when I is the injective hull $E(M)$ of a right module M, and collect the conditions characterizing the decomposability of I into a finite direct sum of indecomposable modules (expanding upon (6.12), etc.). Let us first recall the definitions for some of the terms to be used in the theorem below. A ring H is called *semilocal* (cf. *FC*–(20.1)) if $H/\text{rad } H$ is semisimple; H is called *semiperfect* (cf. *FC*–(23.1)) if H is semilocal and the idempotents of $H/\text{rad } H$ can be lifted to H. A module N_R is called *strongly indecomposable* (cf. *FC*–(19.12)) if $\text{End}(N_R)$ is a local ring.

(13.3) Theorem. *Let $I_R = E(M_R)$, where M is any R-module, and let $H = \text{End}(I_R)$. The following are equivalent:*

(1) $\text{u.dim } M_R < \infty$.
(2) $\text{u.dim } I_R < \infty$.
(3) *I is a finite direct sum of indecomposable modules.*
(4) *I is a finite direct sum of strongly indecomposable modules.*
(5) *H is a semiperfect ring.*
(6) *H is a semilocal ring.*

Proof. The proof is just a matter of piecing together various earlier results. In fact, (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3) is (6.12); (3)$\Longleftrightarrow$(4) follows from (3.52); (4)$\Longleftrightarrow$(5) is *FC*–(23.8) (which holds for any module I_R); and (5)$\Longleftrightarrow$(6) follows from (13.1)(4). □

In connection with the Theorem, note that the case where the endomorphism ring H is *local* corresponds to $\text{u.dim } I = \text{u.dim } M = 1$; that is, I is indecomposable. In case M is QI, this is also equivalent to M itself being indecomposable, according to Exercise (6.32).

[82]Note that the first conclusion of (5) has been proved before, by a different method, in (7.52)(1).

(13.4) Corollary. *Let R be any right self-injective ring. Then the following are equivalent:*

(1) u. dim $R_R < \infty$.
(3) *R_R is a finite direct sum of indecomposable right ideals.*
(5) *R is semiperfect.*
(6) *R is semilocal.*
(7) *R has no infinite orthogonal set of nonzero idempotents.*

Proof. The equivalence of (1), (3), (5) and (6) follows by applying the theorem to $I = R_R$ (upon noting that $\text{End}(R_R) \cong R$). By (6.30)′, (1) means ACC on complements in R_R. Since R_R is injective, complements in R_R are just its direct summands (cf. (6.32)). Therefore, (1) amounts to the ACC on direct summands of R_R, which is just (7) according to (6.59). □

It is interesting that the three finiteness conditions (1), (3), and (7) above (cf. §6E) are equivalent for any right self-injective ring, but in general they are still pretty weak as far as finiteness conditions go. If we add the further condition that $\text{soc}(R_R) \subseteq_e R_R$, then the right self-injective ring R becomes what is known as a *right pseudo-Frobenius ring*: we shall encounter these rings again (albeit very briefly) in (19.25), when we study the concept of an "injective cogenerator". If we assume the much stronger condition that R be right (or left) noetherian, then the right self-injective ring R becomes a *quasi-Frobenius ring*: these rings are very important, and there are many interesting results about their structure. In the next chapter, we shall take up the detailed study of these quasi-Frobenius rings.

In a future section (see (19.24)), we shall construct an example of a commutative nonnoetherian, self-injective, local (and hence semilocal) ring. Such a ring is then a pseudo-Frobenius, but not quasi-Frobenius, ring. On the other hand, there are many rings that are right self-injective, but do not satisfy any of the finiteness conditions in (13.4). For instance, $\mathbb{Q} \times \mathbb{Q} \times \cdots$ is such an example (by (3.11B)). The opposite ring of the ring E constructed in (3.74A) is another.

To conclude this subsection, we shall present one more result on the endomorphism ring of a QI module. This one deals with double-annihilator conditions in a QI module M_R. For any subset $X \subseteq M$, we shall write

$$\text{ann}^R(X) = \{r \in R : Xr = 0\}.$$

This is a right ideal in R. Similarly, for any subset $A \subseteq R$, we shall write

$$\text{ann}^M(A) = \{m \in M : mA = 0\}.$$

Note that this annihilator is always an H-submodule of M, where, as before, $H = \text{End}(M_R)$. As usual, any set $X \subseteq M$ is contained in $\text{ann}^M(\text{ann}^R(X))$, and equality holds iff X is an annihilator in the above sense. The following important result offers a good supply of annihilators in a quasi-injective module M.

(13.5) Theorem (Johnson-Wong). *Let M_R be any QI module, with $H = \mathrm{End}(M_R)$. If X is an annihilator in M, then so is $X + Hm$ for any $m \in M$. (From this, it follows that any f.g. H-submodule of M is an annihilator, in the above sense.)*

Proof. As we have observed above, the job is only to prove that

$$\mathrm{ann}^M(\mathrm{ann}^R(X + Hm)) \subseteq X + Hm.$$

Let m' belong to the LHS, and let $A := \mathrm{ann}^R(X)$. We define a map $\varphi : mA \to m'A$ by $\varphi(ma) = m'a$ for any $a \in A$. Note that this map is well-defined for, if $ma = 0$, then

$$a \in \mathrm{ann}^R(X) \cap \mathrm{ann}^R(m) = \mathrm{ann}^R(X + Hm)$$

implies that $m'a = 0$. Clearly, φ is an R-homomorphism, so there exists an endomorphism $h \in H$ extending φ (by the quasi-injectivity of M). Now $h(ma) = m'a$ (for any $a \in A$) yields

$$m' - h(m) \in \mathrm{ann}^M(A) = \mathrm{ann}^M(\mathrm{ann}^R(X)) = X,$$

so we have $m' \in X + h(m) \subseteq X + Hm$, as desired. The last statement of the Theorem follows from the above by a simple induction (on the number of generators of a given H-submodule), starting with $X = (0)$. □

Note that the above theorem applies to all semisimple modules M since such modules are always quasi-injective. In this case, the theorem was noted by Artin and Tate; it may be viewed as a variant of the Density Theorem of Jacobson and Chevalley (cf. (11.16) in *First Course*). For a good illustrative example, the reader should take a look at the case when M is the simple right module k^n over $R = \mathbb{M}_n(k)$ where k is a division ring: what does the last part of the theorem give us in this case?

In the case when $M = R_R$ and R is a right self-injective ring, we retrieve from (13.5) the following classical result about such rings, due to Ikeda and Nakayama. (Note the change of side.)

(13.5)′ Corollary. *For any right self-injective ring R, any f.g. left ideal $\mathfrak{A}$ is a left annihilator; that is, $\mathfrak{A} = \mathrm{ann}_\ell(\mathrm{ann}_r(\mathfrak{A}))$.*

This Corollary will play a useful role in the proof of the later theorem (15.1) which gives the main characterizations of a quasi-Frobenius ring.

§13B. Construction of $Q^r_{\max}(R)$

In this subsection, we shall apply the results of §13A to $I := E(R)$, the injective hull of the right regular module R_R. The following notation will be fixed through the balance of §13.

(13.6) Notation. For $I = E(R_R)$, let $H = \text{End}(I_R)$, operating on the left of I. Furthermore, let $Q = \text{End}({}_H I)$, operating on the right of I, so we have $I = {}_H I_Q$. (The ring $Q \supseteq R$ is the "double commutant" of R constructed from the injective right module I_R.)

Recall that the rational hull $\tilde{E}(R)$ of R_R can be characterized as

$$\tilde{E}(R) = \{i \in I : \forall\, h \in H,\ h(R) = 0 \Longrightarrow h(i) = 0\}. \tag{13.7}$$

Using this information, we shall prove the following basic facts.

(13.8) Proposition. (1) ${}_H I$ *is a cyclic* H*-module generated by* 1. (2) *The map* $\varepsilon : Q \to I$ *defined by* $\varepsilon(q) = 1 \cdot q$ $(q \in Q)$ *is an* R*-isomorphism from* Q *onto* $\tilde{E}(R)$.

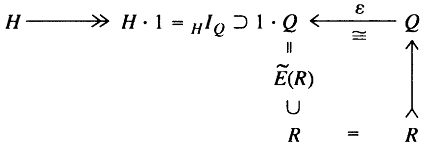
Proof. (1) Let $i \in I$. The R-homomorphism $R_R \to I_R$ sending 1 to i can be extended to some $h \in \text{Hom}_R(I, I) = H$ (by the injectivity of I_R), so $i = h(1) \in H \cdot 1$.

(2) Clearly, ε is an R-homomorphism. If $q \in \ker \varepsilon$, then $1 \cdot q = 0$, and so by (1):

$$0 = H \cdot (1 \cdot q) = (H \cdot 1) \cdot q = I \cdot q. \tag{13.9}$$

Therefore, $q = 0$. This shows that ε is injective. To compute $\text{im}(\varepsilon)$, first note that, for any $h \in H$ such that $h(R) = 0$, we have $h(1 \cdot Q) = (h \cdot 1)Q = 0$, so by (13.7), $\text{im}(\varepsilon) = 1 \cdot Q \subseteq \tilde{E}(R)$. Conversely, let $i \in \tilde{E}(R)$. Then $h \cdot 1 \mapsto h \cdot i$ is a well-defined H-endomorphism of ${}_H I$, since $h \cdot 1 = 0$ implies $h \cdot i = 0$. Therefore, there exists $q \in Q$ such that $h \cdot i = (h \cdot 1)q$ for all $h \in H$. In particular, letting $h = 1$, we get $i = 1 \cdot q = \varepsilon(q)$, so we have proved that $\text{im}(\varepsilon) = \tilde{E}(R)$. □

In summary, we have the following diagram:



To cut down on the formalities, let us henceforth identify Q with $\tilde{E}(R)$ using the isomorphism ε above. This then gives $\tilde{E}(R)$ a ring structure extending its given right R-module structure. We shall denote this ring also by $Q^r_{\max}(R)$, and call it the *maximal right ring of quotients* of R. (The maximal left ring of quotients, $Q^\ell_{\max}(R)$, is defined similarly, by working with $E({}_R R)$.) The multiplication on $\tilde{E}(R) = Q^r_{\max}(R)$ will be denoted as usual by $(i, j) \mapsto ij$.

To justify the terminology of "maximal right ring of quotients", let us first introduce the following definition.

(13.10) Definition. A ring T containing the ring R is called a *(general) right ring of quotients* of R if $R_R \subseteq_d T_R$. (The word "general" is used, whenever necessary, to

distinguish this new kind of rings of quotients from the classical rings of quotients studied before in §10.)

Clearly, $Q^r_{\max}(R) \supseteq R$ is such a general right ring of quotients. The following theorem explicates the sense in which it is the "maximal" one.

(13.11) Theorem. *Let T be any general right ring of quotients of R, and let $Q = Q^r_{\max}(R)$. Then*

(1) *There exists a unique ring homomorphism $g : T \to Q$ extending the identity map on R.*
(2) *The homomorphism g above is one-one.*
(3) *The ring structure on Q is the only one extending the R-module structure on Q_R.*

Proof. By (8.13), there exists a unique R-homomorphism $g : T \to \tilde{E}(R) = Q$ extending the identity map on R, and g is one-one. If we can show that $g(t't) = g(t')g(t)$ for all $t, t' \in T$, clearly all three parts of the Theorem follow. Let $h \in H$ extend the R-homomorphism $\tilde{E}(R) \to \tilde{E}(R)$ given by left multiplication by $g(t't) - g(t')g(t)$. For every $r \in t^{-1}R := \{x \in R : tx \in R\}$, we have

$$\begin{aligned} h(r) &= (g(t't) - g(t')g(t))r \\ &= g(t'tr) - g(t')g(tr) \\ &= g(t')(tr) - g(t')(tr) = 0 . \end{aligned}$$

Therefore, $h(t^{-1}R) = 0$. But $R_R \subseteq_d T_R$ implies that $t^{-1}R \subseteq_d R_R$ (cf. Exercise (8.2)), so (8.6) gives $h(R) = 0$. In particular, $0 = h(1) = g(t't) - g(t')g(t)$. □

Regarding the ring structure on $Q^r_{\max}(R) \subseteq E(R)$, we should recall, from Osofsky's example (3.45), that in general, $E(R)$ itself may not admit *any* ring structure extending the R-module structure on $E(R)$. Therefore, the fact that its R-submodule $Q^r_{\max}(R)$ *has* such a natural ring structure is a valuable piece of information.

We have, also, the following nice consequence of (13.11).

(13.12) Corollary. *If $Q^r_{cl}(R)$ exists,*[83] *then it has a unique embedding in the ring $Q^r_{\max}(R)$ extending the identity map on R.*

Proof. In view of (13.11), it suffices to show that $Q^r_{cl}(R)$ is a general right ring of quotients of R; i.e., $R_R \subseteq_d Q^r_{cl}(R)$. For $x, y \in Q^r_{cl}(R)$, we can write $x = as^{-1}$ and $y = bs^{-1}$ for $a, b \in R$, and s a regular element of R. Then $ys = b \in R$, and $xs = a \neq 0$ if $x \neq 0$. This shows that R_R is dense in $Q^r_{cl}(R)$. □

[83]Recall that this is the case iff R is right Ore.

In the case when R is a commutative domain with quotient field K, we have, of course, $Q^r_{cl}(R) = Q^r_{\max}(R) = K$. The following example shows that, in general, $Q^r_{\max}(R)$ may be bigger than $Q^r_{cl}(R)$, if the latter exists.

(13.13) Example. Let R be the ring of upper triangular $n \times n$ matrices over a semisimple ring k. Since R is artinian, all regular elements are units (cf. (11.6)(2)), so $R = Q^r_{cl}(R)$. On the other hand, by (3.43A), $E(R_R) = \mathbb{M}_n(k)$. *We claim that* $R_R \subseteq_d \mathbb{M}_n(k)$. Once we have shown this, it will follow that $Q^r_{\max}(R) = \mathbb{M}_n(k)$ as rings, by (13.11)(3). To show the denseness of R, let $x = (x_{ij})$, $y = (y_{ij})$, be $n \times n$ matrices, where $x \neq 0$. Choose $s \in R$ with last column $(a_1, \ldots, a_n)^t$ and all other columns zero. Clearly $ys \in R$, and, choosing $(a_1, \ldots, a_n)$ to be the jth unit vector where $x_{ij} \neq 0$ for some i, we also have $xs \neq 0$. This proves our claim that $R_R \subseteq_d \mathbb{M}_n(k)$. Similarly, it can be shown that ${}_RR \subseteq_d \mathbb{M}_n(k)$. Since $\mathbb{M}_n(k)$ is also the injective hull of ${}_RR$ (see the paragraph following (3.44)), we deduce as before that $Q^\ell_{\max}(R) = \mathbb{M}_n(k)$, while $Q^\ell_{cl}(R) = R$.

Our next result gives a sufficient condition for $Q^r_{\max}(R)$ to be equal to $Q^r_{cl}(R)$, in case $Q^r_{cl}(R)$ exists.

(13.14) Proposition. *Suppose $Q^r_{cl}(R)$ exists, and every dense right ideal of R contains a regular element. Then $Q^r_{\max}(R) = Q^r_{cl}(R)$.*

Proof. We shall show that every $q \in Q = Q^r_{\max}(R)$ belongs to $Q^r_{cl}(R)$. Since $R_R \subseteq_d Q_R$, $q^{-1}R \subseteq_d R_R$ by Exercise (8.2). By hypothesis, $q^{-1}R$ contains a regular element s of R. Then $a := qs \in R$, and, since $s \in \mathrm{U}(Q^r_{cl}(R))$, we have $q = as^{-1} \in Q^r_{cl}(R)$, as desired. □

In view of (11.13), we have in particular:

(13.15) Corollary. *If R is a semiprime right Goldie ring, then $Q^r_{\max}(R) = Q^r_{cl}(R)$, and this is a semisimple ring.*

Note that, by the example given in (13.13), this Corollary is not true in general if R is not assumed to be semiprime. However, if we restrict ourselves to commutative rings, we can prove a corresponding result *without* the semiprime condition. (Recall that, in the commutative case, $Q^r_{cl}(R)$ always exists.)

(13.16) Corollary (Small). *If R is a commutative ring with* ACC *on annihilator ideals, then $Q^r_{\max}(R) = Q^r_{cl}(R)$.*

Proof. By (8.31)(1), every dense ideal of R contains a regular element, so (13.14) applies. □

Recall that, if a ring R is Ore, so that $Q^r_{cl}(R)$ and $Q^\ell_{cl}(R)$ both exist, then they are isomorphic over R. If R is also a domain, then (13.15) and (11.20) imply

$Q^r_{cl}(R) = Q^r_{\max}(R)$ and $Q^\ell_{cl}(R) = Q^\ell_{\max}(R)$, so

$$Q^r_{\max}(R) \cong Q^\ell_{\max}(R) \quad \text{over } R. \tag{13.17}$$

For a domain that is not Ore, however, this need not be the case, as we shall show later in (13.28).

§13C. Another Description of $Q^r_{\max}(R)$

In this subsection, we shall give another useful description for $Q^r_{\max}(R)$ (for any ring R) in terms of the dense right ideals of R. First let us prove the following criterion for dense R-submodules of $Q = Q^r_{\max}(R)$.

(13.18) Proposition. *An R-submodule $D_R \subseteq Q_R$ is dense in Q_R iff, for any $h \in H$ (see* (13.6)*), $h(D) = 0$ implies that $h(1) = 0$.*

Proof. Assume first $D \subseteq_d Q$, and suppose $h \in H$ is such that $h(D) = 0$. Then

$$h: \ Q \longrightarrow I = E(R) = E(Q)$$

must be zero by (8.6). In particular, $h(1) = 0$. Conversely, suppose $h(D) = 0 \Longrightarrow h(1) = 0$, for any $h \in H$. Let P be any right R-module between D and Q. Any R-homomorphism $f : P \to Q$ is the restriction of some $h \in H$, by the injectivity of I_R. Thus, if $f(D) = 0$, we have $h(D) = 0$ and so $h(1) = 0$. Therefore,

$$f(P) = h(1 \cdot P) = h(1)P = 0. \tag{13.19}$$

This shows that $\mathrm{Hom}_R(P/D, Q) = (0)$, so by (8.6), $D \subseteq_d Q$, as desired. □

(13.20) Proposition. *Let D, D' be R-submodules of Q_R such that $D \subseteq_d Q$. Then* $\mathrm{Hom}_R(D, D')$ *is isomorphic (as a group) to*

$$E = E_{D,D'} := \{q \in Q : \ qD \subseteq D'\}.$$

In particular, $\mathrm{End}(D_R)$ *is isomorphic to the subring $E_{D,D} \subseteq Q$, and each R-homomorphism from D to R_R is given by left multiplication by a suitable $q \in Q$.*

Proof. Define $\varphi : E \to \mathrm{Hom}_R(D, D')$ by $\varphi(q)(d) = qd$ where $q \in E$ and $d \in D$. Clearly, $\varphi(q)$ is a (right) R-homomorphism. If $\varphi(q) = 0$, then $qD = 0$. Write $q = h \cdot 1$ where $h \in H$. Then

$$h(D) = h(1 \cdot D) = (h \cdot 1)D = qD = 0.$$

Since $D \subseteq_d Q$, (13.18) implies that $q = h \cdot 1 = 0$. To show that φ is *onto*, consider any $f \in \mathrm{Hom}_R(D, D')$. We may assume that f is the restriction of some $h \in H$. Now let $q = h \cdot 1 \in I$. *We claim that* $q \in Q$. Indeed, for every $h' \in H$ such that $h'(R) = 0$ we have

$$(h'h)(D) = h'(hD) = h'(f(D)) \subseteq h'D' \subseteq h'Q = 0.$$

As before, this implies $0 = (h'h)(1) = h'(q)$. Therefore, by (13.7), we have $q \in Q$, and now

$$f(d) = h(d) = (h \cdot 1)d = qd = \varphi(q)(d)$$

for all $d \in D$, so $f = \varphi(q)$. In the case when $D' = D$, the map φ is clearly a ring homomorphism; this gives the last conclusion in the Proposition. □

Note that, in the above, if we let R be a commutative ring for which $Q^r_{\max}(R) = Q^r_{cl}(R)$, then (13.20) would give back our earlier equation (2.16′). However, (13.20) applies more generally to any ring R. With its help, we arrive at the following alternative description of $Q^r_{\max}(R)$.

(13.21) Theorem. *$Q^r_{\max}(R)$ can be identified as a ring whose elements are classes of R-homomorphisms $\mathfrak{A} \to R$ where $\mathfrak{A}$ is any dense right ideal of R. Two such R-homomorphisms $f : \mathfrak{A} \to R$, $f' : \mathfrak{A}' \to R$ are regarded to be in the same class if $f = f'$ on $\mathfrak{A} \cap \mathfrak{A}'$. The classes of $f : \mathfrak{A} \to R$ and $g : \mathfrak{B} \to R$ are added by taking the class of $f + g : \mathfrak{A} \cap \mathfrak{B} \to R$, and they are multiplied by taking the class of $fg : g^{-1}(\mathfrak{A}) \to R$.*

Proof. We shall only give a sketch of the proof here. To see that the description of multiplication is meaningful, note that since $\mathfrak{A} \subseteq_d R_R$, the preimage $g^{-1}(\mathfrak{A})$ is dense in $\mathfrak{B}$ by Exercise (8.1), and hence dense in R_R by (8.7)(2). Therefore $fg : g^{-1}(\mathfrak{A}) \to R$ does define a class. For $q \in Q$, Exercise (8.1) also implies that $q^{-1}R \subseteq_d R_R$, so left multiplication by q gives a right R-homomorphism $q^{-1}R \to R$. Conversely, for $\mathfrak{A} \subseteq_d R_R$, any R-homomorphism $f : \mathfrak{A} \to R$ is given by left multiplication by a unique $q \in Q$, according to (13.20). For such an element q, we have $q\mathfrak{A} \subseteq R$ so $\mathfrak{A} \subseteq q^{-1}R$. The remaining details are easy to complete, and will be left to the reader. □

Another important application of (13.20) is given by the following result of Utumi.

(13.22) Theorem. *Suppose R has a minimal dense right ideal D (e.g. R is right artinian). Then:*

(1) *$D \subseteq D'$ for any right ideal $D' \subseteq_d R_R$.*
(2) *D is an ideal of R containing the right socle* $\operatorname{soc}(R_R)$.
(3) *$Q^r_{\max}(R) \cong \operatorname{End}(D_R)$ as rings.*

Proof. (1) follows from the observation that $D \cap D' \subseteq_d R_R$ (see (8.7)(1)). Consider any $q \in Q$. Since $D \subseteq_d Q_R$, Exercise (8.1) gives

$$q^{-1}D = \{r \in R : qr \in D\} \subseteq_d R_R. \tag{13.23}$$

By (1), we have $D \subseteq q^{-1}D$, so $qD \subseteq D$. In particular, $R \cdot D \subseteq D$ so D is an ideal of R. Since $D \subseteq_e R_R$, any minimal right ideal of R is contained in D; hence $\operatorname{soc}(R_R) \subseteq D$. Finally, (13.20) gives (3). □

Note that if we identify $Q^r_{\max}(R)$ with $\mathrm{End}(D_R)$ in (13.22), then R embeds in $\mathrm{End}(D_R)$ by a $\mapsto$ left multiplication by a. The fact that this is an embedding is double-checked by (8.3)(4).

(13.24) Corollary. *Let R be a right Kasch ring (in the sense of (8.26)). Then $Q^r_{\max}(R) = R$. (In particular, if R is a commutative Kasch ring, then $Q_{\max}(R) = Q_{cl}(R) = R$.)*

Proof. By (8.28), the only dense right ideal in R is R itself. Thus, we can apply (13.22) to $D = R$. For this choice of D, $Q^r_{\max}(R) \cong \mathrm{End}(D_R) \cong R$. (Of course, this conclusion can be obtained directly: for $q \in Q^r_{\max}(R)$ we have as before $q^{-1}R \subseteq_d R_R$, so $q^{-1}R = R$. Therefore, $1 \in q^{-1}R$, whence $q = q \cdot 1 \in R$.) The last statement of the Corollary now follws from (13.12). □

(13.25) Corollary. *Let R be a right nonsingular right artinian ring. Then $S := \mathrm{soc}(R_R)$ is the smallest dense right ideal of R, and $Q^r_{\max}(R) \cong \mathrm{End}(S_R)$.*

Proof. Recall that, for the right artinian ring R, the right singular ideal $\mathcal{Z}(R_R)$ is $\mathrm{ann}_\ell(S)$ by (7.13). Therefore, $\mathrm{ann}_\ell(S) = (0)$ here, which implies that $S \subseteq_d R_R$ by (8.3)(4). In view of (13.22)(2), it follows that S is the *smallest* dense right ideal of R, and (13.22)(3) applies with $D = S$. □

The two results above enable us to give a few more explicit computations of $Q = Q^r_{\max}(R)$.

(13.26) Examples.

(1) Let $(R, \mathfrak{m})$ be a local ring where $\mathfrak{m} = \mathrm{rad}\, R$ is nilpotent. Then, by (8.29)(3), R is a (right) Kasch ring, so by (13.24), $Q^r_{\max}(R) = R$. If we take R to be, for instance, the local 3-dimensional k-algebra in (6.13) where k is a field, then we have

$$R = Q^r_{cl}(R) = Q^r_{\max}(R) \subsetneq E(R), \tag{13.27}$$

since $\dim_k E(R) = 6$.

(2) Let $R = \begin{pmatrix} A & 2A \\ 0 & A \end{pmatrix}$ where $A = \mathbb{Z}_4$, as in Osofsky's example (3.45). This ring of 32 elements is not (left, right) nonsingular by (7.6)(6), but it is Kasch by Exercise (8.16). Thus, we conclude from (13.24) and (3.45) that $R = Q^r_{\max}(R) \subsetneq E(R_R)$. This implies that *no ring properly containing R can be a general right ring of quotients of R*. Take, for instance, the ring $T = \begin{pmatrix} A & 2A \\ A & A \end{pmatrix} \supsetneq R$. For

$$x = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{in } T,$$

it is clear that $yr \in R$ implies $a = 0$ and hence necessarily $xr = 0$. This shows that R_R is not dense in T_R, so T is indeed *not* a general right ring of quotients of R.

The same is true already for the smaller ring $T' = \begin{pmatrix} A & 2A \\ 2A & A \end{pmatrix} \supsetneq R$. We choose $y = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ in this case, and note that $yr \in R \Longrightarrow a \in 2A \Longrightarrow xr = 0$.

(3) Let R be the (artinian) ring of upper triangular $n \times n$ matrices over a semisimple ring k. This ring is right nonsingular (by (7.14b)), so (13.25) applies. In (7.14b), we have observed that $S := \text{soc}(R_R)$ is the ideal of all matrices with zeros on all but the last column. Identifying such matrices with their last columns, we may view S as k^n. Here, any $a = (a_{ij}) \in R$ acts on the right of a column vector by right multiplication by a_{nn}. Therefore,

$$Q^r_{\max}(R) \cong \text{End}(S_R) \cong \text{End}(k^n)_k \cong \mathbb{M}_n(k).$$

Under the composite isomorphism, elements of R do correspond to themselves as upper triangular matrices. Of course, this computation of $Q^r_{\max}(R)$ is in agreement with the earlier one given in (13.13). A similar computation with $\text{soc}({}_R R)$ shows that $Q^\ell_{\max}(R) \cong \mathbb{M}_n(k)$, again with elements of R corresponding to themselves.

(4) Let $R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, where k is as above. By (7.14c), R is again artinian, nonsingular, with

$$S = \text{soc}(R_R) = \begin{pmatrix} 0 & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}, \qquad \text{soc}({}_R R) = \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the latter, we get $Q^\ell_{\max}(R) \cong \mathbb{M}_3(k)$ as before. From the former, we get a decomposition $S_R = \mathfrak{A} \oplus \mathfrak{B}$ where

$$\mathfrak{A} = \begin{pmatrix} 0 & k & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{B} = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}.$$

After computing the actions of R on $\mathfrak{A}$ and $\mathfrak{B}$, we see that $\mathfrak{A}$, $\mathfrak{B}$ are the isotypic components of the semisimple module S_R, and deduce that

$$\begin{aligned} Q^r_{\max}(R) &\cong \text{End}(\mathfrak{A} \oplus \mathfrak{B})_R \\ &\cong \text{End}\ \mathfrak{A}_R \times \text{End}\ \mathfrak{B}_R \\ &\cong \text{End}\ \mathfrak{A}_k \times \text{End}\ \mathfrak{B}_k \\ &\cong \mathbb{M}_2(k) \times \mathbb{M}_2(k). \end{aligned}$$

Thus, as long as $k \neq 0$, the two rings $Q^r_{\max}(R)$, $Q^\ell_{\max}(R)$ are not isomorphic. (For a somewhat different approach to these computations, see Exercise 14.)

(5) (Zelmanowitz and Li) Let k be a nonzero semisimple ring and $V = k \oplus \cdots \oplus k$ (n copies), viewed in the natural way as a (k, k)-bimodule. Let R be the triangular

ring $\begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$, which is, of course, artinian. We have here

$$S = \mathrm{soc}(R_R) = \begin{pmatrix} 0 & V \\ 0 & k \end{pmatrix} \quad \text{and} \quad S' = \mathrm{soc}({}_R R) = \begin{pmatrix} k & V \\ 0 & 0 \end{pmatrix},$$

so by (7.13), R is nonsingular. Using (13.25), it follows as in the last example that

$$Q^r_{\max}(R) \cong \mathrm{End}(S_R) \cong \mathrm{End}(k^{n+1})_k \cong \mathbb{M}_{n+1}(k), \quad \text{and}$$
$$Q^\ell_{\max}(R) \cong \mathrm{End}({}_R S') \cong \mathrm{End}_k(k^{n+1}) \cong \mathbb{M}_{n+1}(k).$$

While these two maximal rings of quotients are isomorphic as rings, *they are not isomorphic over R if $n \geq 2$.* In fact, if $\varphi : Q^r_{\max}(R) \to \mathbb{M}_{n+1}(k)$ and $\psi : Q^\ell_{\max}(R) \to \mathbb{M}_{n+1}(k)$ are the isomorphisms given above, we can check easily that, for $a, b \in k$ and $v = (v_1, \ldots, v_n) \in k^n$:

$$\varphi\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} & & v_1 \\ & aI_n & \vdots \\ & & v_n \\ 0 & \ldots 0 & b \end{pmatrix} \quad \text{and} \quad \psi\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & v_1 \ldots v_n \\ 0 & \\ \vdots & bI_n \\ 0 & \end{pmatrix}.$$

(Of course, these matrix representations depend on a specific labeling of the usual basis of k^{n+1}.) Let $A = \varphi(R) \cong R$ and $B = \psi(R) \cong R$. By (3.43) (and its left analogue), we know that

$$E(A_A) = \mathbb{M}_{n+1}(k) \quad \text{and} \quad E({}_B B) = \mathbb{M}_{n+1}(k).$$

Moreover, it is easy to see that $A \subseteq_d \mathbb{M}_{n+1}(k)$ as right A-modules. In fact, if $x, y \in \mathbb{M}_{n+1}(k)$ with $x \neq 0$, there exists $\begin{pmatrix} v \\ b \end{pmatrix} \in k^{n+1}$ with $x \cdot \begin{pmatrix} v \\ b \end{pmatrix} \neq 0$. For $\alpha := \begin{pmatrix} 0 & v \\ 0 & b \end{pmatrix} \in A$, we have then $x \cdot \alpha \neq 0$, and automatically $y \cdot \alpha \in A$. It follows that $Q^r_{\max}(A) = \mathbb{M}_{n+1}(k)$, and similarly $Q^\ell_{\max}(B) = \mathbb{M}_{n+1}(k)$. This provides alternative computations to $Q^r_{\max}(R)$ and $Q^\ell_{\max}(R)$. (Actually, even the density check above can be omitted, since R is a nonsingular ring; see (13.39)(1).) On the other hand, it is easy to check that B is not essential in $\mathbb{M}_{n+1}(k)_B$, so $\mathbb{M}_{n+1}(k) \supseteq B$ is *not* a maximal right ring of quotients of B. Similarly, $\mathbb{M}_{n+1}(k) \supseteq A$ is *not* a maximal left ring of quotients of A. There is no contradiction here; this simply means that $Q^r_{\max}(R)$ and $Q^\ell_{\max}(R)$ cannot be isomorphic *over* R.

In the last three examples above ((3), (4), and (5)), the ring R is right nonsingular, and we have $Q^r_{\max}(R) = E(R_R)$. This turns out to be true for *any* right nonsingular ring R: see (13.36) below.

We shall include one more example below which shows that, if R is a domain (resp. R is countable), $Q^r_{\max}(R)$ may not be a domain (resp. may not be countable). In this example, it turns out also that $Q^r_{\max}(R)$ and $Q^\ell_{\max}(R)$ are not isomorphic over R.

(13.28) Example. Let $R = \mathbb{Q}\langle a, b\rangle$ (the free $\mathbb{Q}$-algebra on a, b), and let D be the *ideal* generated by b in R. Since R is a domain, (8.4)(3) implies that $D \subseteq_d R_R$. As a right R-module,

(13.29) $$D_R = \bigoplus_{i=0}^{\infty} a^i bR$$

is free of countable infinite rank (cf. the proof of (10.22)). Therefore, by (13.20), $Q = Q^r_{\max}(R)$ contains a subring isomorphic to End(D_R), which is the ring of column-finite $\mathbb{N} \times \mathbb{N}$ matrices over R. In particular, *Q is uncountable and has many 0-divisors* (even nilpotent elements), although R itself is countable and is a domain. Similarly, since D is contained in

(13.30) $$D_n := \bigoplus_{i=0}^{n} a^i bR \oplus a^{n+1}R \cong R_R^{n+2},$$

we have $D_n \subseteq_d R_R$, and so Q also contains (as a subring) a copy of End$(R_R^{n+2}) \cong \mathbb{M}_{n+2}(R)$ (for each n). Thus, Q contains (as subrings) copies of *all* finite matrix rings over R.

Next, we claim that *$Q^r_{\max}(R)$ and $Q^\ell_{\max}(R)$ cannot be isomorphic over R*. Indeed, consider the right R-homomorphism α from $D_0 = aR \oplus bR$ to R defined by $\alpha(af + bg) = f$. By (13.20), there exists $q \in Q$ such that $q \cdot (af + bg) = f$, for all f, $g \in R$. In particular, $qa = 1$ and $qb = 0$. Assume, for the moment, that Q is also the maximal left ring of quotients of R. Then there would also exist a $q' \in Q$ such that $(fa + gb)q' = g$ for all f, $g \in R$. In particular, $aq' = 0$ and $bq' = 1$. But now

$$q' = (qa)q' = q(aq') = 0,$$

contradicting $bq' = 1$ (and hence proving our claim).

§13D. Theorems of Johnson and Gabriel

Let us now return to the general theory, and prove a few more results about $Q = Q^r_{\max}(R)$. Again, the notations introduced in (13.6) will remain in force. The following Proposition confirms that the association of Q to R is a sort of "closure operation". (See also Exercises (10) and (11).)

(13.31) Proposition.

(1) *I_Q is the injective hull of Q_Q.*
(2) $\mathrm{Hom}_Q(I_Q, I_Q) = H$.
(3) $Q^r_{\max}(Q) = Q$.

Proof. Since any Q-endomorphism of I is an R-endomorphism, (2) follows. If we can prove (1), clearly (3) will follow from (1) and (2). To prove (1), it suffices to show that I_Q is Q-injective, for $R_R \subseteq_e I_R$ implies $Q_Q \subseteq_e I_Q$. Let $A \subseteq B$ be right Q-modules and $\varphi_0 : A \to I$ be a Q-homomorphism. We can extend φ_0 to an R-homomorphism $\varphi : B \to I$. We are done if we can show that φ is a

Q-homomorphism. Let $b \in B$ and define $\sigma_0 : Q \to I$ by

$$\sigma_0(q) = \varphi(bq) - \varphi(b)q \qquad (\forall\, q \in Q).$$

For any $r \in R$, we have

$$\sigma_0(qr) = \varphi(bqr) - \varphi(b)qr = (\varphi(bq) - \varphi(b)q)r = \sigma_0(q)r.$$

Therefore, σ_0 is an R-homomorphism, and it can be extended to $\sigma : I_R \to I_R$. Since φ is an R-homomorphism, we have $\sigma_0(R) = 0$ and hence $\sigma(R) = 0$. But $\sigma \in H$, so by (13.7), $\sigma(Q) = 0$ too, and this implies that $\varphi(bq) = \varphi(b)q$ for all $b \in B$ and $q \in Q$. Therefore, $\varphi \in \text{Hom}_Q(B_Q, I_Q)$, as desired. □

(13.32) Corollary. *For $R, I, Q, \ldots$ as in* (13.6), *the following are equivalent:*

(1) $Q = I$.
(2) Q_R *is injective.*
(3) Q_Q *is injective.*
(4) *The surjection* $\varphi : {}_H H \to {}_H I$ *defined by* $\varphi(h) = h \cdot 1$ *is an isomorphism.*

If these conditions hold, then $H \cong Q$ *as rings.*

Proof. The equivalence of (1), (2), and (3) follows easily from (13.31). Assume (1). To show that φ in (4) is injective, suppose $h \cdot 1 = 0$ where $h \in H$. By (13.7), $h \cdot Q = 0$ too, and hence $h \cdot I = 0$ by (1). Therefore, $h = 0$. Conversely, assume (4). Then, for any $h \in H$:

$$hR = 0 \implies h \cdot 1 = 0 \implies h = 0 \implies h \cdot I = 0.$$

By (13.7) again, we see that $I = Q$. Moreover, from (4), we have

$$Q = \text{End}({}_H I) \cong \text{End}({}_H H) \cong H$$

as rings. (Alternatively, one can show directly that φ is a ring isomorphism, after we identify I with the ring Q by (1).) □

In general, of course, Q may be smaller than I so (13.32) may not apply; see, for instance, Examples (1) and (2) in (13.26). But even in the general case, we can identify Q as a "subquotient" of the ring H, by the following considerations. Let K be the kernel of the map φ defined in (13.32); this is a left ideal of H. Let

(13.33) $$P = \{h \in H : Kh \subseteq K\};$$

this is the biggest subring of H in which K is an ideal.

(13.34) Proposition. *We have a ring isomorphism $P/K \cong Q$. If R is commutative, then $Q \cong Z(H)$ (the center of H); in particular, Q is also commutative in this case.*

Proof. By (13.7), $Q = \{i \in I : K \cdot i = 0\}$. Thus, for $h \in H$:

$$\begin{aligned}\varphi(h) \in Q &\Longleftrightarrow K \cdot \varphi(h) = 0\\ &\Longleftrightarrow (K \cdot h)(1) = 0\\ &\Longleftrightarrow K \cdot h \subseteq K.\end{aligned}$$

This shows that the subring P defined in (13.33) is exactly $\varphi^{-1}(Q)$. Therefore, the surjection $\varphi : H \to I$ induces a group isomorphism $P/K \cong Q$. To show that this is a ring isomorphism, we must verify that $\varphi(hh') = \varphi(h)\varphi(h')$ for $h, h' \in P$. Let $\varphi(h) = h \cdot 1 = q$ and $\varphi(h') = h' \cdot 1 = q'$. The product qq' is identified with

$$1 \cdot qq' = (1q)q' = (h \cdot 1)q' = h(1q') = h(h'1) = (hh')(1),$$

so indeed $\varphi(hh') = qq' = \varphi(h)\varphi(h')$. Now let $C = Z(H)$. For $h \in C$, we have $Kh = hK \subseteq K$, so by (13.33), $C \subseteq P$. Also, for $h \in C \cap K$:

$$h \cdot I = h \cdot H1 = Hh \cdot 1 = 0 \Longrightarrow h = 0.$$

Thus, $C \cap K = (0)$, so φ defines an inclusion of the ring C into the ring Q. *Assuming now that R is commutative*, let us show that $\varphi(C) = Q$. Let $q \in Q$. Consider $f \in \mathrm{End}_{\mathbb{Z}} I$ given by $f(i) = iq$ $(i \in I)$. For any $h \in H$, we have

$$(fh)(i) = f(h(i)) = h(i)q = h(iq) = h(f(i)) = (hf)(i)$$

for any $i \in I$, so

(13.35) $$fh = hf \quad (\forall h \in H).$$

For any fixed element $r \in R$, define $h \in \mathrm{End}_{\mathbb{Z}} I$ by right multiplication by r. Since R is commutative, we have $h \in H$, and (13.35) yields $f(ir) = f(i)r$. This shows that $f \in H$, and (13.35) further shows that $f \in C$. Finally, $\varphi(f) = f(1) = q$, as desired. □

From (13.34), we see easily that, for any commutative ring R, $Q^r_{\max}(R) \cong Q^\ell_{\max}(R)$ over R. For another proof of the fact that $Q = Q^r_{\max}(R)$ is commutative if R is (as well as a characterization for the center of Q for a general R), see Exercise 5.

We now come to an important theorem of R. E. Johnson on right nonsingular rings. The main part of this theorem says that *a ring R is right nonsingular iff Q, its maximal right ring of quotients, is von Neumann regular.* Actually, Johnson proved this result *before* the discovery of the general theory of maximal right rings of quotients. In Johnson's original paper, the injective hull $I = E(R_R)$ was used in the place of Q. As the reader will see below, Q is just I for right nonsingular rings R.

(13.36) Johnson's Theorem. *For any ring R, the following are equivalent:*

(1) *R is right nonsingular.*
(2) *$I_R = E(R_R)$ is a nonsingular R-module.*
(3) *$H = \mathrm{End}(I_R)$ is Jacobson semisimple (i.e.,* rad $H = 0$*).*

(4) $Q = Q^r_{\max}(R)$ *is von Neumann regular.*

If these conditions hold, then $Q = I$, *and* $Q \cong H$ *are right self-injective rings.*

Proof. (1)$\Longrightarrow$(2). Since $R_R \subseteq_e I_R$, this follows from (7.6)(2).

(2)$\Longrightarrow$(3) follows from (1) and (4) of (13.1).

(3)$\Longrightarrow$(4). Consider the H-homomorphism $\varphi : H \to I$ defined by $\varphi(h) = h \cdot 1$. By (13.8)(1), φ is *onto*. We claim that φ is also *one-one*. In fact, if $h \in \ker(\varphi)$, then $h(R) = h(1)R = (0)$. Since $R_R \subseteq_e I_R$, (13.1) gives $h \in N = (0)$. Applying (13.32), we get $Q \cong H = H/N$, which is von Neumann regular by (13.1)(2). The other conclusions in the last statement of the Theorem also follow from (13.32).

(4)$\Longrightarrow$(1). Let $b \in \mathcal{Z}(R_R)$. Then $\mathfrak{A} := \text{ann}_r(b) \subseteq_e R_R$, so we also have $\mathfrak{A} \subseteq_e Q_R$. Pick $q \in Q$ such that $b = bqb$. If $b \neq 0$, then $qb \neq 0 \in Q$, so there exists $c \in R$ such that $qbc \in \mathfrak{A}\backslash\{0\}$. But then $bc = (bqb)c \in b\mathfrak{A} = (0)$, a contradiction. □

To illustrate the power of Johnson's Theorem, let us record a few important special cases of it, as follows.

(13.37) Corollary. *Assume that R is a reduced ring, or a right Rickart ring (e.g., a Baer ring or a right semihereditary ring), or an integral group ring $\mathbb{Z}G$. Then $Q^r_{\max}(R)$ is a right self-injective von Neumann regular ring.*

In the case when R is a domain, a bit more can be said. The following lemma and its corollary are due to S. K. Jain.

(13.38) Lemma. *Let $R \subseteq S$ be such that R is a domain and S is a von Neumann regular ring. If $R_R \subseteq_e S_R$, then S is a simple ring.*

Proof. Let I be a nonzero ideal of S. Then there exists a nonzero $a \in R \cap I$. Since R is a domain, $R \cap \text{ann}^S_r(a) = 0$. Therefore, $\text{ann}^S_r(a) = 0$. Let $s \in S$ be such that $a = asa$. Then $a(1 - sa) = 0$ implies that $1 = sa \in I$, so $I = S$. This shows that S is a simple ring.[84] □

(13.38)′ Corollary. *For any domain R, $Q^r_{\max}(R)$ is a simple (right self-injective and von Neumann regular) ring.*

Proof. This follows by applying (13.37), together with (13.38). □

Johnson's Theorem suggests that, for a right nonsingular ring R, the structure of $Q^r_{\max}(R)$ is particularly susceptible to analysis since it is a right self-injective, von

[84]If we interpret the "1" in this proof as 1_S, we do not need to assume $1_R = 1_S$ in this result. On the other hand, it is easy to see that the assumption $R_R \subseteq_e S_R$ actually *implies* that $1_R = 1_S$.

Neumann regular ring. Also, for right nonsingular R, $Q^r_{\max}(R)$ is often "easy" to identify, in view of following statement, which is handy to have in explicit form, although it is just a recap of earlier results.

(13.39) Proposition. *Let $R \subseteq S$ be rings such that R is right nonsingular. If either* (1) $S_R = E(R_R)$, *or* (2) $R_R \subseteq_e S_R$ *and S is right self-injective, then $S = Q^r_{\max}(R)$ (as rings).*

Proof. Let $Q = Q^r_{\max}(R)$. We first work in Case (1). Here, $Q = S$ as right R-modules by (13.36). Therefore, $Q = S$ as rings by (13.11)(3). Next, we work in Case (2). Here, $R_R \subseteq_e S_R$ implies that $\mathcal{Z}(S_R) = 0$ by (7.6)(2), which in turn implies that $R_R \subseteq_d S_R$ by (8.7)(3). Thus, by (13.11)(2), we can identify S as a subring of Q (over R). In particular, we may view Q as a right module over S. Since S_S is injective, $Q = S_S \oplus X$ for some S-submodule $X \subseteq Q_S$. But then $X \cap R = 0$ and $R_R \subseteq_e Q_R$ imply that $X = 0$, so $S = Q$. □

The above additional observations (13.37)–(13.39) conclude our discussions on Johnson's Theorem (13.36). While this theorem gives characterizations for the rings R for which $Q^r_{\max}(R)$ is *von Neumann regular*, we can try to specialize it further by asking for characterizations for those rings R for which $Q^r_{\max}(R)$ is *semisimple*. This brings us to the next result, which is usually attributed to Gabriel, although half of it (the implication (2)$\Longrightarrow$(1)) was also proved by Johnson.

(13.40) Gabriel's Theorem. *For any ring R, the following are equivalent:*

(1) $Q = Q^r_{\max}(R)$ *is semisimple.*
(2) *R is right nonsingular and* u. dim $R_R < \infty$.

Proof. (1)$\Longrightarrow$(2). Certainly Q is von Neumann regular, so (13.36) shows that R is right nonsingular, and that $H \cong Q$. Let $1 = e_1 + \cdots + e_m$ where the e_i's are pairwise orthogonal primitive idempotents in the semisimple ring H. Then $I = e_1 I \oplus \cdots \oplus e_m I$ is a decomposition of I into a direct sum of m indecomposable (injective) modules. By (6.12), this implies that u.dim $R_R = m < \infty$.

(2)$\Longrightarrow$(1). By (6.12) again, u.dim $R_R < \infty$ implies that I_R is a finite direct sum of indecomposables. Therefore, by (13.3), $H/\text{rad } H$ is semisimple. On the other hand, since R is right nonsingular, (13.36) yields $Q \cong H$ and rad $H = 0$, so Q is now semisimple. □

It behooves us to say a few words on the relationship between Gabriel's Theorem proved above and Goldie's Theorem proved earlier in (11.13). The former gives a characterization of rings R for which $Q^r_{\max}(R)$ is semisimple, while the latter gives various characterizations of rings R for which $Q^r_{cl}(R)$ exists and is semisimple. The relationship between the two theorems may be seen in part from the following observation.

(13.41) Proposition. *The conditions* (1) *and* (2) *in* (13.40) *imply that R is right Goldie. If R is semiprime, the converse holds, and we have $Q^r_{\max}(R) = Q^r_{cl}(R)$.*

Proof. Assume (1) and (2) in (13.40). Then, being a semisimple ring, Q has ACC on right annihilators, and by (6.61), R has the same property. Since we also have u.dim $R_R < \infty$, R is right Goldie. Conversely, if R is semiprime right Goldie, then R is right nonsingular by (7.19), so we have (2) in (13.40). (Cf. (2)$\Longleftrightarrow$(4) in (11.13).) In this case, we have $Q^r_{\max}(R) = Q^r_{cl}(R)$ by (13.15). □

Note that the "semiprime" assumption is essential for the second statements in (13.41). If R is a right Goldie ring, the conditions (1) and (2) in (13.40) need not hold in general. For instance, a right noetherian ring R is always right Goldie, but R certainly need not be right nonsingular (see, e.g., (7.6)(6)). Therefore, we might think of Gabriel's Theorem as a result extending Goldie's Theorem. In the case of semiprime right Goldie rings, the two theorems basically give the same conclusions. For right Goldie rings R that are *not* semiprime, Goldie's Theorem does not yield any information, but Gabriel's Theorem implies that $Q^r_{\max}(R)$ is semisimple iff R is right nonsingular. The germane examples here are the right nonsingular right artinian rings. For these rings, (2) in (13.40) is satisfied, and according to (13.25),

(13.42) $$Q^r_{\max}(R) \cong \text{End}(S_R), \quad \text{where } S = \text{soc}(R_R).$$

Since S_R is a f.g. semisimple R-module, the rings in (13.42) are indeed semisimple, as predicted by (13.40). For explicit examples of this, see (13.26)((3), (4), and (5)). In these examples, R is not semiprime, and (since R is artinian), $Q^r_{cl}(R) = R$ is not semisimple.

We now conclude this section with the following special case of Gabriel's Theorem.

(13.43) Corollary. *For $Q = Q^r_{\max}(R)$, the following are equivalent:*

(1) *Q is a division ring.*
(2) *R is a right Ore domain.*
(3) *R is a domain and Q is reduced.*

Proof. (1)$\Longrightarrow$(2) follows from (13.40) and (10.22).

(2)$\Longrightarrow$(3). Under (2), $Q = Q^r_{\max}(R) = Q^r_{cl}(R)$ is a division ring, hence reduced.

(3)$\Longrightarrow$(1). Under (3), Q is simple by (13.39) and hence prime. Since Q is also reduced, it is a domain (see *FC*–Exer. (10.3)). The fact that Q is von Neumann regular (by (13.37)) now implies (1). □

It follows from (13.43) that, for any domain R which is not right Ore, $Q^r_{\max}(R)$ must have nonzero nilpotent elements. We have already seen an explicit example of this in (13.28) by taking $R = \mathbb{Q}\langle a, b\rangle$.

For R a nonzero *right nonsingular* ring, there are also results characterizing, in terms of R, the conditions for $Q = Q^r_{\max}(R)$ to be (1) a prime ring, (2) a simple ring, or (3) a simple artinian ring. These results are due to Johnson, Goodearl, Handelman, Jain, Lam, Leroy, and others. For instance, Q is a prime ring iff R is "right irreducible" in the sense of Johnson; that is, for any ideal $A \subseteq R$, $A = \text{ann}_\ell(\text{ann}_\ell A)$ and $A \cap \text{ann}_\ell(A) = 0$ imply that $A = 0$ or $A = R$. There is also a general theory for the direct product decomposition of Q into prime (von Neumann regular, right self-injective) rings: see Goodearl [91: Ch. 10] and Jain-Lam-Leroy [98]. In particular, in the latter, it is shown that Q is a direct product of n prime rings iff n is the largest integer for which there exist nonzero ideals $A_1, \ldots, A_n \subseteq R$ such that $A_1 \oplus \cdots \oplus A_n$ is right essential in R.

Exercises for §13

(In the following Exercises, the notations in (13.6) are in force.)

1. (Utumi) Show that $Q^r_{\max}(\prod_j R_j) \cong \prod_j Q^r_{\max}(R_j)$ for any family of rings $\{R_j\}$.

2. Let $b \in R$ be right regular in R; i.e., $\text{ann}^R_r(b) = 0$. Show that (1) b remains right regular in Q, and (2) if R is right nonsingular, b has a left inverse in Q. (3) If b is regular in R, is it necessarily regular in Q?

3. Show that an element $b \in R$ is a unit in Q iff b is right regular in R and $bR \subseteq_d R_R$.

4. For $q \in Q$, show that the following are equivalent:
(1) $q \in \text{U}(Q)$;
(2) q is right regular in Q and $qQ \subseteq_d Q_Q$;
(3) for $i \in I$, $iq = 0 \Longrightarrow i = 0$, and, for $r \in R$, $qr = 0 \Longrightarrow r = 0$.

5. Show that an element $q \in Q$ is central in Q iff it commutes with all elements of R. Deduce from this that, if R is commutative, so is Q. (The first part will be proved later in (14.15); look up the proof there only if you are absolutely stuck.)

6. (Utumi) Let R be a prime ring with $S = \text{soc}(R_R) \neq 0$. Show that $Q = Q^r_{\max}(R)$ is isomorphic to $\text{End}(V_D)$ for a suitable right vector space V over some division ring D.

7. (1) Show that a ring R has the form $\text{End}(V_D)$ where V is a right vector space over a division ring D iff R is prime, right self-injective, and with $\text{soc}(R_R) \neq 0$. (2) Show that a prime, self-injective ring R with $\text{soc}(R_R) \neq 0$ must be semisimple.

8. Show that if R is simple (resp. prime, semiprime), so is every general right ring of quotients of R.

9. Let $\mathfrak{B} \subseteq R \subseteq S$, where R, S are rings, and $\mathfrak{B}$ is a left ideal of S with $\text{ann}^S_\ell(\mathfrak{B}) = 0$. Let $I = E(S_S)$. Show that

(1) For $i \in I$, $i\mathfrak{B} = 0 \Longrightarrow i = 0$;
(2) $I = E(R_R)$;
(3) $\text{End}(I_R) = \text{End}(I_S)$; and finally,
(4) $Q^r_{\max}(R) = Q^r_{\max}(S)$.

(Note that this exercise can be used to compute $Q^r_{\max}(R)$ for many rings, for instance those in Examples (3.43A) through (3.43F).)

10. (This exercise provides a more general view, and a new proof, for the fact that $R \mapsto Q^r_{\max}(R)$ is a "closure operation": cf. (13.31)(3).) Let $R \subseteq S \subseteq T$ be rings such that S (resp. T) is a general right ring of quotients of R (resp. S). Show that T is a general right ring of quotients of R.

11. Let T be a general right ring of quotients of R.

    (1) Show that $T = Q^r_{\max}(R)$ iff $Q^r_{\max}(T) = T$; and that,
    (2) in general, $Q^r_{\max}(R) = Q^r_{\max}(T)$.

12. Let $k \subsetneq K$ be fields and let $R = \begin{pmatrix} k & K \\ 0 & K \end{pmatrix} \subseteq S = \mathbb{M}_2(K)$. Show that $S = Q^r_{\max}(R)$, but S is not a general *left* ring of quotients of R.

13. For the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, compute $Q^r_{cl}(R)$ and $Q^r_{\max}(R)$.

14. (Cf. (13.26)(4).) Let $R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, where k is a semisimple ring. Determine $Q^\ell_{\max}(R)$ and $Q^r_{\max}(R)$ directly by computing the injective hulls of ${}_RR$ and R_R. (**Hint.** To compute $E(R_R)$, use the embedding $\varphi : R \to A = \mathbb{M}_2(k) \times \mathbb{M}_2(k)$ defined by

$$\varphi\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} = \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & e \end{pmatrix}\right) \in A,$$

and use (3.42) to show that $E(R_R) = A$.)

15. Show that any automorphism of a ring R extends uniquely to an automorphism of $Q = Q^r_{\max}(R)$.

16. (R. E. Johnson) Let $Q = Q^r_{\max}(R)$, where R is a right nonsingular ring. Show that any closed R-submodule of Q_R is a principal right ideal of Q (and conversely). Using this, show that there is a one-one correspondence between the closed right ideals of R and the principal right ideals of Q.

17. Show that a commutative ring R is reduced iff $Q_{\max}(R)$ is reduced, iff $Q_{\max}(R)$ is von Neumann regular.

18. Show that R is a Boolean ring iff $Q^r_{\max}(R)$ is a Boolean ring.

19. A subset X in a set W is said to be *cofinite* if $W \backslash X$ is finite. Show that

$$R = \{X \subseteq W : X \text{ is either finite or cofinite}\}$$

is a Boolean subring of the Boolean ring S of all subsets of W, and show that $Q_{\max}(R) = S$.

20. Let R be a domain and $Q = Q^r_{\max}(R)$.
(1) For any nonzero idempotent $e \in Q$, show that $eQ \cong Q_Q$.
(2) Show that any nonzero f.g. right ideal $\mathfrak{A} \subseteq Q$ is isomorphic to Q_Q.
(3) If Q is Dedekind-finite, show that R is a right Ore domain and Q is its division ring of right fractions.

21. Let R be a semiprime ring with only finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, and let $\bar{R}_i = R/\mathfrak{p}_i$. Show that $Q^r_{\max}(R) \cong \prod_i Q^r_{\max}(\bar{R}_i)$. (**Hint.** Use the ideal $\mathfrak{B} \subseteq \prod_i \bar{R}_i$ constructed in Exercise (11.17), and apply Exercise 1.)

22. (Cf. Exercise (11.19).) Let R be the (commutative) ring $\mathbb{Q}\langle\{x_i : i \geq 1\}\rangle$ with the relations $x_i x_j = 0$ for all unequal i, j. (1) Show that $Q_{\max}(R)$ is isomorphic to the direct product $T = \prod_{i \geq 1} \mathbb{Q}(x_i)$, where R is embedded in T by sending $a \in \mathbb{Q}$ to $(a, a, \ldots)$ and sending x_i to $(0, \ldots, x_i, 0, \ldots)$ with x_i in the ith coordinate. (2) Show that $Q_{cl}(R) \subsetneq Q_{\max}(R)$.

23. Let $T = k \times k \times \ldots$ where k is any right self-injective ring, and let R be the subring of T consisting of all sequences $(a_1, a_2, \ldots) \in T$ which are eventually constant. Show that $T = Q^r_{\max}(R)$.

24. Give an example of a pair of commutative local artinian rings $R \subseteq S$ such that $R \subseteq_e S_R$, but S is *not* a general (right) ring of quotients of R.

25. (Nicholson-Yousif) Recall that a ring R is *right principally injective* if, for any $a \in R$, any $f \in \text{Hom}_R(aR, R)$ extends to an endomorphism of R_R (see Exercise (3.44)). Show that, for such a ring R, $\text{rad}\, R = \mathcal{Z}(R_R)$. (This generalizes (13.2)(1).)

26. Show that a ring R is semisimple iff R has the following properties: (1) R is semiprime, (2) R is right principally injective, and (3) R satisfies ACC on right annihilators of elements. (**Hint.** Use (7.15)(1), (13.2)(5), *FC*–(10.29), and Exercise 25.) Note that, from this exercise, it follows that any right principally injective semiprime right Goldie ring is semisimple.

27. For any right principally injective ring R, show that $\text{soc}(R_R) \subseteq \text{soc}(_R R)$. (In particular, equality holds for any 2-sided principally injective ring.)

28. Let I_R be a QI module, and $I' = E(I)$. Let $H = \text{End}(I_R)$, $N = \text{rad}\, H$, and $H' = \text{End}(I'_R)$, $N' = \text{rad}\, H'$. Show that there is a natural ring isomorphism $H'/N' \cong H/N$.

29. (Ikeda-Nakayama) Show that a ring R is von Neumann regular iff, for any $a \in R$ and any right ideal $I \subseteq R$, any $f \in \mathrm{Hom}_R(aR, R/I)$ is induced by left multiplication by an element of R.

§14. Martindale Rings of Quotients

§14A. Semiprime Rings Revisited

After discussing maximal rings of quotients in §13, it is natural to include an introduction to the idea of Martindale rings of quotients. This kind of rings of quotients was introduced by W. S. Martindale [69] for prime rings in 1969, and by S. A. Amitsur [72] for semiprime rings in 1972. A more precise term for such rings of quotients would have been the Martindale-Amitsur rings of quotients as, for instance, in the book of Rowen [88]. In the interest of brevity, however, we shall refer to them simply as Martindale's rings of quotients.

Since the theory of Martindale's rings of quotients works only for semiprime rings, we shall begin our discussion by stating and proving a few properties of semiprime rings which are crucial for the development of this theory. This short subsection may thus be viewed as a continuation of §11D on semiprime rings.

We begin with a basic observation on left annihilators in any ring R. Note that for any left ideal $A \subseteq R$, the left annihilator, $\mathrm{ann}_\ell(A)$, is always an ideal in R.

(14.1) Proposition. *For any ideal A in a ring R with the property that $A \cap \mathrm{ann}_\ell(A) = 0$, the following are equivalent:*

(1) $\mathrm{ann}_\ell(A) = 0$.
(2) $A \subseteq_e {}_RR_R$ *(A is 2-sided essential in R).*
(3) $A \subseteq_e R_R$ *(A is right essential in R).*
(4) $A \subseteq_d R_R$ *(A is right dense in R).*

Proof. (4)$\Longrightarrow$(3)$\Longrightarrow$(2) are clear.

(2)$\Longrightarrow$(1). Assume that $A \subseteq_e {}_RR_R$. Since $\mathrm{ann}_\ell(A)$ is an ideal with zero intersection with A, we have $\mathrm{ann}_\ell(A) = 0$.

(1)$\Longrightarrow$(4) follows from (8.3)(4). □

We shall now specialize to semiprime rings. Recall that, if R is semiprime, then for any ideal $A \subseteq R$, we have $\mathrm{ann}_r(A) = \mathrm{ann}_\ell(A)$. This common annihilator is denoted simply by $\mathrm{ann}(A)$, and is called an *annihilator ideal* in R. Since $(A \cap \mathrm{ann}(A))^2 = 0$, the assumption that R is semiprime forces $A \cap \mathrm{ann}(A) = 0$. Therefore, (14.1) boils down to the following simpler statement, which provides the beginning point for Martindale's theory of rings of quotients for semiprime rings.

(14.2) Corollary. *For an ideal A in a semiprime ring R, we have* $\text{ann}(A) = 0$ *iff* $A \subseteq_e {}_R R_R$, *iff* $A \subseteq_e R_R$, *iff* $A \subseteq_d R_R$.

(14.3) Notation. For a given *semiprime* ring R, we write $\mathcal{F} = \mathcal{F}(R)$ for the set of ideals $A \subseteq R$ satisfying the (equivalent) conditions in (14.2).

The family $\mathcal{F}$ will play an important role in the following, so the notation $\mathcal{F} = \mathcal{F}(R)$ will be fixed throughout §14. The fact that, for any essential ideal A, $b \cdot A = 0 \Longrightarrow b = 0$ suggests that, for a semiprime ring R, the module R_R "comes close" to being nonsingular. (The latter requires that $b \cdot B = 0$ for an essential *right ideal* B implies that $b = 0$.) However, in general, a semiprime (or even prime) ring need not be right nonsingular: see, for instance, (11.21)(4).

(14.4) Proposition. *If $A, B \in \mathcal{F}$ (in the semiprime ring R), then AB, $A \cap B$ and A^n ($n \geq 0$) all belong to $\mathcal{F}$.*

Proof. It suffices to prove that $AB \in \mathcal{F}$. Suppose $x \cdot AB = 0$, where $x \in R$. Then $xA = 0$ (since $\text{ann}(B) = 0$), and hence $x = 0$ (since $\text{ann}(A) = 0$). This shows that $\text{ann}(AB) = 0$, and hence $AB \in \mathcal{F}$. □

A typical way of exploiting the properties of the ideals in $\mathcal{F}$ is as follows.

(14.5) Proposition. *Suppose $A \in \mathcal{F}$ in the semiprime ring R and let $C \subseteq R$ be a right ideal containing A. For $f, g \in \text{Hom}_R(C, R_R)$, $f|_A = g|_A \Longrightarrow f = g$.*

Proof. We may assume that $g \equiv 0$, and start with $f|A = 0$. Consider any element $c \in C$. From $cA \subseteq A$, we have $0 = f(cA) = f(c)A$, and hence $f(c) = 0$ (since $\text{ann}(A) = 0$). This shows that $f \equiv 0$ on C. □

We close this subsection with the following important observation which indicates why the case of prime rings is particularly easy and pleasant to work with.

(14.6) Remark. Suppose R is a prime ring. Then any nonzero ideal $A \subseteq R$ has $\text{ann}(A) = 0$ (by one of the definitions of primeness). On the other hand, the zero ideal has annihilator $R \neq 0$. Therefore, $\mathcal{F} = \mathcal{F}(R)$ *is simply the set of all nonzero ideals in R.*

§14B. The Rings $Q^r(R)$ and $Q^s(R)$

In this subsection, we shall introduce the definitions of Martindale's right rings of quotients and symmetric rings of quotients. *Throughout the exposition, unless otherwise specified, R denotes a semiprime ring.* Martindale's original theory was introduced for prime rings, and was designed for applications to rings satisfying a polynomial identity. As we have observed earlier in §14A, the generalization to semiprime rings was due to Amitsur. These rings of quotients associated with

semiprime rings have since proved to be useful not only for the theory of rings with polynomial identities, but also for the Galois theory of noncommutative rings, and for the study of prime ideals under ring extensions in general. Due to limitation of space, however, we shall not delve into these applications, but shall content ourselves mainly with an exposition of the basic properties of Martindale's rings of quotients $Q^r(R)$ and $Q^s(R)$, with examples along the way.

The quickest way to get our hands on $Q^r(R)$ and $Q^s(R)$ is to define them as subrings of the maximal right ring of quotients $Q^r_{\max}(R)$ (of a semiprime ring R). Recall that $\mathcal{F} = \mathcal{F}(R)$ denotes the set of ideals $A \subseteq R$ which are right (or 2-sided) essential in R.

(14.7) Definition/Proposition. *For R semiprime, we define*

$$Q^r(R) = \{q \in Q^r_{\max}(R) : \ qA \subseteq R \textit{ for some } A \in \mathcal{F}\},$$
$$Q^s(R) = \{q \in Q^r_{\max}(R) : \ qA, \ Bq \subseteq R \textit{ for some } A, \ B \in \mathcal{F}\}.$$

These are subrings of $Q^r_{\max}(R)$, with $R \subseteq Q^s(R) \subseteq Q^r(R)$. The ring $Q^r(R)$ (resp. $Q^s(R)$) is called Martindale's right (resp. symmetric) ring of quotients of R.

Proof. Both $Q^r(R)$, $Q^s(R)$ are closed under addition, since $\mathcal{F}$ is closed under intersection. For multiplication, let $q, q' \in Q^r(R)$; say, $qA \subseteq R$, $q'A' \subseteq R$, where $A, A' \subseteq \mathcal{F}$. Then

$$qq'A'A \subseteq qRA = qA \subseteq R.$$

By (14.4), $A'A \in \mathcal{F}$, so we have $qq' \in Q^r(R)$. Similarly, we can show that $q, q' \in Q^s(R) \Longrightarrow qq' \in Q^s(R)$. Thus, $Q^r(R)$, $Q^s(R)$ are subrings of $Q^r_{\max}(R)$, and it is clear that $R \subseteq Q^s(R) \subseteq Q^r(R)$. □

Note that, since $\mathcal{F}$ is closed under finite intersections, we could have taken A, B to be the same ideal (in $\mathcal{F}$) in the definition of $Q^s(R)$.

Recall that, without any assumptions on R, we have always

(14.8) $$q^{-1}R = \{x \in R : \ qx \in R\} \subseteq_d R_R, \quad \text{and} \quad q(q^{-1}R) \subseteq R.$$

Here, $q^{-1}R$ is in general only a right ideal. The point in the definition of $Q^r(R)$ in (14.7) is that we require the A there to be an *ideal* (essential in R).

Just as we can think of the elements of $Q^r_{\max}(R)$ as equivalence classes of the right R-linear functionals on dense right ideals of R (see (13.21)), we can interpret the elements of $Q^r(R)$ as classes of right R-linear functionals on the ideals in $\mathcal{F}$. The following result is a specialization of (13.21).

(14.9) Proposition. *$Q^r(R)$ consists of equivalence classes $[A, f]$ where $A \in \mathcal{F}$ and $f \in \operatorname{Hom}_R(A_R, R_R)$. Here, two pairs (A, f), (A', f') are defined to be equivalent if $f = f'$ on $A \cap A'$. If we think of the elements of $Q^r(R)$ as equivalence classes of the form $[A, f]$, then addition and multiplication in $Q^r(R)$*

are described by

$$[A, f] + [B, g] = [A \cap B, f + g],$$
$$[A, f] \cdot [B, g] = [BA, fg].$$

Proof. (Sketch) The relation we defined on the pairs (A, f) is clearly reflexive and symmetric. To show that it is also transitive, assume that $(A, f) \sim (A', f')$ and $(A', f') \sim (A'', f'')$, where $A, A', A'' \in \mathcal{F}$, and f, f', f'' are the respective functionals. Certainly, f, f', f'' are all equal on $A \cap A' \cap A''$. Since $A \cap A' \cap A'' \in \mathcal{F}$ by (14.4), it follows from (14.5) that f and f'' are equal on $A \cap A''$. Therefore, we have $(A, f) \sim (A'', f'')$. Now it is safe to talk about the equivalence classes $[A, f]$.

For each $q \in Q^r(R)$, we have $qA \subseteq R$ for some $A \in \mathcal{F}$. To such q, we can associate the class $[A, f]$ where $f : A \to R$ is given by left multiplication by q. Conversely, given $f \in \mathrm{Hom}_R(A_R, R_R)$ where $A \in \mathcal{F}$, we know from (13.20) that f is given by left multiplication by a unique $q \in Q^r_{\max}(R)$. (Note that (14.1) guarantees that $A \subseteq_d R_R$!) Since $qA \subseteq R$, we have by definition $q \in Q^r(R)$. It is easy to check that the above procedure establishes a one-one correspondence[85] between $Q^r(R)$ and the set of equivalence classes of the form $[A, f]$.

The interpretation of addition on equivalence classes is immediate, upon noting that $A \cap B \in \mathcal{F}$ by (14.4). For multiplication, note that by (14.4), $BA \in \mathcal{F}$. Since $g(BA) \subseteq g(B)A \subseteq A$, fg is defined on BA. If $[A, f]$ and $[B, g]$ correspond respectively to $q, q' \in Q^r(R)$, the class $[BA, fg]$ clearly corresponds to qq'. (**Note**. The choice of BA as the domain for fg is somewhat arbitrary. We could have chosen, for instance, the more symmetrical expression $(A \cap B)^2$.) □

It is possible to give a more axiomatic definition for $Q^r(R)$. In this definition, we obtain $Q^r(R)$ as an extension ring of R with certain quotient ring properties. This approach will be presented later in §14D. Similarly, the symmetric Martindale ring of quotients $Q^s(R)$ can also be described by axiomatic properties. Aside from their theoretical interest, these alternative descriptions do turn out to be of practical use. Indeed, if we work with the maximal *left* ring of quotients $Q^{\ell}_{\max}(R)$, then we can define the Martindale left ring of quotients $Q^{\ell}(R)$, and define inside it *another* symmetric Martindale ring of quotients. The axiomatic characterizations developed in §14D below will show that the two symmetric Martindale rings of quotients are indeed the same, thereby justifying the terminology. For now, $Q^s(R)$ continues to denote the symmetric Martindale ring of quotients defined within $Q^r_{\max}(R)$.

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[85] The injectivity of the map $q \mapsto [A, f]$ corresponds to the fact that, if $qA = 0$ where $A \in \mathcal{F}$, then $q = 0$. This is a consequence of $A \subseteq_d R_R$, and should be viewed in the light of (13.20). This fact will be used over and over again in the following.

(14.10) Examples.

(a) Let the semiprime ring R be *right duo*, i.e. such that right ideals in R are ideals. Then $\mathcal{F} = \mathcal{F}(R)$ is the family of all dense right ideals. In this case, $Q^r(R)$ boils down to the entire maximal right ring of quotients $Q^r_{\max}(R)$, by (13.21). This is the case, for instance, if R is any commutative ring. Here, we also have $Q^s(R) = Q^r_{\max}(R)$.

(b) Let R be any simple ring. Then clearly $\mathcal{F}(R)$ consists of a single ideal, namely, R. In this case, the definition of $Q^r(R)$ shows that $Q^r(R) = R$, and therefore $Q^s(R) = R$ too.

We shall give below more examples for the Martindale rings of quotients. For later reference, let us first note the following special property of $Q^r(R)$.

(14.11) Lemma. *Let $q \in Q^r(R)$ where R is semiprime, and let $A \in \mathcal{F}(R)$. If either $qA = 0$ or $Aq = 0$, then $q = 0$.*

Proof. If $qA = 0$, then we know $q = 0$ (as long as $q \in Q^r_{\max}(R)$). Now assume $Aq = 0$ instead. Since $q \in Q^s(R)$, there exists $B \in \mathcal{F}$ such that $qB \subseteq R$. From $Aq = 0$, we get $A(qB) = 0$, and so $qB = 0$ since $\operatorname{ann}(A) = 0$ in R. Therefore, we are back to the first case, and can conclude that $q = 0$. □

(14.12) Proposition. *Suppose R is a domain (resp. prime, semiprime). Then so is $Q^s(R)$.*

Proof. First, suppose R is a domain. Let $q,\ q' \in Q^s(R)$ be such that $qq' = 0$. There exist $A,\ A' \in \mathcal{F}$ such that $Aq \subseteq R$ and $q'A' \subseteq R$. Then

$$0 = A(qq')A' = (Aq)(q'A')$$

implies that $Aq = 0$ or $q'A' = 0$. By (14.11), it follows that $q = 0$ or $q' = 0$. Therefore, $Q^s(R)$ is also a domain. Next assume that R is prime. Let $q,\ q' \in Q^s(R)$ be such that $qQ^s(R)q' = 0$. Choose $A,\ A' \in \mathcal{F}$ as before and note that

$$qQ^s(R)q' = 0 \implies qRq' = 0 \implies (Aq)R(q'A') = 0.$$

Since $Aq,\ q'A' \subseteq R$ and R is prime, we have $Aq = 0$ or $q'A' = 0$, and so $q = 0$ or $q' = 0$ as before. The case when $Q^s(R)$ is semiprime follows similarly, since

$$qQ^s(R)q = 0 \implies (Aq)R(Aq) \subseteq A(qRq) = 0 \implies Aq = 0,$$

and hence again $q = 0$. □

In the case of "prime" and "semiprime", (14.12) is also true with $Q^s(R)$ replaced by $Q^r(R)$. The proofs are similar, and are left as exercises in this section. However, in the "domain" case, the example below shows that we cannot replace $Q^s(R)$ by $Q^r(R)$ in (14.12).

(14.13) Example. Let R be the free algebra $k\langle x_1, \ldots, x_n\rangle$ over a field k, where $n \geq 2$. Then $Q^s(R)$ is a domain by (14.12), but we shall show that the x_i 's *are* 0*-divisors in* $Q^r(R)$, *so* $Q^r(R)$ *is not a domain.* (In particular, we must have $Q^s(R) \subsetneq Q^r(R)$.) Consider the ideal $A \subseteq R$ consisting of (noncommuting) polynomials in the x_i 's with zero constant term. Since R is a prime ring, $A \in \mathcal{F}$. As a right R-module, $A = x_1 R \oplus \cdots \oplus x_n R$ is free with basis $\{x_1, \ldots, x_n\}$. Let $f : A \to R$ be the right R-module homomorphism defined by $f(x_1) = 1$ and $f(x_2) = \cdots = f(x_n) = 0$. Let $q \in Q^r(R)$ be the unique element such that f is given by left multiplication by q. Then $qx_1 = 1$ and $qx_2 = \cdots = qx_n = 0$, so $x_2, \ldots, x_n$ are right 0-divisors in $Q^r(R)$. Similarly, x_1 is a right 0-divisor in $Q^r(R)$ too (and $Q^r(R)$ is not Dedekind-finite). According to a result of V. K. Kharchenko [79], $Q^s(R)$ is in fact equal to R for the free algebra $R = k\langle x_1, \ldots, x_n\rangle$. For a self-contained proof of this fact, see D. S. Passman [87].

To close this subsection, we shall prove a useful result which enables us to conclude that a semiprime ring R has the same Martindale rings of quotients as some of its subrings. This result is, for instance, very convenient in generating new examples from old ones. (Cf. Exercise 9 in §13.)

(14.14) Theorem. *Let R be a ring, and $I \subseteq R$ be an ideal with* $\operatorname{ann}_r(I) = \operatorname{ann}_\ell(I) = 0$. *Let T be any subring of R containing I. Then*

(1) *R is prime (resp. semiprime) iff T is.*
(2) *If R is semiprime, then $Q^r(R) = Q^r(T)$ and $Q^s(R) = Q^s(T)$.*

Proof. (1) We shall only work with the "prime" case since the "semiprime" case is similar. Assume R is prime, and consider any nonzero ideal B of T. Then IBI is an ideal of R lying in B, and $IBI \neq 0$ by the assumptions on I. Therefore, if B' is another nonzero ideal of T, then

$$BB' \supseteq (IBI)(IB'I) \neq 0,$$

so T is prime. Conversely, if T is prime, then again, for nonzero ideals A, A' of R, IAI, $IA'I$, are nonzero ideals of T (in fact, of R too), and so

$$AA' \supseteq (IAI)(IA'I) \neq 0.$$

This shows that R is prime.

(2) Here we only give a sketch of the proof. Since R (and hence also T) is semiprime, we can use the notations $\mathcal{F}(R)$ and $\mathcal{F}(T)$. We shall define mutually inverse homomorphisms:

$$\varphi : Q^r(R) \longrightarrow Q^r(T), \qquad \psi : Q^r(T) \longrightarrow Q^r(R).$$

To define φ, take $f : A_R \to R_R$, where $A \in \mathcal{F}(R)$. By (14.4), $IAI \in \mathcal{F}(R)$. Since $IAI \subseteq T$, we have $IAI \in \mathcal{F}(T)$. We then define

$$\varphi[A, f] = [IAI, f'] \in Q^r(T),$$

where f' is the restriction of f to IAI. Similarly, if $g : B_T \to T_T$ where $B \in \mathcal{F}(T)$, then $IBI \in \mathcal{F}(R)$. Indeed, for $x \in R$:

$$x(IBI) = 0 \implies (xI)B = 0 \implies xI = 0 \implies x = 0.$$

Since $IBI \subseteq B$, we can define

$$\psi[B, g] = [IBI, g'] \in Q^r(R),$$

where g' is the restriction of g to IBI. Here, we do have to verify that g' is a right R-homomorphism. Take $i, i' \in I$, $b \in B$, and $x \in R$. We have

$$g'(ibi' \cdot x) = g(ib \cdot (i'x)) = g(ib)i'x = g'(ibi')x,$$

as desired. It is routine to check that φ, ψ are well-defined, mutually inverse ring homomorphisms. Therefore, we can use them to identify $Q^r(R)$ with $Q^r(T)$. If $q \in Q^s(R) \subseteq Q^r(R)$, then $Aq \subseteq R$ for some $A \in \mathcal{F}(R)$. But then $(IAI)q \subseteq IAq \subseteq T$ shows that $q \in Q^s(T)$. Similarly, we can show that $Q^s(T) \subseteq Q^s(R)$, so equality holds. □

Remark. In the context of (14.14), I must (somewhat grudgingly) concede that the formulation of the theorem would have been easier if we had admitted throughout rings without identities. In that case, I itself would be a ring, for which we could form the Martindale rings of quotients. Then (14.14) could have been formulated by saying that $Q^r(I) = Q^r(R)$ and $Q^s(I) = Q^s(R)$, without the use of rings T between R and I.

The following typical application of (14.14) is worth mentioning. Here we go back to rings with identities!

Example. Let $R = k\langle x, y\rangle$ where k is a field. We can take the ideal $I = RxR \subseteq R$, and form the subdomain $T = k+I \subsetneq R$. Then $Q^r(T) = Q^r(R)$, and, if we assume Kharchenko's result (mentioned in (14.13)) that $Q^s(R) = R$, then (14.14) shows that $Q^s(T) = R = k\langle x, y\rangle$.

§14C. The Extended Centroid

We begin our considerations here with a result describing the elements in $Z(Q)$, the center of the maximal right ring of quotients $Q := Q^r_{\max}(R)$. For this result (which was actually given earlier as Exercise 5 in §13), no assumptions are needed on the base ring R.

(14.15) Lemma. *Let $q_0 \in Q$ where R is any ring. Then $q_0 \in Z(Q)$ iff q_0 commutes with every element of R.*

Proof. ("If" part) Assume that q_0 commutes with every element of R, but $q_0q \neq qq_0$ for some $q \in Q$. Then there exists $a \in R$ such that $(q_0q - qq_0)a \neq 0$ and

$qa \in R$. But then

$$q_0(qa) = (qa)q_0 = q(aq_0) = qq_0a,$$

a contradiction. □

Coming back to semiprime rings, we further observe the following.

(14.16) Lemma. *Let $Q = Q^r_{\max}(R)$ where R is semiprime. If $q_0 \in Z(Q)$, then $q_0 \in Q^s(R)$ (the symmetric Martindale ring of quotients).*

Proof. We know by Exer. (8.1) that $B := q_0^{-1}R = \{x \in R : q_0x \in R\}$ is a dense right ideal in R. Now B is also a left ideal. Indeed, for any $x \in B$ and $t \in R$, we have

$$q_0(tx) = (q_0t)x = tq_0x \in tR \subseteq R,$$

so $tx \in B$. Therefore, we have $B \in \mathcal{F} = \mathcal{F}(R)$. Moreover, $q_0 \in Z(Q)$ implies that

$$Bq_0 = q_0B = q_0(q_0^{-1}R) \subseteq R,$$

so $q_0 \in Q^s(R)$. □

The lemma leads to the following important fact.

(14.17) Proposition. *For any semiprime ring R, we have*

$$Z(Q^s(R)) = Z(Q^r(R)) = Z(Q), \quad \textit{and} \quad Z(Q) \cap R = Z(R).$$

Proof. If $q_0 \in Z(Q)$, then $q_0 \in Q^s(R)$ by (14.16), so q_0 is central in both $Q^s(R)$ and $Q^r(R)$. Conversely, a central element in $Q^s(R)$ or in $Q^r(R)$ commutes elementwise with R, and so is also central in Q by (14.15). This shows the first part of the Proposition, and the second part follows similarly. □

(14.18) Definition. The (commutative) ring $Z(Q^s(R)) = Z(Q^r(R)) = Z(Q)$ is called the *extended centroid* of R.

It is of interest to give an alternative description of the extended centroid $Z(Q)$ in terms of the equivalence classes $[A, f]$ $(A \in \mathcal{F}$, and $f \in \mathrm{Hom}_R(A_R, R_R))$.

(14.19) Proposition. *Upon interpreting the elements of $Q^r(R)$ as the equivalence classes $[A, f]$ as in (14.9), the elements of the extended centroid $Z(Q)$ are precisely those of the form $[A, f]$ where $A \in \mathcal{F}$ and $f : A \to R$ is an (R, R)-bimodule homomorphism.*

Proof. First consider a class $[A, f]$ where A and f are as described above. Let q_0 be the corresponding element in $Q^r(R)$, so f is left multiplication by q_0. Since

f is a left (as well as right) R-module homomorphism, $(q_0t - tq_0)A = 0$ for any $t \in R$. This implies that q_0 commutes with every element of R, so by (14.15) $q_0 \in Z(Q)$. Conversely, suppose a class $[A, f]$ $(A \in \mathcal{F},\ f \in \mathrm{Hom}_R(A_R, R_R))$ corresponds to an element $q_0 \in Z(Q^r(R)) = Z(Q)$. For any $a \in A$ and $t \in R$, we have $f(ta) = q_0ta = tq_0a = tf(a)$, so f is a left (as well as right) R-module homomorphism, as desired. □

Next, we state and prove a very pleasant property of the extended centroid, due to S. Amitsur. (We continue to assume that R is semiprime.)

(14.20) Proposition. *$Z(Q)$ is a von Neumann regular ring.*

Proof. Recall that if M is a semisimple module, End(M) is always a von Neumann regular ring: see *FC*–(4.27). The proof of the present proposition uses the same idea as in that proof. Let $[A, f] \in Z(Q)$, so $A \in \mathcal{F}$, and $f : A \to R$ is an (R, R)-bimodule homomorphism. Then $B := \ker(f) \subseteq A$ is an ideal of R. Let C be a 2-sided complement to B in A (i.e., C is an ideal of R contained in A maximal with respect to $C \cap B = 0$). Then $B \oplus C \subseteq_e {}_RA_R$, so upon replacing A by $B \oplus C$ (which clearly belongs to $\mathcal{F}$), we may assume that $B \oplus C = A$. Consider the ideal $C' := f(A) = f(C) \subseteq R$, which is isomorphic to C as an (R, R)-bimodule. Let $B' := \mathrm{ann}(C')$, for which we have $B' \oplus C' \subseteq_e {}_RR_R$ by (11.38). Therefore, $A' := B' \oplus C' \in \mathcal{F}$. Now define $g : A' \to C \subseteq R$ by

$$g(b' + f(c)) = c \qquad (b' \in B',\ \ c \in C).$$

We check easily that g is an (R, R)-bimodule homomorphism. Note that g is defined on im(f) and f is defined on im(g). For a general element $b + c \in A$ $(b \in B$ and $c \in C)$, we have

$$(fgf)(b + c) = fg(f(c)) = f(g(f(c))) = f(c) = f(b + c),$$

so $fgf = f$ as bimodule homomorphisms from A to R. Here $[A', g] \in Z(Q)$ since $A' \in \mathcal{F}$ and g is a bimodule homomorphism. Therefore, we have checked that every element $f \in Z(Q)$ has a "pseudo-inverse" $g \in Z(Q)$. □

(14.21) Remark. If R is not assumed to be semiprime, then the conclusion of (14.20) may not hold. For instance, if R is a nonreduced commutative ring, then so is $Q = Q^r_{\max}(R)$ and hence $Z(Q) = Q$ cannot be von Neumann regular. On the other hand, in case R is a semiprime ring which happens to be also right nonsingular, then Q is von Neumann regular by (13.36), and therefore $Z(Q)$ is also von Neumann regular by *FC*–Exer. 21.7. This gives an alternative proof of (14.20) in a special case.

(14.22) Corollary. *If R is a prime ring, then the extended centroid $Z(Q)$ is a field.*

Proof. First, by (14.12), the fact that R is prime implies that $Q^s(R)$ is prime. Now, clearly, the center of a prime ring is a domain. Therefore, $Z(Q) = Z(Q^s(R))$ is a domain. Since $Z(Q)$ is von Neumann regular by (14.20), it follows readily that $Z(Q)$ is a field. In fact, a much more direct argument is possible. Let $0 \neq c \in Z(Q)$. As in (14.19), we can think of c as the class of an (R, R)-bimodule homomorphism $f : A \to R$ given by left multiplication by c, where $A \in \mathcal{F}$. Now f must be injective. For, if $0 = f(a) = ca$, then

$$(14.23) \qquad c(AaA) = AcaA = 0 \implies AaA = 0 \implies a = 0.$$

If $g : cA \to A \subseteq R$ is the inverse of $f : A \to cA$, then $[cA, g]$ is the inverse of $[A, f] = c$ in $Z(Q)$. □

As a follow-up on this corollary, consider $Z(R)$, the center of the prime ring R. This is a domain, contained in the field $Z(Q)$. In general, $Z(Q)$ contains, but may not be equal to, the field of quotients of $Z(R)$. However, since the elements of $S := Z(R)\backslash\{0\}$ are all invertible in $Z(Q) \subseteq Q^s(R)$, the Martindale symmetric ring of quotients $Q^s(R)$ contains the central localization $S^{-1}R$ of R.

We close this subsection with a remark on terminology. Why is $Z(Q)$ called the *extended centroid* of the semiprime ring R? In the study of rings possibly without identities, there is a useful notion of the "centroid": for a ring R which may not have an identity, the centroid C of R is defined to be the ring of all endomorphisms of R as an (R, R)-bimodule. Under reasonable assumptions on R (for instance, $R \cdot R = R$), C can be shown to be a commutative ring (and it coincides with the center of R if R already has an identity). Via (14.19), C may be viewed as a subring of $Z(Q)$. Thus, $Z(Q)$ is a somewhat bigger object than the centroid, and it makes sense to call it the "extended centroid" of R.

§14D. Characterizations of $Q^r(R)$ and $Q^s(R)$

We return now to give axiomatic characterizations of the Martindale rings of quotients. As we shall see later in this section, these characterizations are quite useful for the purposes of computations. Our exposition here follows closely that of Passman [87].

We start with the following characterization of Martindale's right ring of quotients, $Q^r(R)$.

(14.24) Proposition. *Let R be a semiprime ring, and let S be a ring containing R as a subring. Let $\mathcal{F} = \mathcal{F}(R)$ be the set of ideals in R with zero annihilators. Then S is R-isomorphic to $Q^r(R)$ iff S has the following properties:*

1. *For any $q \in S$, there exists $A \in \mathcal{F}$ such that $qA \subseteq R$.*
2. *For $q \in S$ and $A \in \mathcal{F}$, $qA = 0 \implies q = 0$.*
3. *For any $A \in \mathcal{F}$ and $f \in \mathrm{Hom}_R(A_R, R_R)$, there exists $q \in S$ such that $f(a) = qa$ for all $a \in A$.*

Proof. Since (1), (2) and (3) are known properties of $Q^r(R)$, we need only prove the "if" part. Assume that S has the above properties. For any $q \in S$, take $A \in \mathcal{F}$ such that $qA \subseteq R$. Let $f \in \mathrm{Hom}_R(A_R, R_R)$ be defined by left multiplication by q. Then $[A, f] \in Q^r(R)$ depends only on q. (For if $B \in \mathcal{F}$ is such that $qB \subseteq R$ and g is defined by left multiplication by q on B, then f, g clearly agree on $A \cap B \in \mathcal{F}$.) Defining $\varphi : S \to Q^r(R)$ by $\varphi(q) = [A, f]$, we check easily that φ is a ring homomorphism over R. By (2), φ is injective and by (3), φ is surjective, so we have $S \cong Q^r(R)$ over R, as desired. □

Needless to say, there is a left analogue of (14.24) giving the characterization of $Q^\ell(R)$. Next, we come to the characterization of $Q^s(R)$, the symmetric Martindale ring of quotients defined within $Q^r_{\max}(R)$.

(14.25) Proposition. *Let $R \subseteq S$ and $\mathcal{F}$ be as in* (14.24). *Then S is R-isomorphic to $Q^s(R)$ iff S has the following properties:*

(1) *For any $q \in S$, there exist $A, B \in \mathcal{F}$ such that $qA, Bq \subseteq R$.*
(2) *For $q \in S$ and $A, B \in \mathcal{F}$, $qA = 0$ or $Bq = 0 \implies q = 0$.*
(3) *Let $A, B \in \mathcal{F}$ and $f \in \mathrm{Hom}_R(A_R, R_R)$, $g \in \mathrm{Hom}_R({}_RB, {}_RR)$ be such that $b(fa) = (bg)a$ for all $a \in A$ and $b \in B$. Then there exists $q \in S$ such that $fa = qa$ and $bg = bq$ for all $a \in A$ and $b \in B$.*

(Note that since g is a homomorphism between left R-modules, we write it on the right of the arguments. Also, to simplify notations, we have suppressed the usual parentheses around the arguments.)

Proof. First assume $S = Q^s(R)$. Then (1) follows by definition, and (2) follows from (14.11). To verify (3), let f, g be as given. Then there exists $q \in Q^r(R)$ such that $fa = qa$ for all $a \in A$. Plugging this equation into $b(fa) = (bg)a$, we get $(bq - bg)A = 0$, so by (2), $bg = bq$ for all $b \in B$, as desired.

Conversely, suppose a ring $S \supseteq R$ satisfies (1), (2), and (3). For any $q \in S$, fix $A, B \in \mathcal{F}$ such that $qA, Bq \subseteq R$. Let $f \in \mathrm{Hom}_R(A_R, R_R)$ (resp. $g \in \mathrm{Hom}_R({}_RB, {}_RR)$) be defined by left (resp. right) multiplication by q. Then

$$(14.26) \qquad b(fa) = b(qa) = (bq)a = (bg)a \qquad (\forall a \in A,\ b \in B).$$

By what we have shown in the first part, there exists (a unique) $q' \in Q^s(R)$ such that $fa = q'a$ and $bg = bq'$ for all $a \in A$ and $b \in B$. It is easy to check that $\varphi(q) = q'$ gives a well-defined ring homomorphism φ from S to $Q^s(R)$. As in the proof of (14.24), (2) implies that φ is injective, and (3) implies that φ is surjective, so we have $S \cong Q^s(R)$ over R, as desired. □

(14.27) Remarks.

(A) Since the conditions (1), (2) and (3) above are left-right symmetric, an immediate consequence of (14.25) is that (as we have pointed out earlier) the symmetric

Martindale ring of quotients defined within $Q^r_{\max}(R)$ is R-isomorphic to that defined within $Q^\ell_{\max}(R)$.

(B) By the argument given in the proof of (14.11), we see that, in the presence of (1) in (14.25), the condition (2) there could have been weakened to $qA = 0 \Longrightarrow q = 0$, or to $Bq = 0 \Longrightarrow q = 0$. However, this would have destroyed the symmetry of the condition. Therefore, it seemed best to formulate the condition (2) as we did in (14.25) and just let the redundancy there stand.

An application of the characterization theorems in this subsection will be given later in §14F.

§14E. X-Inner Automorphisms

In this subsection, we introduce briefly the notion of an X-inner automorphism. This notion plays an interesting role in the Galois theory of prime and semiprime rings, and has also applications to other areas in ring theory. However, we introduce the notion of X-inner automorphisms mainly as an illustration of the utility of Martindale's rings of quotients. Our exposition here is therefore limited to a rendering of ideas, definitions, and examples.

We begin with the notion of a normalizing element.

(14.28) Definition. Let $R \subseteq S$ be rings (with the same identity). An element $x \in S$ is said to *normalize* R if $xR = Rx$. In this case, we shall also say that $x \in S$ is an R-*normalizing* element.

Some immediate properties of R-normalizing elements are the following.

(14.29) Lemma. *Let N be the set of R-normalizing elements in S. Then*

(1) *N is closed under multiplication.*
(2) *The set $R \cdot N$ consisting of finite sums $\sum r_i x_i$ ($r_i \in R, x_i \in N$) is a subring of S containing R (called the normal closure of R in S).*
(3) *If $x \in N$ and A is an ideal of R such that $xA \subseteq R$, then xA is also an ideal of R.*
(4) *Suppose $x \in N$ is such that, for any $r \in R$, $xr = 0$ or $rx = 0$ implies that $r = 0$. Then there exists a unique automorphism φ of R such that $xa = \varphi(a)x$ for all $a \in R$.*

Proof. (1) Let $x, y \in N$. Then $xyR = xRy = Rxy$, so $xy \in N$.

(2) For $x, y \in N$ and $r, s \in R$, we have

$$(rx)(sy) = r(s'x)y = (rs')(xy)$$

for some $s' \in R$, so $R \cdot N$ is closed under multiplication. Therefore, $R \cdot N$ is a ring containing R. (It is just the subring of S generated by R and N.)

(3) We have $(xA)R = x(AR) \subseteq xA$ and

$$R(xA) = (Rx)A = xRA \subseteq xA,$$

so xA is an ideal in R.

(4) For $a \in R$, there exists $a' \in R$ such that $xa = a'x$. Such an element $a' \in R$ is unique, by our assumptions on x in (4). We can therefore define $\varphi : R \to R$ by taking $\varphi(a) = a'$. A routine calculation shows that φ is a ring endomorphism of R. If $\varphi(a) = 0$, then $xa = a'x = 0$ implies that $a = 0$. For any given $a' \in R$, there exists $a \in R$ such that $a'x = xa$, so $a' = \varphi(a)$. Therefore, φ is an automorphism of R. The uniqueness of φ is clear (again by our assumptions on x). (Of course, if $x \in \mathrm{U}(S)$, then φ is just the restriction (to R) of the inner automorphism $s \mapsto xsx^{-1}$ of S.) □

Returning now to semiprime rings, we study below the R-normalizing elements in the maximal right ring of quotients. The following result shows immediately the relevance of Martindale's symmetric ring of quotients $Q^s(R)$.

(14.30) Theorem. *Let R be a semiprime ring, and let N be the set of R-normalizing elements in $Q^r_{\max}(R)$. Then $N \subseteq Q^s(R)$.*

Proof. For $x \in N$, let $A := \{a \in R : xa \in R\}$. This is a dense right ideal in R. We claim that A is also a *left* ideal of R. In fact, if $a \in A$ and $t \in R$, then

$$x(ta) \in xRa = R(xa) \subseteq R,$$

so indeed $ta \in A$. Since $A \subseteq_d R_R$, we have $A \in \mathcal{F}(R)$. Now let $B = \{b \in R : bx \in xA\}$. This is a left ideal in R since xA is an ideal by (14.29)(3). We claim that B is also a *right* ideal of R. In fact, if $b \in B$ and $t \in R$, then $(bt)x \in bxR \subseteq xA$, so indeed $bt \in B$. Note that $Bx \subseteq xA$ is an equality (since $xA \subseteq Rx$). Using this, we can check that $B \in \mathcal{F}(R)$. For if $t \in \mathrm{ann}^R(B)$, then $0 = tBx = txA$ shows that $tx = 0$. The latter implies, in particular, that $t \in B$, so now $t \in B \cap \mathrm{ann}^R(B) = 0$. From $Bx = xA \subseteq R$ and $A, B \in \mathcal{F}(R)$, we conclude that $x \in Q^s(R)$. □

(14.31) Corollary. *The normal closure*[86] *$R \cdot N$ of R in $Q^r_{\max}(R)$ is contained in $Q^s(R)$.*

(14.32) Proposition. *In the notation of* (14.30), *let $x \in N$ be such that, for $c \in R$, $xc = 0 \Longrightarrow c = 0$. Then $x \in \mathrm{U}(Q^s(R))$.*

Proof. Let $A \in \mathcal{F}(R)$ be as in the proof of (14.30). By (14.29)(3), $xA \subseteq R$ is an ideal. In fact, $xA \in \mathcal{F}(R)$ also. For, if $t \in R$ is such that $(xA)t = 0$, then $At = 0$

[86]There is a notion of "central closure" also. Let $C = Z(Q^r_{\max}(R))$ be the extended centroid. Then the central closure of R is defined to be $R \cdot C$. Since clearly $C \subseteq N$, we have $R \cdot C \subseteq R \cdot N \subseteq Q^s(R)$.

(by the hypothesis on x) and thus $t = 0$. Now let $f : A_R \to R_R$ be the R-homomorphism given by left multiplication by x. Then f is an injection, and we have an inverse map $g : xA \to A \subseteq R$ which defines an element $[xA, g] \in Q^r(R)$ inverse to $x = [A, f]$. Therefore, $x \in \mathrm{U}(Q^r(R))$. From $xR = Rx$, we have $Rx^{-1} = x^{-1}R$, so x^{-1} is also R-normalizing. By (14.30), $x^{-1} \in Q^s(R)$ also, so we have $x \in \mathrm{U}(Q^s(R))$. □

(14.33) Corollary. *If R is a prime ring, then*

$$0 \neq x \in N \Longrightarrow x \in \mathrm{U}(Q^s(R)).$$

(In particular, $N^* := N\backslash\{0\}$ is a multiplicative group.)

Proof. It suffices to check that, for $c \in R$, $xc = 0 \Longrightarrow c = 0$. For the ideal $A \in \mathcal{F}(R)$ in the above proof, we have $(xA)(cR) \subseteq Rx(cR) = 0$. Since $xA \subseteq R$ is a nonzero right ideal by (14.11) and R is prime, $c = 0$. The parenthetical statement is clear since $x \neq 0$ and $Rx = xR$ now imply $Rx^{-1} = x^{-1}R$. □

Of course, this result implies, once again, that the extended centroid is a field in case R is a prime ring.

The following special cases of (14.32) and (14.33) are already worth noting.

(14.34) Corollary. *Let R be a semiprime (resp. prime) ring. Then any regular (resp. nonzero) element $x \in R$ such that $xR = Rx$ is invertible in $Q^s(R)$.*

We shall now point out the relevance of R-normalizing elements to the study of automorphisms of a semiprime ring R. We shall write $\mathrm{Aut}(R)$ for the group of all automorphisms on R. For any $\varphi \in \mathrm{Aut}(R)$, we define

$$\mathrm{X}(\varphi) = \{x \in Q^r_{\max}(R) : \ xa = \varphi(a)x \ \text{ for all } \ a \in R\}.$$

This is clearly an additive group. In fact it is a module over the extended centroid $Z(Q^r_{\max}(R))$. In the special case when $\varphi = \mathrm{Id}_R$ (the identity automorphism),

$$\mathrm{X}(\mathrm{Id}_R) = \{x \in Q^r_{\max}(R) : \ xa = ax \ \forall\, a \in R\}$$

is exactly the extended centroid, according to (14.15).

(14.35) Lemma. *For any $\varphi \in \mathrm{Aut}(R)$ (R semiprime), we have $\mathrm{X}(\varphi) \subseteq N \subseteq Q^s(R)$.*

Proof. Let $x \in \mathrm{X}(\varphi)$. For any $a \in R$, $xa = \varphi(a)x \in Rx$, and similarly, $ax = x\varphi^{-1}(a) \in xR$. Thus, $x \in N$, and $N \subseteq Q^s(R)$ is just (14.30). □

We can now define the notion of an X-inner automorphism of a semiprime ring. This interesting notion was introduced by the Russian mathematician V. K. Kharchenko, although, as S. Montgomery has pointed out to me, similar ideas had been used earlier by analysts and by researchers working with orders in semisimple

algebras. The "X" in "X-inner" is taken from the first letter of Kharchenko's Russian name.

(14.36) Definition. An automorphism φ of a semiprime ring R is said to be X-*inner* if $\mathrm{X}(\varphi) \neq 0$; that is, if there exists a nonzero x (necessarily in $N \subseteq Q^s(R)$) such that $xa = \varphi(a)x$ for any $a \in R$. The set of all X-inner automorphisms on R is denoted by X-Inn(R).

For a semiprime ring R, X-Inn(R) *may not be a subgroup of* Aut(R). The problem is the following. If $\varphi, \varphi' \in$ X-Inn(R), say, $0 \neq x \in \mathrm{X}(\varphi)$ and $0 \neq x' \in \mathrm{X}(\varphi')$, then for any $a \in R$,

$$xx'a = x\varphi'(a)x' = \varphi\varphi'(a)xx', \tag{14.37}$$

so $xx' \in \mathrm{X}(\varphi\varphi')$. Unfortunately, xx' may be zero, so we cannot conclude that $\varphi\varphi' \in$ X-Inn(R). The following offers an explicit example (even over a commutative ring) of a pair of X-inner automorphisms whose product is *not* X-inner.

(14.38) Example. Let R be a *commutative* semiprime (i.e., reduced) ring, and let $Q = Q_{\max}(R)$. Then, for any $\varphi \in \mathrm{Aut}(R)$,

$$\mathrm{X}(\varphi) = \mathrm{ann}^Q\{\varphi(a) - a : \ a \in R\},$$

so φ *is* X-*inner iff this annihilator is nonzero.* This makes it pretty "easy" to decide if $\varphi \in$ X-Inn(R). For instance, let $R = k \times k \times k$, where k is any field. Then $R = Q_{\max}(R)$. We define $\varphi, \varphi' \in \mathrm{Aut}(R)$ as follows, where $a = (a_1, a_2, a_3) \in R$:

$$\varphi(a) = (a_1, a_3, a_2), \quad \text{and} \quad \varphi'(a) = (a_3, a_2, a_1).$$

An easy computation shows that

$$\mathrm{X}(\varphi) = k \times 0 \times 0 \quad \text{and} \quad \mathrm{X}(\varphi') = 0 \times k \times 0,$$

so both φ and φ' are X-inner. However, since

$$\varphi\varphi'(a) = \varphi(a_3, a_2, a_1) = (a_3, a_1, a_2),$$

we have $\mathrm{X}(\varphi\varphi') = 0$. Therefore, the composition $\varphi\varphi'$ is *not* X-inner!

Fortunately, the failure of multiplicative closure of X-Inn(R) cannot happen over a prime ring. This is a consequence of (14.33), which we shall spell out explicitly. The notation $N^* = N\backslash\{0\}$ in (14.33) will remain in force. Recalling that the extended centroid $C := Z(Q^s(R))$ is a field in case R is a prime ring, let us also write $C^* = C\backslash\{0\}$ for the multiplicative group of C.

(14.39) Proposition. *Let R be a prime ring. Then* X-Inn(R) *is a group, isomorphic to N^*/C^*. Also,* X-Inn(R) *contains* Inn(R) *(the group of inner automorphisms of R) as a normal subgroup.*

Proof. By (14.33) for any $\varphi \in$ X-Inn(R), any nonzero $x \in X(\varphi)$ is a unit in $Q^s(R)$; so φ *is just the restriction to R of the inner automorphism* $t \mapsto xtx^{-1}$ of $Q^s(R)$. If φ, $\varphi' \in$ X-Inn(R) and $0 \neq x \in X(\varphi)$, $0 \neq x' \in X(\varphi')$, then x, x' and hence xx' are units of $Q^s(R)$; hence (14.37) implies that $\varphi\varphi' \in$ X-Inn(R). Also, $x^{-1}a = \varphi^{-1}(a)x^{-1}$ $(\forall a \in R)$ shows that $x^{-1} \in X(\varphi^{-1})$, so $\varphi^{-1} \in$ X-Inn(R). This verifies that X-Inn(R) is a group, which clearly contains Inn(R). There is a surjective group homomorphism $\pi : N^* \to$ X-Inn(R) given by $\pi(x)(a) = xax^{-1}$ (for any $a \in R$). If $\pi(x) = \mathrm{Id}_R$, then $xa = ax$ for all $a \in R$, and hence $x \in C^*$ by (14.15). This shows that X-Inn$(R) \cong N^*/C^*$.

Quite generally, Inn(R) is a normal subgroup of Aut(R). Therefore Inn(R) is also normal in X-Inn(R). (If $\psi \in$ Inn(R) is induced by $u \in U(R)$ and $\varphi \in$ X-Inn(R) is induced by $x \in U(Q^s(R))$, then $\varphi\psi\varphi^{-1}$ is induced by $\varphi(u) = xux^{-1} \in U(R)$.) □

We mention in passing the following nice result of Kharchenko.

(14.40) Theorem. *For any* $\varphi \in$ X-Inn(R) *over a semiprime ring* R, $X(\varphi)$ *is a cyclic module over the extended centroid* C.

We shall prove this only in the case when R is a prime ring. Here C is a field, so the result means that $X(\varphi)$ is a 1-dimensional C-vector space. This is easy: if $0 \neq x \in X(\varphi)$, then for any $y \in X(\varphi)$,

$$x^{-1}ya = x^{-1}\varphi(a)y = ax^{-1}y \qquad (\forall a \in R).$$

This shows that $x^{-1}y \in C$ by (14.15), and hence $y \in xC$. □

For a proof of (14.40) in the semiprime case, see Cohen-Montgomery [79].

If R happens to be a simple ring, then $Q^s(R) = R$ by (14.10)(b). Therefore, X-Inn(R) = Inn(R). Some examples of X-inner automorphisms that are not inner over noncommutative domains are given in the following. In each of these examples, we can actually compute the quotient group X-Inn(R)/Inn(R).

(14.41) Example. Let k be a field, and σ be an automorphism on k. Let $R = k[x;\sigma]$ be the skew polynomial domain over k, with elements of the form $\sum a_i x^i$ $(a_i \in k)$ and with multiplication defined via the twist equation $xa = \sigma(a)x$ for $a \in k$. Note that we can extend σ to an automorphism φ of R by taking $\varphi(x) = x$; in other words, we define

$$\varphi\left(\sum a_i x^i\right) = \sum \sigma(a_i)x^i.$$

To check that φ is an automorphism of R depends mainly on checking that φ "respects" the twist equation $xa = \sigma(a)x$. This is indeed the case since $\varphi(\sigma(a))\varphi(x) = \sigma^2(a)x$ and $\varphi(x)\varphi(a) = x\sigma(a) = \sigma^2(a)x$ also. Since

$$x\left(\sum a_i x^i\right) = \sum \sigma(a_i)x^{i+1} = \varphi\left(\sum a_i x^i\right)x,$$

we have $x \in \mathrm{X}(\varphi)\backslash\{0\}$. Therefore, $\varphi \in$ X-Inn(R). On the other hand, $\mathrm{U}(R) = k^*$. Using this, it is easy to show that $\varphi \notin \mathrm{Inn}(R)$ provided $\sigma \neq \mathrm{id}_k$.

If we assume that $\sigma \in \mathrm{Aut}(k)$ has infinite order, it is also possible to compute explicitly the various rings of quotients of R, as well as the quotient group X-Inn$(R)/\mathrm{Inn}(R)$. First, by the discussion at the beginning of §10C, R is an Ore domain, with a division ring of fractions Q. Then $Q = Q^r_{\max}(R)$ by (13.15). To compute $Q^r(R)$ and $Q^s(R)$, we use the fact that $\mathcal{F}(R) = \{x^m R : m \geq 0\}$ (see *FC*–(11.12); this is where we need the fact that σ has infinite order.) From this, we have

$$\begin{aligned} Q^r(R) &= \{q \in Q : qA \subseteq R \text{ for some } A \in \mathcal{F}(R)\} \\ &= \{q \in Q : qx^m \in R \text{ for some } m \geq 0\} \\ &= \bigcup_{m \geq 0} Rx^{-m}. \end{aligned}$$

This is precisely the (twisted) Laurent polynomial ring $k[x, x^{-1}; \sigma]$. Since we already know that $x^{-1} \in Q^s(R)$ (by (14.33)), it also follows that $Q^s(R) = k[x, x^{-1}; \sigma]$. An easy computation shows that $\mathrm{U}(Q^s(R)) = k^* \cdot \langle x \rangle$: this is a semidirect product with normal subgroup k^* and with x acting on k^* by σ. Here, x induces the X-inner automorphism φ on R, while elements of k^* induce inner automorphisms. Therefore, we have

$$\text{X-Inn}(R)/\mathrm{Inn}(R) \cong \langle \varphi \rangle \cong \langle \sigma \rangle \cong \mathbb{Z},$$

where the isomorphism $\langle \varphi \rangle \cong \langle \sigma \rangle$ is due to the fact that $\varphi^n \mid k = \sigma^n$ for all n.

The X-inner automorphisms in the above example are all induced by elements in R or by their inverses. An easy modification of the construction, however, will lead to examples of X-inner automorphisms that are *not* induced by elements in the base ring or by their inverses.

(14.42) Example. Keep the notations above, and let m be a fixed positive integer. Let $R_m = k + x^m R$, which is the subdomain of R consisting of skew polynomials of the form

$$a_0 + a_m x^m + a_{m+1} x^{m+1} + \cdots + a_n x^n \qquad (n \geq m, \ \ a_i \in k).$$

Since R and R_m have a common nonzero ideal $x^m R$, we have by (14.14)

$$Q^r(R_m) = Q^s(R_m) = Q^s(R) = k[x, x^{-1}; \sigma].$$

Exactly as before, X-Inn$(R_m)/\mathrm{Inn}(R_m) \cong \langle \varphi_m \rangle \cong \langle \sigma \rangle \cong \mathbb{Z}$, where $\varphi_m := \varphi | R_m$. Here, the X-inner automorphisms $\varphi_m^{\pm m}$, $\varphi_m^{\pm(m+1)}$, ... are induced by x^m, x^{m+1}, ... $\in R_m$ and their inverses, while $\varphi_m^{\pm 1}$, $\varphi_m^{\pm 2}$, ..., $\varphi_m^{\pm(m-1)}$ are induced by $x^{\pm 1}$, $x^{\pm 2}$, ..., $x^{\pm(m-1)}$ which are *outside of* R_m.

In the two examples above, the various Martindale rings of quotients (and the normal closures of the base ring) are all given by $S = k[x, x^{-1}; \sigma]$. Now, since σ

is assumed to have infinite order, this latter ring is a simple domain by *FC*–(3.19). Therefore, we have

$$Q^r(S) = Q^\ell(S) = Q^s(S) = S$$

by (14.10)(b). However, in general, Q^r, Q^ℓ and Q^s are *not* "closure operations"; in other words, $Q^s(Q^s(R))$ may not be equal to $Q^s(R)$, etc. The formation of the normal closure (of a semiprime ring) is not a closure operation either. All of these contrast sharply with the fact, proved earlier in (13.31)(3), that $R \mapsto Q^r_{\max}(R)$ *is* a closure operation. (There is a "silver lining": see Exercise 17.)

(14.43) Example. Let $R = \mathbb{Z}1 \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$ be the domain of integer quaternions. This is a 2-sided order in the division ring Q of rational quaternions. Therefore,

$$Q^r_{\max}(R) = Q^r_{cl}(R) = Q,$$

and the same holds for the left quotient rings. The extended centroid C is $Z(Q) = \mathbb{Q}$. Each automorphism of R induces an automorphism of $\mathrm{U}(R) = \{\pm 1, \pm i, \pm j, \pm k\}$ (the quaternion group) and the map $\mathrm{Aut}(R) \to \mathrm{Aut}(\mathrm{U}(R))$ is a group isomorphism. In fact, it is well known that both groups are $\cong S_4$ (see, e.g., Zassenhaus' *Theory of Groups*, p. 148). We just give a quick explanation for the fact that $|\mathrm{Aut}(R)| = 24$. Each automorphism φ of R is determined by $\varphi(i)$ and $\varphi(j)$ which must be from $\{\pm i, \pm j, \pm k\}$. There are six choices for $\varphi(i)$, and after $\varphi(i)$ is chosen, there are four choices for $\varphi(j)$. (If $\varphi(i)$ is chosen from $\{\pm j\}$, then $\varphi(j)$ must be chosen from $\{\pm i, \pm k\}$, etc.) Therefore, the total number of choices is $6 \cdot 4 = 24$ (and all choices are possible). Using the notations in (14.39), we have

$$N^*/C^* \xrightarrow{\pi} \text{X-Inn}(R) \subseteq \mathrm{Aut}(R) \cong S_4.$$

We claim that $\text{X-Inn}(R) = \mathrm{Aut}(R)$; i.e., all automorphisms are X-inner. To see this, let us introduce the bigger domain of Hurwitz quaternions:

$$\bar{R} = \{(a + bi + cj + dk)/2 : \ a,\ b,\ c,\ d \in \mathbb{Z} \text{ all even or all odd}\} \supseteq R.$$

It is known that

$$\mathrm{U}(\bar{R}) = \{\pm 1,\ \pm i,\ \pm j,\ \pm k,\ (\pm 1 \pm i \pm j \pm k)/2\},$$

where the signs are arbitrarily chosen. This is the binary tetrahedral group,[87] which is an extension of $\mathrm{U}(R)$ by the cyclic group $\langle x \rangle$ of order 3 where $x = -(1 + i + j + k)/2$. By easy calculation, we have

$$xix^{-1} = j, \quad xjx^{-1} = k, \quad \text{and} \quad xkx^{-1} = i.$$

Thus, $\mathrm{U}(\bar{R}) \subseteq N^*$; e.g., x induces an X-inner automorphism φ given by $\varphi(i) = j$, $\varphi(j) = k$, and $\varphi(k) = i$. We have an injection

$$\pi : \ \mathrm{U}(\bar{R})/\mathrm{U}(\bar{R}) \cap C^* = \mathrm{U}(\bar{R})/\{\pm 1\} \longrightarrow \text{X-Inn}(R),$$

[87] As an abstract group, this is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$.

whose image is of order 12. Now $\mathrm{U}(\bar{R}) \cdot C^*$ is not yet the whole group N^*. For instance, the element $y = 1 + i$ clearly satisfies

$$yi = iy, \quad yj = ky \quad \text{and} \quad yk = -jy,$$

so we have $y \in N^* \backslash \mathrm{U}(\bar{R}) \cdot C^*$. Thus, y induces a new X-inner automorphism ψ of R given by $\psi(i) = i$, $\psi(j) = k$ and $\psi(k) = -j$. This shows that $|\text{X-Inn}(R)| > 12$ and therefore we must have

$$\text{X-Inn}(R) = \mathrm{Aut}(R) \cong S_4, \quad \text{and} \quad \text{X-Inn}(R)/\mathrm{Inn}(R) \cong S_3.$$

(Note that $\mathrm{Inn}(R) \cong \mathrm{U}(R)/\{\pm 1\}$ is the Klein 4-group.) This also enables us to determine the group N^*, which has $\mathrm{U}(\bar{R}) \cdot C^*$ as a subgroup of index 2, with a nontrivial coset given by the representative $y = 1 + i$.

Since the extended centroid C is the field $\mathbb{Q}$, it goes without saying that $Q^r(R)$, $Q^\ell(R)$, $Q^s(R)$, and the normal closure are all equal to Q.

We close this subsection with an example of a trivial X-inner automorphism group. This example assumes, however, a result mentioned earlier without proof.

(14.44) Example. Let R be the free algebra $k\langle x_1, \ldots, x_n \rangle$ over a field k, where $n \geq 2$. By a result of Kharchenko mentioned in (14.13), $Q^s(R) = R$. Therefore, by (14.33), the group N^* of nonzero R-normalizing elements is already in $\mathrm{U}(R)$. Since $\mathrm{U}(R) = k^*$, it follows that $\text{X-Inn}(R) = \{1\}$. But of course, the automorphism group $\mathrm{Aut}(R)$ itself is much larger.

§14F. A Matrix Ring Example

In this final subsection, we present a clever example, due to D. Passman [87], of a prime ring R for which we can compute all three of the Martindale rings of quotients $Q^r(R)$, $Q^\ell(R)$ and $Q^s(R)$ (and these turn out to be all different, and all bigger than R). In this example, we can also give substantial information on the X-inner automorphisms of R. What makes the computations possible here is the fact that R has only one ideal A besides 0 and R, so that the family $\mathcal{F} = \mathcal{F}(R)$ consists of only A and R.

(14.45) Example. Let $\mathbb{M}_\infty(k)$ be the additive group of all $\mathbb{N} \times \mathbb{N}$ matrices over a field k and let A be the set of finite matrices in $\mathbb{M}_\infty(k)$ (i.e., matrices with only a finite number of nonzero entries). Let $R = k + A$, where k is identified with $\{\mathrm{diag}(a, a, \ldots) : a \in k\}$. Then R is a prime k-algebra with $\mathcal{F} = \{A, R\}$ and we have:

(A) $Q^\ell(R) = \{\text{row finite matrices}\}$;
(B) $Q^r(R) = \{\text{column finite matrices}\}$; and
(C) $Q^s(R) = \{\text{row and column finite matrices}\} = Q^\ell(R) \cap Q^r(R)$.

First note that A is a ring without identity. Thus $R = k + A$ is a k-algebra, consisting of matrices with only finitely many nonzero elements off the diagonal,

and with diagonal entries eventually constant. Clearly, A is a maximal ideal in R. Let $a \in R\backslash\{0\}$. We shall verify below that *RaR is either A or R*. This will show that $\mathcal{F} = \{A, R\}$, and since A^2 (as well as $A \cdot R$, $R \cdot A$, and R^2) is nonzero, *R is a prime ring*.

Case 1. $a \in A$. Then $a \in \mathbb{M}_n(k)$ for all large n. Since $\mathbb{M}_n(k)$ is a simple ring, $\mathbb{M}_n(k)\, a\, \mathbb{M}_n(k) = \mathbb{M}_n(k)$. Therefore, all finite matrices lie in $\mathbb{M}_n(k)\, a\, \mathbb{M}_n(k)$ for some large n, and therefore in RaR. In this case, we have $RaR = A$.

Case 2. $a \notin A$. In this case, we will show that Ra contains a nonzero matrix $a_0 \in A$. If so, then $RaR \supseteq Ra_0R = A$ by Case 1, and therefore $RaR = R$ since A is a maximal ideal of R. Write the matrix a in the form

$$(14.46) \qquad \begin{pmatrix} L & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \ddots \end{pmatrix}, \quad \text{where } L \in \mathbb{M}_r(k) \text{ and } \lambda \in k\backslash\{0\}.$$

Left multiplying a by the matrix unit $E_{r+1,r+1}$, we see that Ra contains the nonzero finite matrix $a_0 = \lambda E_{r+1,r+1}$, as desired.

We have now confirmed that R is a prime ring with $\mathcal{F} = \{A, R\}$. To verify (B), *let S be the k-algebra of all column finite matrices*, which contains our prime algebra R. To see that $S \cong Q^r(R)$, it suffices to check that S has the properties (1), (2) and (3) in (14.24). In the following, we shall work with the matrix units $\{E_{ij}\}$ in R.

(1) Let $q \in S$; i.e., q is a column finite matrix. By direct inspection, each matrix in qA is column finite and has only finitely many nonzero columns. Therefore, we have $qA \subseteq A \subseteq R$. Since $A \in \mathcal{F}$, this verifies (1) in (14.24).

(2) Suppose $qB = 0$ where $q \in S$ and $B \in \mathcal{F}$. Since B is either R or A, we have in any case $qE_{ii} \in qA = 0$. This means that the ith column of q is zero (for all i); hence $q = 0$.

(3) Let $f \in \text{Hom}_R(B_R, R_R)$, where $B \in \mathcal{F}$. If $B = R$, then f is left multiplication by an element of $R \subseteq S$. Now assume $B = A$. Again using the fact that $E_{ii} \in A \subseteq R$, we have

$$f(E_{ii}) = f(E_{ii}E_{ii}) = f(E_{ii})E_{ii} \in R.$$

This means that $f(E_{ii})$ is a finite matrix with nonzero elements only in the ith column. Therefore, the infinite sum $q := \sum_j f(E_{jj})$ makes sense, and is an element of S. We have clearly $qE_{ii} = f(E_{ii})$ for each i and, for $j \neq i$,

$$f(E_{ij}) = f(E_{ii}E_{ij}) = f(E_{ii})E_{ij} = qE_{ii}E_{ij} = qE_{ij}.$$

Since f is a k-vector space homomorphism and the E_{ij}'s span A over k, we see that f coincides with left multiplication by q on A, checking the property (3) in (14.24).

We can now identify $Q^r(R)$ with S, the ring of all column finite matrices. Similarly, we can identify $Q^\ell(R)$ with S', the ring of all row finite matrices. Finally, we compute $Q^s(R)$, using the definition

$$\begin{aligned} Q^s(R) &= \{q \in Q^r(R) : \ Bq \subseteq R \ \text{ for some } \ B \in \mathcal{F}\} \\ &= \{q \in S : \ Aq \subseteq R\}. \end{aligned}$$

Since $E_{ij}q$ has its ith row given by the jth row of q, and has zeros elsewhere, the condition $Aq \subseteq R$ (for any $q \in \mathbb{M}_\infty(k)$) means simply that q is a row finite matrix. Therefore, the expression for $Q^s(R)$ above yields

$$Q^s(R) = S \cap S' = \{\text{row and column finite matrices}\},$$

as asserted in (C).

It is also possible to give some information on the group of X-inner automorphisms of R. As it turns out, the quotient group X-Inn(R)/Inn(R) is quite large. To see this, we first construct some "large" groups of units of $T := Q^s(R)$. For any fixed integer n, we have a k-algebra embedding $\theta : \mathbb{M}_n(k) \to T$ defined by

$$\theta(M) = \text{diag}(M, M, \dots) \in T.$$

Therefore U(T) contains $\theta(GL_n(k))$. It is easy to see that each matrix $\theta(M)$ ($M \in GL_n(k)$) is R-normalizing. (Use the fact that each matrix in R has the form (14.46) and choose r there to be a multiple of n.) Moreover, $\theta(M)$ commutes with all matrices in R iff M is a scalar matrix. Therefore, $\theta(GL_n(k))$ induces a group G of X-inner automorphisms of R which is isomorphic to $GL_n(k)/k^* = PGL_n(k)$. It turns out that $G \cap \text{Inn}(R) = \{1\}$. In fact, any matrix in R commutes with all matrix units $E_{N+i,N+j}$ for large N. If $\theta(M)$ commutes with all these matrix units, M must commute with all $n \times n$ matrix units and hence M is scalar. This shows that $G \cap \text{Inn}(R) = \{1\}$, and therefore that X-Inn(R)/Inn(R) *contains a copy of the group* $G \cong PGL_n(k)$ for any n. This is significant for the Galois theory of prime rings as G provides an example of a so-called "Noether group" of non-inner automorphisms of R.

By using group ring constructions, Passman [87] has shown that, in fact, *any* group can be realized as X-Inn(R)/Inn(R) for a prime ring (or even a domain) R.

Exercises for §14

Throughout the following exercises, $\mathcal{F} = \mathcal{F}(R)$ denotes the family of ideals with zero annihilators in a semiprime ring R.

1. For any semiprime ring R, show that $\mathcal{F}$ contains any prime ideal of R which is not a minimal prime ideal.

2. Let $S = Q^r(R)$, where R is a prime ring. If $I,\ I' \subseteq S$ are nonzero right (resp. left) R-submodules of S, show that $II' \neq 0$. (In particular, S is also a prime ring.)

3. Let $S = Q^r(R)$, where R is a semiprime ring.

   (1) If $I \subseteq S$ is a nonzero right or left R-submodule of S, show that $I^2 \neq 0$. (In particular, S is also a semiprime ring.)
   (2) If $J \subseteq S$ is an (R, R)-subbimodule of S, show that $\mathrm{ann}_r^S(J) = \mathrm{ann}_\ell^S(J)$.

4. Let R be any reduced ring. Show that $Q^s(R)$ is also a reduced ring. How about $Q^r(R)$?

5. Let R be a semiprime ring. If the extended centroid of R is a field, show that R must be a prime ring.

6. Let $\mathbb{M}_\infty(k)$ be the additive group of $\mathbb{N} \times \mathbb{N}$ matrices over a field k and let E be the ring of matrices in $\mathbb{M}_\infty(k)$ which are both row finite and column finite. Show that E is a prime ring, and determine the Martindale rings of quotients $Q^r(E)$, $Q^\ell(E)$, and $Q^s(E)$.

7. (Martindale) Let R be a prime ring and let $a, b, c, d \in R\backslash\{0\}$. Show that the following are equivalent:

   (1) $arb = crd$ for all $r \in R$.
   (2) There exists a unit q in the extended centroid $C = Z(Q^s(R))$ such that $c = qa$ and $d = q^{-1}b$.

8. Let $R \subseteq S$ be rings, and $N \subseteq S$ be the set of R-normalizing elements in S. Show that if $e = e^2 \in N$, then e commutes with every element of R. Deduce that any R-normalizing idempotent in $Q^r_{\max}(R)$ is central.

9. Let $\varphi \in \mathrm{Aut}(R)$, where R is a semiprime ring. By Exer. (13.15), φ extends uniquely to an automorphism of $Q^r_{\max}(R)$, which we denote by Φ. Show that

   (1) $\Phi | Q^r(R)$ (resp. $\Phi | Q^s(R)$) is the unique extension of φ to $Q^r(R)$ (resp. $Q^s(R)$).
   (2) For any $\sigma \in \mathrm{Aut}(R)$, $\Phi(\mathrm{X}(\sigma)) = \mathrm{X}(\varphi\sigma\varphi^{-1})$. (In particular, $\Phi(\mathrm{X}(\varphi)) = \mathrm{X}(\varphi)$).
   (3) The set X-Inn(R) is closed under conjugation in $\mathrm{Aut}(R)$.
   (4) $\mathrm{X}(\varphi) = \mathrm{X}(\Phi)$. (Basically, this requires proving that, whenever $x \in \mathrm{X}(\varphi)$, $xq = \Phi(q)x$ for any $q \in Q^r_{\max}(R)$.)

10. Keep the notations in the last exercise, and let $x \in \mathrm{X}(\varphi)$.

    (1) Show that $\Phi(x^2) = x^2$.
    (2) Show that $\Phi(x) = x$ if R is either commutative or prime.

11. Keeping the notations in the last exercise, show that $\Phi(x) = x$ always holds (for any $x \in \mathrm{X}(\varphi)$). (**Hint.** Write $z = \Phi(x)$, and consider any $q \in Q := Q^r_{\max}(R)$. Using $\Phi(x^2) = x^2$, show that $zqx = xqx = zqz$. From these equations (and semiprimeness), deduce that $(z - x)Q(z - x) = 0$.)

12. Let C denote the extended centroid of a semiprime ring R, and let G be any subgroup of $\mathrm{Aut}(R)$. Recall that, for any $\varphi \in G$:

$$\mathrm{X}(\varphi) = \{x \in Q^r_{\max}(R) : \; xa = \varphi(a)x \;\; (\forall\, a \in R)\},$$

and let $\mathrm{X}(G) := \sum_{\varphi \in G} \mathrm{X}(\varphi)$. Show that $\mathrm{X}(G)$ is a C-subalgebra of the normal closure of R in $Q^r_{\max}(R)$, and deduce that $\mathrm{X}(G) \subseteq Q^s(R)$.

13. Compute $\mathrm{Inn}(\bar{R})$, $\mathrm{X}\text{-}\mathrm{Inn}(\bar{R})$, and $\mathrm{X}\text{-}\mathrm{Inn}(\bar{R})/\mathrm{Inn}(\bar{R})$ for the Hurwitz ring of quaternions $\bar{R}$. Also, determine the group of nonzero normalizing elements for $\bar{R}$ in its division ring of quotients.

14. Let A be a commutative unique factorization domain with quotient field K, and let $R = \mathbb{M}_n(A)$. Show that $Q^r_{\max}(R) = \mathbb{M}_n(K)$ and that the group of nonzero R-normalizing elements N^* in $Q^r_{\max}(R)$ is exactly $K^* \cdot \mathrm{U}(R)$. What are the normal and central closures of R in this example?

15. Show that the conclusion $N^* = K^* \cdot \mathrm{U}(R)$ in the last exercise may not hold if the commutative domain A there is *not* a unique factorization domain.

16. Let A be a commutative domain with a nonidentity automorphism φ_0 and let φ be the automorphism on $R = \mathbb{M}_n(A)$ defined by $\varphi((a_{ij})) = (\varphi_0(a_{ij}))$. Show that φ is not an X-inner automorphism.

17. Let R be a semiprime ring with extended centroid C and let $R \cdot C$ be its central closure. Show that the central closure of $R \cdot C$ is itself.

Chapter 6

Frobenius and Quasi-Frobenius Rings

The class of rings that are self-injective (as a left or right module over themselves) has been under close scrutiny by ring theorists. There is a vast literature on the structure of self-injective rings satisfying various other conditions. In a book of limited ambition such as this, it would be difficult to do justice to this extensive literature. As a compromise, we focus our attention in this chapter on a special class of such rings called *quasi-Frobenius* (QF) *rings*, and the subclass of *Frobenius rings*. It will be seen that the finite-dimensional Frobenius algebras discussed in §3B are examples of the latter.

A QF *ring is, in short, a right noetherian ring that is right self-injective.* There is no need to use the term "right QF", since, miraculously, this definition turns out to be left-right symmetric. Moreover, a QF ring is always (2-sided) artinian. There is a very rich structure theory for both the 1-sided ideals of R and the left/right modules over R, including a very elegant duality theory for f.g. modules. The basic features of these theories, as well as some of the many interesting characterizations of QF rings, are given in §15.

The second section of this chapter, §16, is devoted to the study of Frobenius rings. Historically, Frobenius rings made their first appearance, in the work of R. Brauer, C. Nesbitt, T. Nakayama, and others, in the form of Frobenius algebras. These are finite-dimensional algebras over a field that have a certain "self-dual" property (with respect to the field). The study of such algebras was motivated by the representation theory of finite groups, since group algebras of finite groups provided a large source of examples. Later, the notion of Frobenius algebras evolved into that of Frobenius rings and quasi-Frobenius rings (which, in turn, spawned a plethora of generalizations, such as the QF-1, QF-2, QF-3 rings of Thrall, and the PF (pseudo-Frobenius) rings of Azumaya, etc.). For maximum efficiency in our exposition, we have chosen to reverse the historical order, so Frobenius rings will come only in §16, after a full discussion of quasi-Frobenius rings in §15. In order not to lose sight of the historical perspective, however, we have intentionally included a discussion of finite-dimensional Frobenius algebras in §3, in our introductory treatment of injective modules. This was our early start on QF rings and Frobenius rings. We shall return to Frobenius algebras midway in §16, and show that these are just the finite-dimensional algebras which happen to be Frobenius

rings. Some remarkable characterizations of Frobenius algebras in terms of certain dimension equations involving dual modules and annihilator ideals will be given in §16D. This is followed by a short introduction to symmetric algebras, their characterizations, and some of their special properties. The chapter concludes with a historical subsection (§16G: "Why Frobenius?") in which we put the material in perspective by backtracking to the classical approach to the subject pioneered by Frobenius, Brauer, and others.

Besides the connection to group representation theory, Frobenius rings appear also in other branches of algebra. For instance, commutative local Frobenius rings are precisely the zero-dimensional local Gorenstein rings: these rings play an interesting role in number theory, algebraic geometry, and combinatorics. Frobenius algebras have also shown up in the recent study of Hopf algebras and Koszul algebras. Today, the use of Frobenius rings has reached way beyond the realm of pure algebra and ring theory. For instance, some applications of Frobenius rings to coding theory are presented in J. A. Wood's recent article [97]. In topology and geometry, Frobenius algebras occur as cohomology rings of compact oriented manifolds and as quantum cohomology rings of certain compact Kähler manifolds, and they have also shown up in the recent work on the solutions of the Yang-Baxter equation. In March/April 1996 I attended the series of Hitchcock Lectures on geometry and physics given by Chern Professor Sir Michael F. Atiyah at Berkeley, and was delighted to see that one of his transparencies in Lecture 3 displayed the impressive equation

"TOP QFT ($d = 1$) = Frobenius Algebra".

(Here, "TOP QFT ($d = 1$)" referred to "1-dimensional topological quantum field theory.") By Lecture 6, Sir Michael has upgraded this equation to "2-dim. TOP QFT = Frobenius Algebra", and was well on way to expound

"3-dim. TOP QFT $\rightarrow$ 2-dim. QFT (Frobenius Algebra)"!

§15. Quasi-Frobenius Rings

§15A. Basic Definitions of QF Rings

At the beginning of §13, we have had our first glimpse into the structure of a right self-injective ring. In particular, in (13.5), we saw that, for a right self-injective ring R, two of the finiteness conditions, namely, u.dim $R_R < \infty$ and the nonexistence of an infinite orthogonal set of nonzero idempotents, are equivalent, and these amount to the fact that R is semilocal (or even semiperfect). However, this is still a far cry from R being right noetherian.

To get the strongest structure theorems on right self-injective rings, we follow the lead of Emmy Noether and impose the ascending chain condition on right ideals. By adding this condition, we are led to one of the most important classes of rings, called QF *(quasi-Frobenius) rings.* Without further ado, we now embark on the main result of this subsection.

(15.1) Theorem. *For any ring R, the following are equivalent:*

(1) *R is right noetherian and right self-injective.*
(2) *R is left noetherian and right self-injective.*
(3) *R is right noetherian and satisfies the following double annihilator conditions:*
   (3a) $\operatorname{ann}_r(\operatorname{ann}_\ell A) = A$ *for any right ideal* $A \subseteq R$.
   (3b) $\operatorname{ann}_\ell(\operatorname{ann}_r \mathfrak{A}) = \mathfrak{A}$ *for any left ideal* $\mathfrak{A} \subseteq R$.
(4) *R is (2-sided) artinian and satisfies* (3a) and (3b).

If R satisfies any of (1)–(4), we say that R is a QF ring. Of course, the Frobenius algebras studied in §3B are QF rings, by (3.14).

Note that (4) above is left-right symmetric. Thus, the theorem implies the same for (1), (2), and (3). In particular, QF rings may also be defined as left (resp. right) noetherian left self-injective rings, or as left noetherian rings satisfying the double annihilator conditions. The fact that we can so randomly combine these conditions and end up with the same notion is quite remarkable. Indeed, this single theorem summarizes, and presents in a nutshell, a large quantity of work done by various authors over a considerable period of time spanning the 1940s and 50s.

Before beginning the proof of (15.1), we make the following remark about (3ab). *The condition* (3a) (*resp.* (3b)) *simply amounts to the assumption that any right (resp. left) ideal is a right (resp. left) annihilator.* In the proof of (15.1), it will be important, of course, to distinguish right ideals from left ideals. To this end, we shall write $A, B, \ldots$ to denote right ideals, and $\mathfrak{A}, \mathfrak{B}, \ldots$ to denote left ideals in the ring R.

Proof of (15.1). We shall show (4)$\Longrightarrow$(3)$\Longrightarrow$(1)$\Longrightarrow$(2)$\Longrightarrow$(4).

(4)$\Longrightarrow$(3). If R is right artinian, it is right noetherian by the Hopkins-Levitzki Theorem (*FC*–(4.15)).

(3)$\Longrightarrow$(1). Given (3), consider any right ideals A, B. We have

(15.2) $$\operatorname{ann}_r(\operatorname{ann}_\ell A + \operatorname{ann}_\ell B) = \operatorname{ann}_r(\operatorname{ann}_\ell A) \cap \operatorname{ann}_r(\operatorname{ann}_\ell B) = A \cap B.$$

Taking left annihilators, we deduce that

(15.3) $$\operatorname{ann}_\ell A + \operatorname{ann}_\ell B = \operatorname{ann}_\ell (A \cap B).$$

To see that R_R is injective, we apply Baer's Test. Let $g \in \operatorname{Hom}_R(C, R_R)$, where C is any right ideal, say, $C = \sum_{i=1}^n c_i R$. To show that g is a left multiplication by some element of R, we induct on n. For $n = 1$, let $d = g(c_1)$. Then $d \cdot \operatorname{ann}_r(c_1) = 0$ implies that $d \in \operatorname{ann}_\ell(\operatorname{ann}_r(Rc_1)) = Rc_1$ (by (3b)), so $d = yc_1$ for some $y \in R$. Therefore, left multiplication by y extends the given g. Now assume $n > 1$, and let $B = \sum_{i=2}^n c_i R$. Then $g|c_1 R$ is left multiplication by some $y \in R$, and (by the inductive hypothesis) $g|B$ is left multiplication by some $x \in R$. On $c_1 R \cap B$, left multiplication by $x - y$ is then the zero map, so

(15.4) $$x - y \in \operatorname{ann}_\ell(c_1 R \cap B) = \operatorname{ann}_\ell(c_1 R) + \operatorname{ann}_\ell B \qquad \text{(by (15.3))}.$$

Write $x - y = x' - y'$ where $x' \in \text{ann}_\ell B$ and $y' \in \text{ann}_\ell(c_1 R)$. Letting $z := x - x' = y - y'$, we can then extend g to R_R by using left multiplication by z.

(1)$\Longrightarrow$(2). The quotient ring $R/\text{rad}\, R$ is right noetherian, and von Neumann regular by (13.2)(2), so it is semisimple by *FC*–(4.25). On the other hand, by (7.15)(2) and (13.2)(1), rad R is a nilpotent ideal. Therefore, R is a semiprimary ring in the sense of *FC*–(4.15), and the latter implies that R is right artinian. To show that R is *left noetherian*, it is enough to check the ACC for f.g. left ideals $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots$. By the DCC on right ideals, we have $\text{ann}_r\, \mathfrak{A}_n = \text{ann}_r\, \mathfrak{A}_{n+1} = \ldots$ for some n, and therefore

$$\text{(15.5)} \qquad \text{ann}_\ell(\text{ann}_r\, \mathfrak{A}_n) = \text{ann}_\ell(\text{ann}_r\, \mathfrak{A}_{n+1}) = \ldots$$

By Step 2 below, we conclude that $\mathfrak{A}_n = \mathfrak{A}_{n+1} = \ldots$, as desired.

We now come to (2)$\Longrightarrow$(4), which is the hardest part of the theorem. We break up the proof into a sequence of steps.

Step 1. *If R_R is injective, then* (15.3) *always holds.* We need only prove the inclusion "$\supseteq$". Let $x \in \text{ann}_\ell(A \cap B)$. Define $f : A + B \to R_R$ by $f(a + b) = xb$ where $a \in A$ and $b \in B$. This is a well-defined map (hence an R-homomorphism) since, in the event that $a + b = 0$, we have $b \in A \cap B$ and hence $xb = 0$. Since R_R is injective, there exists an element $y \in R$ such that $xb = y(a + b)$ for all $a \in A$, $b \in B$. For $b = 0$, this shows that $y \in \text{ann}_\ell A$; for $a = 0$, this shows that $x - y \in \text{ann}_\ell B$. Adding these two elements, we see that $x \in \text{ann}_\ell A + \text{ann}_\ell B$.

Step 2. *If R_R is injective, then* $\text{ann}_\ell(\text{ann}_r\, \mathfrak{A}) = \mathfrak{A}$ *for any f.g. left ideal* $\mathfrak{A}$. This property of a right self-injective ring was noted before as a consequence of the Johnson-Wong Double Annihilator Theorem; see (13.5)′. In order to make the proof of (15.1) more self-contained, we shall give an ad hoc (and slightly different) proof for the desired equation here, banking on Step 1. First consider the special case $\mathfrak{A} = Rc$ ($c \in R$). We need only prove that any $d \in \text{ann}_\ell(\text{ann}_r(Rc))$ belongs to Rc. Define $g : cR \to R_R$ by $g(cx) = dx$ ($\forall x \in R$). This map is well defined (and hence an R-homomorphism) since, if $cx = 0$, then $x \in \text{ann}_r(Rc)$ and hence $dx = 0$. By the injectivity of R_R again, there exists $y \in R$ such that $dx = ycx$ ($\forall x \in R$). In particular, $d = yc \in Rc$. For *any* $\mathfrak{A} = \sum_{i=1}^n Rc_i$, we have therefore:

$$\begin{aligned} \text{ann}_\ell(\text{ann}_r\, \mathfrak{A}) &= \text{ann}_\ell\left(\bigcap_{i=1}^n \text{ann}_r(Rc_i)\right) \\ &= \sum_{i=1}^n \text{ann}_\ell(\text{ann}_r(Rc_i)) \quad \text{(by Step 1)} \\ &= \sum_{i=1}^n Rc_i = \mathfrak{A}\,. \end{aligned}$$

In the following, we shall assume that R is left noetherian as well as right self-injective. Then Step 2 gives (3b) for *all* left ideals $\mathfrak{A}$: we are 1/4 done toward proving (4).

Step 3. *R is left artinian.* By the Hopkins-Levitzki Theorem again, it suffices to show that R is semiprimary. Since $R/\text{rad } R$ is left noetherian, and von Neumann regular, it is semisimple as before. It remains to show the nilpotency of $J := \text{rad } R$. Here we cannot use the argument in (1)$\Longrightarrow$(2). Instead, look at the chain $\text{ann}_r(J) \subseteq \text{ann}_r(J^2) \subseteq \ldots$. Since the J^i 's are ideals, so are their annihilators. Viewing the latter as *left* ideals, we have then $\text{ann}_r(J^n) = \text{ann}_r(J^{n+1}) = \ldots$ for some n. Taking left annihilators (and using Step 2), we have then $J^n = J^{n+1}$, and Nakayama's Lemma (*FC*–(4.22)) shows $J^n = 0$, as desired.

Step 4. *R is left Kasch* (in the sense of (8.26)). This follows by applying the left analogue of (8.27), and using (3b) for maximal left ideals.

Step 5. *R is also right Kasch.* To see this, let $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ be the isotypic ("homogeneous") components of $\text{soc}({}_R R)$. These are easily seen to be ideals of R. We claim that the ideal

$$\mathfrak{B}_i := \text{ann}_\ell(J) \cap \mathfrak{A}_i \neq 0 \quad (\forall i),$$

where $J = \text{rad } R$. Indeed, fix $0 \neq a_i \in \mathfrak{A}_i$. If $\mathfrak{B}_i = 0$, $a_i x_1 \neq 0$ for some $x_1 \in J$. Then $a_i x_1 x_2 \neq 0$ for some $x_2 \in J$. This eventually leads to a contradiction since J is nilpotent. From $\mathfrak{B}_i J = 0$, we see therefore that $\mathfrak{B}_i$ contains a minimal right ideal, say, V_i. (Recall that R/J is semisimple.) For $i \neq j$, we have $V_i \not\cong V_j$ as right R-modules. For, if $V_i \cong V_j$, such an isomorphism is induced by *left* multiplication by some $r \in R$ (since R_R is injective). But then

$$V_j = rV_i \subseteq r\mathfrak{A}_i \subseteq \mathfrak{A}_i$$

gives a contradiction. Since R/J has m simple left modules and therefore m simple right modules, each simple right module is isomorphic to some $V_i \subseteq R_R$.

Step 6. *For any nonzero right module M_R,* $\text{Hom}_R(M, R_R) \neq 0$. To see this, note that $MJ \neq M$ for $J = \text{rad } R$ (since J is nilpotent). Viewing M/MJ as a right module over the semisimple ring R/J, we can map it onto a simple module V_R. Embedding V into R_R (by Step 5), we get a nonzero homomorphism from M to R.

Step 7. (3a) *holds for any right ideal A.* Indeed, let $M = \text{ann}_r(\text{ann}_\ell A)/A$, and consider any $f \in \text{Hom}_R(M, R_R)$. We may view f as a homomorphism $\text{ann}_r(\text{ann}_\ell A) \to R_R$ vanishing on A. Since R_R is injective, this is given by left multiplication by some $y \in R$. But $yA = 0$ implies that $yx = 0$ for any $x \in \text{ann}_r(\text{ann}_\ell A)$, so $f \equiv 0$. Step 6 then gives $M = 0$, so we have proved (3a).

Step 8. Since R has ACC on left annihilators, it has DCC on right annihilators (see (6.57)). But by (3a), any right ideal is a right annihilator, so R is right artinian. This is our last step! □

Some of the properties obtained for QF rings in Steps 1–8 above are worth celebrating over. The fact that QF rings are Kasch is already highly significant for Frobenius algebras. Indeed, for group algebras kG of a finite group G over a field k (which are Frobenius algebras by (3.15E)), this translates into the fact that *any irreducible k-representation of G is afforded by a minimal 1-sided ideal of kG.* This is an especially interesting fact in the case of "modular representations" (when char k divides $|G|$).

From the "double annihilator conditions" (3a), (3b) (and from (3.17)), we deduce immediately the following.

(15.6) Corollary. *For any* QF *ring R, the maps*

$$\mathfrak{A} \mapsto \mathrm{ann}_r\, \mathfrak{A} \quad \textit{and} \quad A \mapsto \mathrm{ann}_\ell\, A$$

define mutually inverse lattice anti-isomorphisms between the left ideals and the right ideals of R.

(15.7) Corollary. *For any* QF *ring R with $J = \mathrm{rad}\, R$:*

$$\mathrm{ann}_\ell(J) = \mathrm{soc}(R_R) = \mathrm{soc}({}_R R) = \mathrm{ann}_r(J).$$

Proof. Use the notations in Step 5 above. If V is any right ideal of R isomorphic to V_i, we must have $V = rV_i$ for some $r \in R$, so $V \subseteq r\mathfrak{A}_i \subseteq \mathfrak{A}_i$. This shows that $A_i \subseteq \mathfrak{A}_i$ for the isotypic component A_i of $\mathrm{soc}(R_R)$ containing V_i. By left-right symmetry, we must have $A_i = \mathfrak{A}_i$ and a fortiori $\mathrm{soc}(R_R) = \mathrm{soc}({}_R R)$. The other equalities follow easily from the fact that R is artinian (cf. *FC*–Exer. (4.20)). □

(15.8) Remark. In fact, the equation $\mathrm{soc}(R_R) = \mathrm{soc}({}_R R)$ holds more generally in any (2-sided) principally injective ring R: see Exercise (13.27).

What are some examples of QF rings? As a starter, we can take any semisimple ring, or $\mathbb{Z}/n\mathbb{Z}$ ($n \neq 0$), or $k[t]/(f(t))$ ($f \neq 0$), where k is any field (see (3.13)). Among finite-dimensional algebras over a field k, *Frobenius algebras are prominent examples of* QF *rings* (see (3.14)). This includes all the examples from (3.15A) through (3.15F) (not counting, of course, (3.15B′)). However, as we shall see later, not all QF algebras are Frobenius algebras. More examples of QF rings will be given below in §15D.

§15B. Projectives and Injectives

Here, we offer another very charming characterization of QF rings, in terms of their projective and injective modules.

(15.9) Theorem. *For any ring R, the following are equivalent:*

(1) *R is* QF.

(2) *A right R-module is projective iff it is injective.*

Proof. (1)$\Longrightarrow$(2). Say I_R is injective. Since R is right noetherian, $I \cong \bigoplus_i I_i$ for suitable indecomposable injective modules $\{I_i\}$ (see (3.49)). Fix $0 \neq a_i \in I_i$. Since R is right artinian, $a_i R$ certainly contains a simple submodule V_i. The indecomposability of I_i then implies that $E(V_i) = I_i$. Viewing I_i as embedded in the injective module R_R (cf. Step 5 above), we see that $I_i = E(V_i)$ is isomorphic to a direct summand of R_R. Therefore, I_i is projective, and so is $I \cong \bigoplus_i I_i$. Next, consider any projective module P_R. Then $P \oplus Q$ is a free module $R^{(J)}$ for some indexing set J. Since R is right noetherian, $R^{(J)} = \bigoplus_{j \in J} R_R$ is injective, and so is P_R.

(2)$\Longrightarrow$(1). Since R_R is projective, it is injective by (2). Also, since any direct sum of projectives is projective, (2) implies that any direct sum of injectives is injective (for right modules). By (3.46), R is a right noetherian ring, and hence QF by (15.1). □

(15.10) Remark. In fact, the theorem above can be further improved. Faith and Faith-Walker have shown that *R is* QF *iff right projective R-modules are injective, iff right injective R-modules are projective.* To prove this would require more work, so we have chosen to prove the simpler form of the Faith-Walker Theorem in (15.9). A proof for the full form of this theorem can be found in Faith [76: p. 209].

We shall develop a few more properties of QF rings below. Recall that a module M_R is called *torsionless* if, for any $m \neq 0$ in M, there exists $f \in M^* = \mathrm{Hom}_R(M, R_R)$ such that $f(m) \neq 0$. (In other words, the natural map $M \to M^{**}$ is injective: see (4.65)(a).) A torsionless module M_R is said to be *reflexive* if the injection $M \to M^{**}$ is in fact an isomorphism (of right R-modules).

(15.11) Theorem. *Let R be a* QF *ring. Then*

(1) *Any module M_R can be embedded in a free module, and is torsionless.*
(2) *Any f.g. module M_R is reflexive.*[88]
(3) *A (left or right) R-module M is f.g. iff M^* is f.g.*

Proof. (1) The injective hull $E(M)$ is projective by (15.9), so it embeds into a free module F. As we have observed in (4.65)(b), $M \subseteq E(M) \subseteq F$ implies that M is torsionless (since F is).

(2) Assume M_R is f.g. Fix an exact sequence $0 \to K \to F \to M \to 0$, where $F = R^n$. Since R_R is injective, the duality functor $\mathrm{Hom}_R(-, R_R)$ is exact from $\mathfrak{M}_R$ to ${}_R\mathfrak{M}$. Similarly, $\mathrm{Hom}_R(-, {}_RR)$ from ${}_R\mathfrak{M}$ to $\mathfrak{M}_R$ is also exact. Therefore,

[88]The converse of (2) is true too for a QF ring R: we shall prove this later in §19.

we have an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & F^{**} & \xrightarrow{\beta} & M^{**} & \longrightarrow & 0 \end{array}$$

Since $F = R^n$, α is an isomorphism. The surjectivity of β therefore implies the surjectivity of γ. Since M is torsionless, γ is an isomorphism.

(3) It suffices to verify this for right modules M_R. If M_R is f.g., we can take a surjection $F \to M$ as in (2) where $F = R^n$. Taking the first dual, we have an injection $M^* \to F^* = {}_R(R^n)$. Since R is a left noetherian ring, M^* is is f.g. Conversely, suppose M^* is f.g. Then by the above, M^{**} is f.g. By (1), M embeds into M^{**}. Since R is a right noetherian ring, this implies that M is also f.g. □

§15C. *Duality Properties*

As observed in the proof of (15.11)(2) above, for a QF ring R, the "dual" operator $*$ is a *contravariant* exact functor from $\mathfrak{M}_R$ to ${}_R\mathfrak{M}$, and we have also a similar functor from ${}_R\mathfrak{M}$ to $\mathfrak{M}_R$. Writing $\mathfrak{M}_R^{fg}$ and ${}_R^{fg}\mathfrak{M}$ for the subcategories of f.g. modules, we see that the double-dual functors are naturally equivalent to the identity functors on $\mathfrak{M}_R^{fg}$ and ${}_R^{fg}\mathfrak{M}$. Thus we have a "perfect duality":

(15.12) $$\mathfrak{M}_R^{fg} \underset{*}{\overset{*}{\rightleftarrows}} {}_R^{fg}\mathfrak{M}$$

between the two categories of f.g. modules. As an easy consequence of this, we obtain the following:

(15.13) Corollary. *Let R be a* QF *ring, and M be a right R-module. Then M is simple (resp. f.g. indecomposable) iff M^* is (as a left R-module).*

The term "perfect duality" for the one-one correspondence in (15.12) was coined by Dieudonné [58]. We shall not define this term precisely here, but it will be convenient to use it occasionally in an intuitive way. To explain more explicitly the features of this perfect duality, let us introduce the "$\perp$" notation. Let $M \in \mathfrak{M}_R^{fg}$. For any submodule $A \subseteq M$, let $A^\perp = \{f \in M^* : f(A) = 0\}$, which is a submodule of M^*. Similarly, for any submodule $\mathfrak{A} \subseteq M^*$, let $\mathfrak{A}^\perp = \bigcap_{a\in\mathfrak{A}} \ker(a)$ (a submodule of M). Then we have:

(15.14) $A^\perp \cong (M/A)^*$ and $A^* \cong M^*/A^\perp$.

(15.15) $A^{\perp\perp} = A$ and $\mathfrak{A}^{\perp\perp} = \mathfrak{A}$.

(15.16) $(A + B)^\perp = A^\perp \cap B^\perp$ and $(A \cap B)^\perp = A^\perp + B^\perp$.

(15.17) $(\mathfrak{A} + \mathfrak{B})^\perp = \mathfrak{A}^\perp \cap \mathfrak{B}^\perp$ and $(\mathfrak{A} \cap \mathfrak{B})^\perp = \mathfrak{A}^\perp + \mathfrak{B}^\perp$.

Here, the first formulas in (15.14), (15.16), and (15.17) are true without any assumptions on the modules or the ring, and $A^* \cong M^*/A^\perp$ is true as long as R_R is injective, without any assumptions on $A \subseteq M$. The others are easily deduced from the duality in (15.12), under the assumptions that R is QF and M_R is f.g. In particular, by taking duals, we have, for $A,\ B \subseteq M$:

(15.18) $A \cong B$ in $\mathfrak{M}_R$ iff $M^*/A^\perp \cong M^*/B^\perp$ in ${}_R\mathfrak{M}$.

(15.19) $M/A \cong M/B$ in $\mathfrak{M}_R$ iff $A^\perp \cong B^\perp$ in ${}_R\mathfrak{M}$.

Remarkably, even more is true. We now prove:

(15.20) Proposition. *Let M be f.g. projective over the* QF *ring R, and A, B be submodules of M. Then any R-isomorphism $h : A \to B$ extends to an R-automorphism of M.*

Proof. Since M_R is also injective, h extends to an endomorphism of M, which we continue to denote by h. Let $H = \text{End}_R(M)$, which is a semilocal ring by (13.3). Let $\mathfrak{A}$ be the left ideal of H consisting of all endomorphisms vanishing on A. *We claim that* $H = Hh + \mathfrak{A}$. Once this is proved, Bass' Theorem *FC*–(20.9) implies that there is a unit $u := h + v \in \text{U}(H)$ for some $v \in \mathfrak{A}$. Since $v(A) = 0$, we have $u|A = h|A$, as desired. To prove our claim, it suffices to find a decomposition $\text{Id}_M = f + g$ where $f \in Hh$ and $g \in \mathfrak{A}$. For $C = \ker(h)$, we have $A \cap C = 0$. By the injectivity of M, there exists $g \in H$ such that $g|C = \text{Id}_C$ and $g|A = 0$. Then $g \in \mathfrak{A}$, and $f := \text{Id}_M - g$ is zero on $C = \ker(h)$. By Exercise 3.23, the latter implies that $f \in Hh$, as desired. □

(15.21) Theorem. *Let M be as above, and $A, B \subseteq M$. The following are equivalent:*

(1) $A \cong B$ *in* $\mathfrak{M}_R$.
(1′) $M^*/A^\perp \cong M^*/B^\perp$ *in* ${}_R\mathfrak{M}$.
(2) $M/A \cong M/B$ *in* $\mathfrak{M}_R$.
(2′) $A^\perp \cong B^\perp$ *in* ${}_R\mathfrak{M}$.

Proof. From (15.18) and (15.19), we have (1) $\Longleftrightarrow$ (1′) and (2) $\Longleftrightarrow$ (2′). From (15.20), we have (1) $\Longrightarrow$ (2), and, applying (15.20) to $A^\perp$, $B^\perp$ in the f.g. projective module M^*, we have (2′) $\Longrightarrow$ (1′). □

The theorem above reveals the beginning of an unusually rich module theory for QF rings; we shall return to develop this a little further in §16. At this point, let us note that, in (15.21), both assumptions on M are essential for the truth of the theorem. If M need not be f.g., we can take $M = R \oplus R \oplus \cdots$, and

(15.22) $\qquad A = (0) \oplus R \oplus R \oplus \cdots, \qquad B = (0) \oplus (0) \oplus R \oplus R \oplus \cdots.$

Here, $A \cong B$, but $M/A \cong R$ and $M/B \cong R \oplus R$ are not isomorphic (unless $R = (0)$). Therefore, no isomorphism from A to B can be extended to an automorphism of M. To construct a counterexample in the case when M is f.g. but not projective, we proceed as follows. Let $R_R = P_1 \oplus \cdots \oplus P_r$ be a decomposition of R_R into a direct sum of principal indecomposables, and let $P_i = e_i R$ where $e_i^2 = e_i \neq 0$, and $e_1 + \cdots + e_r = 1$. Assume that $J = \text{rad } R \neq 0$. Then $e_i J \neq 0$ for some i, and Nakayama's Lemma implies that $T := P_i / e_i J \neq 0$. Let

$$(15.23) \qquad M = T \oplus P_i, \quad \text{and} \quad A = (0) \oplus P_i \subseteq M, \quad B = T \oplus e_i J \subseteq M.$$

Then $M/A \cong T \cong M/B$. However, $A \ncong B$ since A is indecomposable but B is not! Therefore, (15.21) holds for all f.g. M over a right artinian ring R only if R is semisimple.

Let us now specialize our general results to the case when $M = R_R$ and $A \subseteq R$ is a right ideal. Here M^* is just ${}_R R$, and $A^\perp = \text{ann}_\ell \, A$. Therefore, we have:

(15.24) Corollary. *For any right ideals A, B in a* QF *ring R, $A \cong B$ iff $R/A \cong R/B$, iff* $\text{ann}_\ell \, A \cong \text{ann}_\ell \, B$, *iff* $R/\text{ann}_\ell \, A \cong R/\text{ann}_\ell \, B$ *(in the appropriate module categories). If A, B are ideals, then $A \cong B$ in $\mathfrak{M}_R$ iff $A = B$.*

(The last statement follows from the (by now familiar) fact that $A_R \cong B_R$ implies $B = rA$ for some $r \in R$. If A, B are *ideals*, then $B = rA \subseteq A$ and similarly $A \subseteq B$.)

To complete our discussion of duality properties, it will be useful to bring in the viewpoint of bimodules. Note that if $A \subseteq R$ is an ideal, then A and R/A are both (R, R)-bimodules. For any (R, R)-bimodule M, we can form the dual $M_R^* = \text{Hom}_R(M_R, R_R)$, and this again carries a natural structure of an (R, R)-bimodule. (The left R-action on M_R^* is defined via the left R-action on R, and the right R-action on M^* is defined via the *left* R-action on M.) Similarly, the other dual, ${}_R M^* = \text{Hom}_R({}_R M, {}_R R)$ is also an (R, R)-bimodule.

Let A be an ideal in a QF ring R. Then, so is $\text{ann}_\ell \, A$, and by what we said above, A_R^* and $R/\text{ann}_\ell \, A$ are both (R, R)-bimodules. By (15.14), the natural map $R/\text{ann}_\ell \, A \to A_R^*$ is an isomorphism of left R-modules. An easy check shows that this is, in fact, an (R, R)-bimodule isomorphism. Let us now apply this observation to deduce a useful fact about the socle of a QF ring.

In a QF ring R, let us write $\text{soc}(R)$ to denote the ideal $\text{soc}(R_R) = \text{soc}({}_R R)$, and recall that $\text{soc}(R) = \text{ann}_r \, J = \text{ann}_\ell \, J$, where $J = \text{rad } R$ (see (15.7)). Let us also write $\bar{R}$ for the (R, R)-bimodule R/J.

(15.25) Proposition. *For any* QF *ring R, we have (R, R)-bimodule isomorphisms*

$$\text{soc}(R) \cong ({}_R \bar{R})^* \cong (\bar{R}_R)^*.$$

Proof. By symmetry, it is sufficient to prove the first bimodule isomorphism. By what we said before the Proposition (applied to $A = \text{soc}(R)$), we get a bimodule

isomorphism

$$(\text{soc } R)_R^* \cong R/\text{ann}_\ell(\text{soc } R) = R/\text{ann}_\ell(\text{ann}_r\ J) = R/J = \bar{R}.$$

Treating these as left R-modules and taking their duals, we get the desired bimodule isomorphism $\text{soc}(R) \cong ({}_R\bar{R})^*$. □

Note that it is not a priori clear that the right module dual $(\bar{R}_R)^*$ and the left module dual $({}_R\bar{R})^*$ are isomorphic as (R, R)-bimodules. We have proved this to be the case, however, over any QF ring R.

§15D. Commutative QF Rings, and Examples

In this last subsection of §15, we would like to understand the structure of commutative QF rings. Before we go over to the commutative case, let us first look at some examples and nonexamples of QF rings, besides the ones already mentioned at the end of §15A.

(15.26) Examples.

(1) Let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, where k is a division ring. This is an artinian ring with $J = \text{rad } R = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$. We have shown in (3.10)(B) that R is not right self-injective, and in (8.29)(5) that R is not right Kasch. In particular, R is not QF. Let us mention a few other properties of a QF ring which fail in R. Consider $\mathfrak{m} = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$, which is an ideal in R. As in (8.29)(5), $\text{ann}_r(\text{ann}_\ell\ \mathfrak{m}) = \mathfrak{m}$. However, an easy computation shows that $\text{ann}_r\ \mathfrak{m} = 0$; in particular, $\text{ann}_\ell(\text{ann}_r\ \mathfrak{m}) = R \neq \mathfrak{m}$, so the double-annihilator condition fails for the left ideal $\mathfrak{m}$. Secondly,

$$\text{soc}({}_R R) = \text{ann}_r\ J = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}, \qquad \text{soc}(R_R) = \text{ann}_\ell\ J = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix},$$

so the two socles are not the same. If $V,\ V'$ denote the two simple right R-modules defined in (8.29)(5) (with $V = R/\mathfrak{m}$), then $\text{soc}(R_R) \cong V \oplus V$. Therefore, $V^* \neq 0$, but $V'^* = 0$.

(2) Let $R = S/\mathfrak{B}$ where S is a Dedekind ring and $\mathfrak{B} \subseteq S$ is a nonzero ideal. By Exercise (3.13), R is self-injective. Since R is clearly noetherian, it is QF.

(3) Let $R = \prod_{i=1}^n R_i$. Then *R is* QF *iff each R_i is* QF. This follows readily from (15.1) and (3.11B).

(4) If R is QF, then so is any matrix ring $\mathbb{M}_n(R)$. This can be proved as a somewhat intriguing exercise; or more conceptually, it can be deduced easily from the theory of category equivalences to be developed in the next chapter.

(5) (Nakayama) If A is a QF ring and G is any finite group, then the group ring AG is also QF: this is part of Exercise 14 below.

(6) We have noted before that a self-injective ring may not be artinian or noetherian, so it may not be QF. However, for certain classes of rings, the chain conditions may be implied by self-injectivity. We mention without proof the following two interesting results in this direction. First, J. Lawrence has shown that any *countable* right self-injective ring is QF. Second, S. P. Smith and J. J. Zhang have shown that any self-injective graded algebra $\bigoplus_{i\geq 0} A_i$ over a field k with $A_0 = k$ (and $A_iA_j \subseteq A_{i+j}$) is also QF.

Let us now go over to the commutative case. While it is true that commutative QF rings do not reveal all the subtleties of general QF rings, the former have the advantage of being much easier to handle. We have the following standard characterization theorem for these rings.

(15.27) Theorem. *For any commutative ring R, the following are equivalent:*

(1) *R is* QF*;*
(2) *R is an artinian ring, and* $\text{soc}(R)$ *is "square-free" (i.e., it contains no more than one copy of each simple R-module);*
(3) *$R \cong R_1 \times \cdots \times R_s$, where each R_i is a local artinian ring with a simple socle.*

Proof. (1) $\Longrightarrow$ (2). Suppose $\text{soc}(R)$ contains $A \oplus B$, where A, B are isomorphic simple R-modules. Let $\varphi : A \longrightarrow B$ be an R-isomorphism. Since R_R is injective, φ is given by multiplication by an element $r \in R$. But then $B = rA \subseteq A$, a contradiction.

(2) $\Longrightarrow$ (3). Let $1 = e_1 + \cdots + e_s$ be a decomposition of 1 into a sum of orthogonal primitive idempotents. Let $R_i = e_iR$. By *FC*–(21.18), each R_i is a local (artinian) ring, and R is isomorphic to the direct product of rings $R_1 \times \cdots \times R_s$. Since R_i has only one simple module, (2) clearly implies that R_i has a simple socle.

(3) $\Longrightarrow$ (1). In view of (15.26)(3), it suffices to show that each R_i is QF. Therefore, we may assume that R is *local,* with $V := \text{soc}(R)$ simple. Since R is artinian, every nonzero ideal contains a minimal ideal, which must be V. Hence $V \subseteq_e R$, and we may assume that R is contained in the injective hull $E := E_R(V)$. Now V is (up to isomorphism) the unique simple R-module, so by (3.64)(2), $\text{length}_R(E) = \text{length}_R(R)$. Therefore, we must have $R = E$. This shows that R is self-injective, hence QF. □

In the above, we have established (1) $\Longleftrightarrow$ (3) in the commutative case. This equivalence turns out to have a noncommutative analogue, but the proof is considerably harder: see (16.4) below (and for the local case, Exercise (16.1)).

From the above theorem, we see that, in the category of commutative rings, *the local artinian rings with simple socles*[89] *are the building blocks for* QF *rings.* Such local artinian rings are of basic importance, and are known to commutative algebraists and algebraic geometers as "zero-dimensional local Gorenstein rings." These are the zero-dimensional noetherian local rings $(R, \mathfrak{m})$ for which the injective hull $E := E_R(R/\mathfrak{m})$ in §3I is free of rank 1: see Exercise (16.1) below. (In §3I, we called E the "standard module" over R; in this case, E is exactly what commutative algebraists call the "canonical module" of the Gorenstein ring R.) Generally, zero-dimensional local Gorenstein rings arise in abundance, for instance, as quotients of regular local rings R modulo an ideal generated by a maximal R-sequence. For details on this and much more related information in the commutative case "with a view toward algebraic geometry", see Chapter 21 in Eisenbud [95].

In the case of commutative finite-dimensional algebras over a field k, the "building blocks" referred to in the last paragraph are precisely the (commutative) local Frobenius k-algebras (see Exercise (3.14)). We close this subsection by making a list of examples of such algebras.

(1) Finite field extensions of k.

(2) Trivial extensions of the type constructed in (3.15C).

(3) Group algebras kG, where G is a finite abelian p-group and k has prime characteristic p. (See (3.15E) and *FC*–(19.11).)

(4) The algebra $R = k[x_1, \ldots, x_r]$ defined by the relations $x_1^{n_1} = \cdots = x_r^{n_r} = 0$, where all $n_i > 0$. (See (3.15B).) Its k-dimension is $n_1 \cdots n_r$. In fact, R is just the tensor product of the local Frobenius k-algebras $k[t_i]/(t_i^{n_i})$.

(5) The algebra $S = k[x, y]$, defined by the relations $xy = x^2 - y^2 = 0$. Note that in S, $x^3 = x \cdot x^2 = xy^2 = 0$, and similarly $y^3 = 0$. Thus, $\dim_k S = 4$, and S is local with maximal ideal (x, y). The socle of S is easily seen to be $k \cdot x^2 = k \cdot y^2$, so S is a local Frobenius k-algebra.

(6) The algebra $W = k[x, y, z]$, defined by the relations

$$x^2 = y^2 = xz = yz = xy - z^2 = 0.$$

This is a 5-dimensional local algebra with maximal ideal (x, y, z), and with a simple socle $k \cdot xy = k \cdot z^2$, so W is again a Frobenius k-algebra; see Exercise 24.

A general method for constructing commutative local Frobenius algebras is sketched in Exercise 21. The examples (4), (5), and (6) above are all special cases of this construction: see Exercise 24.

[89]For readers familiar with the notion of subdirect products (see *FC*–§12), it is worth pointing out that, among commutative artinian rings, the ones with simple socles are precisely the subdirectly irreducible ones.

Exercises for §15

1. Redo Exercise (3.2)(3) ("A right self-injective domain R is a division ring") using the idea from Step 2 in the proof of (2)$\Longrightarrow$(4) in (15.1).)

2. Give an example of a commutative ring R with two ideals A, B such that $\text{ann}(A) + \text{ann}(B)$ is properly contained in $\text{ann}(A \cap B)$.

3. Show that if R has ACC on left annihilators and is right self-injective, then R is QF. (**Comment.** The statement is also true upon replacing "left annihilators" by "right annihilators". The proof is quite a bit harder, requiring Bass' Theorem *FC*–(23.20) for right perfect rings.)

4. Show that R is QF iff every right ideal is the right annihilator of a finite set and every left ideal is the left annihilator of a finite set.

5. Show that a quotient R/I of a QF ring R need not be QF.

6. Let C be a cyclic right R-module, say, $C = R/A$ where A is a right ideal in R.

   (1) Show that $C^* \cong \text{ann}_\ell(A)$ as left R-modules.
   (2) Show that C is torsionless (i.e., the natural map $\varepsilon : C \to C^{**}$ is an injection) iff A is a right annihilator.
   (3) Show that C is reflexive iff A is a right annihilator and every left R-homomorphism $\text{ann}_\ell(A) \to {}_R R$ is given by right multiplication by an element of R.

7. Show that a right noetherian ring R is QF iff every 1-sided cyclic R-module is torsionless.

8. Let $e = e^2 \in R$, and $J \subseteq R$ be a right ideal such that $eJ \subseteq J$. Use Exercise 6 to show that $(eR/eJ)^* \cong \text{ann}_\ell(J) \cdot e$ as left R-modules.

9. Let R be a QF ring and $J = \text{rad}(R)$. In (15.7), a proof is given for $\text{ann}_\ell(J) = \text{ann}_r(J)$ using the equality of the right and left socles of R. Give another proof for $\text{ann}_\ell(J) = \text{ann}_r(J)$ by using the last exercise.

10. (Nakayama) For (R, J) in the last exercise, show that $\text{ann}_\ell(J^n) = \text{ann}_r(J^n)$ for any positive integer n.

11. Show that a QF ring is right semihereditary iff it is semisimple.

12. Assuming the Faith-Walker Theorem (see (15.10)), show that a ring R is QF iff every module M_R embeds into a free R-module.

13. For any QF ring R, show that:

    (1) For any simple module S_R, the injective hull $E(S)$ is a principal indecomposable R-module;
    (2) For any f.g. module M_R, $E(M)$ is also f.g.

14. (Nakayama, Connell) Let R be a group ring AG, where A is a ring and G is a finite group. Show that R is right self-injective (resp. QF) iff A is.

15. Let R be a QF ring. Show that, for any central multiplicative set $S \subseteq R$, the localization RS^{-1} is also a QF ring.

16. Show that an idempotent e in a QF ring R is central iff eR is an ideal of R.

17. For any module M_R over a QF ring R, show that $\mathrm{pd}_R(M)$ (the projective dimension of M) is either 0 or ∞. Prove the same thing for $\mathrm{id}_R(M)$ (the injective dimension of M).

18. (Bass) Let R be a left noetherian ring such that, for any f.g. module M_R, $\mathrm{pd}_R(M)$ is either 0 or ∞. Show that R is a left Kasch ring. (**Hint.** Use the Unimodular Column Lemma (Exercise (1.34)) to show that any left ideal $\sum_{i=1}^{n} Ra_i \subsetneq R$ has a nonzero right annihilator.)

19. Let R be a commutative noetherian ring in which the ideal (0) is meet-irreducible. Show that $Q_{cl}(R)$ is a (commutative) local QF ring.

20. For any field k, let $R = k[u, v]$, with the relations $u^2 = v^2 = 0$, and $S = k[x, y]$, with the relations $xy = x^2 - y^2 = 0$. By (4) and (5) at the end of §15D, R and S are (commutative) 4-dimensional local Frobenius k-algebras. Show that $R \cong S$ as k-algebras iff $-1 \in k^2$ and $\mathrm{char}(k) \neq 2$. (In particular, $R \cong S$ if $k = \mathbb{C}$, and $R \ncong S$ if $k = \mathbb{R}$.)

The next exercise describes a general method for constructing quotients of a polynomial ring that are local Frobenius algebras. The remaining exercises amplify this point, and provide further explicit computational examples for the construction.

21. Let $A = k[x_1, \ldots, x_r]$ (where k is a field), and let $\mathfrak{m} = (x_1, \ldots, x_r)$. Let $\lambda : A \to k$ be a k-linear functional with $\ker(\lambda) \supseteq \mathfrak{m}^n$ for some $n \geq 1$ and let J_λ be the largest ideal of A that is contained in $\ker(\lambda)$. If $\lambda \neq 0$, show that $J_\lambda \subseteq \mathfrak{m}$ and that A/J_λ is a local Frobenius k-algebra. Conversely, if $J \subseteq \mathfrak{m}$ is an ideal in A such that A/J is a local Frobenius k-algebra, show that J has the form J_λ for some nonzero functional λ as described above. (**Hint.** Use the characterization of Frobenius algebras given in (3.15)(3).)

22. Keep the notations in the last exercise, and let N be the space of functionals

$$\{\lambda \in \mathrm{Hom}_k(A, k) : \ \lambda(\mathfrak{m}^n) = 0 \text{ for some } n \geq 1\},$$

viewed as an A-submodule of the A-module $\mathrm{Hom}_k(A, k)$. For any $\lambda \in N$, let $\mathrm{ann}^A(A\lambda)$ denote the A-annihilator of the cyclic submodule $A\lambda \subseteq {}_AN$, and let $\mathrm{Ann}^A(A\lambda)$ denote the space of common zeros of the linear functionals in $A\lambda$. Show that $\mathrm{ann}^A(A\lambda) = \mathrm{Ann}^A(A\lambda) = J_\lambda$, and deduce that $A\lambda \cong A/J_\lambda$ as A-modules.

23. In the notations of the last two exercises, let $\mathrm{ann}^N(J_\lambda)$ be the annihilator of J_λ in the A-module N, and let $\mathrm{Ann}^N(J_\lambda)$ be the space of functionals in

N vanishing on the ideal J_λ. Show that $\text{ann}^N(J_\lambda) = \text{Ann}^N(J_\lambda) = A\lambda$, and deduce that, for any $\lambda, \mu \in N$, $J_\lambda = J_\mu$ iff $A\lambda = A\mu$. (**Comment.** Combining this with Exercise 21, we get a one-one correspondence between the nonzero cyclic A-submodules of N and the ideals $J \subseteq (x_1, \ldots, x_r)$ of A for which A/J is a local Frobenius k-algebra.)

24. Keeping the notations in the above exercises, let us identify the space of functionals N with the A-module of "inverse polynomials" $T = k[x_1^{-1}, \ldots, x_r^{-1}]$, as in (3.91)(1). Take the "functionals"

$$\lambda_1 = x^{-1}y^{-2}, \quad \lambda_2 = x^{-2} + y^{-2}, \quad \lambda_3 = x^{-1}y^{-1} + z^{-2}$$

in $T = k[x^{-1}, y^{-1}]$ and $T = k[x^{-1}, y^{-1}, z^{-1}]$ respectively, and show that

$$J_{\lambda_1} = (x^2, y^3), \quad J_{\lambda_2} = (xy,\ x^2 - y^2) \text{ and } J_{\lambda_3} = (x^2, y^2, xz, yz, xy - z^2).$$

State a generalization for each of these three cases. (For further amusement, make up your own nonzero inverse polynomial $\lambda \in T$, and come up with a local Frobenius algebra A/J_λ that no one has set eye upon before!)

25. For the ideal $J := (y^3, x^2 - xy^2)$, in $A = k[x, y]$, show that A/J is a 6-dimensional local Frobenius k-algebra, and find a linear functional $\lambda \in k[x^{-1}, y^{-1}]$ such that $J = J_\lambda$.

§16. Frobenius Rings and Symmetric Algebras

§16A. The Nakayama Permutation

In order to introduce the notion of Frobenius rings, we shall first develop in more detail some special properties of a QF ring R. Recall from (15.13) that if M is a simple right (resp. left) R-module, then its first dual, M^*, is a simple left (resp. right) R-module. Let us now prove that, for artinian rings, this statement actually *characterizes* a QF ring. This result is due to J. Dieudonné. We begin with a lemma.

(16.1) Lemma. *Let $A \subseteq B$ be right ideals in a ring R such that $(B/A)^*$ is either* (0) *or a simple left R-module. Then* $\text{ann}_\ell(A)/\text{ann}_\ell(B)$ *is either* (0) *or isomorphic to* $(B/A)^*$.

Proof. We can define a map $f : \text{ann}_\ell(A) \to (B/A)^*$ by $f(x)(b + A) = xb$ for $x \in \text{ann}_\ell(A)$ and $b \in B$. This is easily checked to be a left R-homomorphism, and its kernel is $\text{ann}_\ell(B)$. Therefore, $\text{ann}_\ell(A)/\text{ann}_\ell(B)$ embeds into $(B/A)^*$. Since $(B/A)^*$ is either (0) or a simple module, the desired conclusion follows. □

(16.2) Theorem (Dieudonné). *An artinian ring R is QF iff the dual of any simple 1-sided R-module is either* (0) *or simple. In this case, $M \mapsto M^*$ gives a one-one*

correspondence between the isomorphism classes of simple left R-modules and simple right R-modules.

Proof. The "only if" part and the last statement of the theorem are already contained in (15.12) and (15.13). For the converse, assume that R is an artinian ring such that the dual of any simple 1-sided R-module is either (0) or simple. Consider any composition series

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = R$$

of R_R. By assumption, each $(A_{i+1}/A_i)^*$ is either (0) or simple. Therefore, by (16.1), $\text{ann}_\ell(A_i)/\text{ann}_\ell(A_{i+1})$ is either (0) or a simple left R-module. Thus, if we ignore possible collapsings,

$$(*) \qquad 0 = \text{ann}_\ell(A_n) \subseteq \cdots \subseteq \text{ann}_\ell(A_i) \subseteq \cdots \subseteq \text{ann}_\ell(A_0) = R$$

is a composition series for ${}_RR$. Hence we have $\text{length}({}_RR) \leq \text{length}(R_R)$. By symmetry, we also have the reverse inequality, so equality holds. In particular, there is *no* collapsing possible in (*), and it is a composition series for ${}_RR$ as it stands. Applying the same argument on (*), we see that

$$0 = \text{ann}_r(\text{ann}_\ell(A_0)) \subseteq \cdots \subseteq \text{ann}_r(\text{ann}_\ell(A_i)) \subseteq \cdots \subseteq \text{ann}_r(\text{ann}_\ell(A_n)) = R$$

is also a composition series for R_R. Since $A_i \subseteq \text{ann}_r(\text{ann}_\ell(A_i))$, it follows that equality holds for each i. Now any right ideal in R is part of a composition series, so we have proved the double-annihilator property for right ideals. By symmetry, the same holds for left ideals. According to (15.1), R must be QF. □

(16.3) Remark. Clearly, the same argument can be used to give a slightly different characterization of QF rings using only the duals of simple *right* modules. If we assume from the outset that $\text{length}({}_RR) \leq \text{length}(R_R)$, then a criterion for QF is that the dual of any simple right R-module be either (0) or simple.

Recall that, for an artinian ring, a right principal indecomposable R-module is a module of the form eR where e is a primitive idempotent (i.e., a nonzero idempotent which is not the sum of two nonzero orthogonal idempotents). In the following, we shall write $J = \text{rad}\, R$ (the Jacobson radical of R), and $\bar{R} = R/J$ (a semisimple ring). For the principal indecomposable module eR above, it is easy to check that eJ is a (unique) maximal submodule of eR, with $eR/eJ \cong \bar{e}\bar{R}$; furthermore, up to isomorphism, *every* simple right R-module has this form (see *FC*–(21.18), (21.22), or (25.2)). Similar results hold, of course, for left principal indecomposable modules.

Let us now prove the following new characterization of a QF ring.

(16.4) Theorem. *Let R be an artinian ring. Then R is* QF *iff R is Kasch and every 1-sided principal indecomposable R-module has a simple socle.*

Proof. First assume R is QF. By the proof of (15.1), R is Kasch. For any primitive idempotent, consider the principal indecomposable right R-module eR. Since eR is projective, by (15.9)(2) it is also injective. Let M be a simple submodule of eR. Clearly eR must be the injective hull of M, so M is essential in eR. In particular, $\text{soc}(eR) = M$ is simple. By symmetry, $\text{soc}(Re)$ is also simple.

Conversely, assume that R is Kasch and that every 1-sided principal indecomposable R-module has a simple socle. We will show that R is QF by applying the criterion in (16.2). We proceed in a number of steps.

Step 1. $\text{soc}(R_R) = \text{soc}({}_RR)$. Let e be any primitive idempotent. By assumption, $\text{soc}({}_RR)$ contains a copy of the simple module $\bar{R}\bar{e}$, so $e \cdot \text{soc}({}_RR) \neq 0$. Since $\text{soc}({}_RR)$ is an ideal, $e \cdot \text{soc}({}_RR)$ is a nonzero right submodule of eR. The simplicity of $\text{soc}(eR)$ then yields $\text{soc}(eR) \subseteq e \cdot \text{soc}({}_RR) \subseteq \text{soc}({}_RR)$. Now R_R is a direct sum of principal indecomposables of the form eR (for a finite set of primitive idempotents $\{e\}$), so $\text{soc}(R_R)$ is a direct sum of $\text{soc}(eR)$'s (see Exer. (6.12)(6)). This shows that $\text{soc}(R_R) \subseteq \text{soc}({}_RR)$, and by symmetry $\text{soc}({}_RR) \subseteq \text{soc}({}_RR)$.

Step 2. Let M be any simple right R-module. By assumption, M has an embedding into R_R, and hence also an embedding into a suitable principal indecomposable eR. Thus, we may assume that $M = \text{soc}(eR)$ (since $\text{soc}(eR)$ is simple). We claim that

$$M^* = (\text{soc}(eR))^* \cong \bar{R}\bar{e}. \tag{16.5}$$

This will show that M^* is simple, and by symmetry, the R-dual of any simple left R-module will also be simple. Thus, (16.2) *applies to show that R is* QF.

Step 3. Let $\sigma : Re \to M^*$ be the left R-homomorphism sending re to the left multiplication by re (on M). For $J = \text{rad}(R)$, we have (using Step 1):

$$\sigma(Je)(M) = J(eM) \subseteq J \cdot \text{soc}(R_R) = J \cdot \text{soc}({}_RR) = 0.$$

Thus, σ induces a homomorphism $Re/Je \to M^*$. Since $Re/Je \cong \bar{R}\bar{e}$ is simple, (16.5) will follow if we can show that σ is *onto*.

Step 4. To show that σ is onto, let $M \cong \bar{f}\bar{R}$, where f is a primitive idempotent. Say $s \in M$ corresponds to $\bar{f}$ under this isomorphism. Then $M = sR$, and $s = sf$ (since $sf \in M$ corresponds to $\bar{f}\bar{f} = \bar{f}$ also). Given any nonzero $\varphi \in M^*$, let

$$t := \varphi(s) = \varphi(sf) = tf \in \varphi(M) \cong M.$$

Using Step 1, we have

$$s \in Rf \cap M \subseteq Rf \cap \text{soc}(R_R) = Rf \cap \text{soc}({}_RR) \subseteq \text{soc}(Rf),$$

and similarly $t \in \text{soc}(Rf)$. Since $\text{soc}(Rf)$ is simple, we have $Rs = \text{soc}(Rf) = Rt$. In particular, $t = rs$ for some $r \in R$. Recalling that $M \subseteq eR$, we have, for any $x \in R$:

$$\varphi(sx) = \varphi(s)x = tx = rsx = (re)(sx).$$

Thus φ is just left multiplication by re, as desired. □

Remark. The example (15.26)(1) of the triangular matrix ring $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ (where k is a division ring) shows that the Kasch condition in Theorem (16.4) cannot be omitted. Also, the condition on the simplicity of the socles in (16.4) must be imposed on *both* right and left principal indecomposables. There do exist artinian local (necessarily Kasch) rings for which the left socle is simple but the right socle is not. Such rings are *not* QF: see Exercise 2 below.

(16.6) Corollary. *Let e, f be primitive idempotents in a* QF *ring R such that* $\operatorname{soc}(eR) \cong \bar{f}\bar{R}$. *Then:*

(1) $\operatorname{soc}(Rf) \cong \bar{R}\bar{e}$;
(2) $(\bar{f}\bar{R})^* \cong \bar{R}\bar{e}$;
(3) $(\bar{R}\bar{e})^* \cong \bar{f}\bar{R}$.

Proof. (2) is just (16.5), and (3) follows from (2) by taking the dual (and using (15.11)(2)). To prove (1), we use the left module analogue of (16.5) to get $\operatorname{soc}(Rf)^* \cong \bar{f}\bar{R}$. Taking the dual and using (2), we get $\operatorname{soc}(Rf) \cong (\bar{f}\bar{R})^* \cong \bar{R}\bar{e}$. □

For any artinian ring R, let

(16.7) $$1 = e_{11} + \cdots + e_{1n_1} + \cdots + e_{s1} + \cdots + e_{sn_s} \qquad (n_i > 0)$$

be a decomposition of 1 into a sum of orthogonal primitive idempotents, where $e_i := e_{i1}$ $(1 \le i \le s)$ are mutually nonisomorphic (meaning that $e_iR \not\cong e_jR$ or equivalently $Re_i \not\cong Re_j$ for $i \ne j$: see *FC*–(21.20)), but e_i is isomorphic to each $e_{i\ell}$. Let

(16.8) $$U_i = e_iR, \quad S_i = \bar{e}_i\bar{R}; \quad U_i' = Re_i, \quad S_i' = \bar{R}\bar{e}_i.$$

Thus, $\{U_i\}$ (resp. $\{U_i'\}$) is a complete set of right (resp. left) principal indecomposables, and $\{S_i\}$ (resp. $\{S_i'\}$) is a complete set of simple right (resp. left) R-modules. Note that, since

$$R_R \cong n_1U_1 \oplus \cdots \oplus n_sU_s \Longrightarrow \bar{R}_{\bar{R}} \cong n_1S_1 \oplus \cdots \oplus n_sS_s,$$

the Wedderburn-Artin theorem gives a ring isomorphism

(16.8′) $$\bar{R} \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_s}(D_s),$$

where D_i is the division ring $\operatorname{End}_{\bar{R}}(S_i) = \operatorname{End}_R(S_i)$.

In case R is a QF ring, define a map $\pi : \{1, \ldots, s\} \to \{1, \ldots, s\}$ by

(16.9) $$\operatorname{soc}(U_i) \cong S_{\pi(i)} \qquad (1 \le i \le s).$$

Then by (16.6), we have

(16.10) $$\operatorname{soc}(U'_{\pi(i)}) \cong S_i', \qquad S^*_{\pi(i)} \cong S_i', \qquad (S_i')^* \cong S_{\pi(i)}.$$

From the last two isomorphisms (or else), it is clear that π is a permutation of $\{1, \ldots, s\}$. This is called the *Nakayama permutation* (of the QF ring R). Of course, the principal indecomposables and the corresponding simple modules could have been labeled in any way; thus, π is only determined up to a conjugation.

As a bonus of the above considerations, we obtain the following characterization of a QF ring that was actually used as its definition by T. Nakayama (ca. 1941).

(16.11) Corollary. *An artinian ring R is* QF *iff there exists a permutation π of $\{1, \ldots, s\}$ such that* $\mathrm{soc}(U_i) \cong S_{\pi(i)}$ *and* $\mathrm{soc}(U'_{\pi(i)}) \cong S'_i$ *for every i.*

Proof. ("If" part) Suppose π exists with the stated properties. Then $S_{\pi(i)}$ and S'_i can be embedded in R so R is Kasch. Also, the given isomorphisms imply that the principal indecomposables have simple socles. Therefore, R is QF by (16.4). □

After the above discussion on the Nakayama permutation, a small question remains. What happens when we take the R-duals of the principal indecomposables? We call this a small question since it turns out to have a small answer:

(16.12) Proposition. *Over a* QF *ring R, we have $U_i^* \cong U_i'$ and $(U_i')^* \cong U_i$.*

This is a special case of the following more general result.

(16.13) Theorem. (1) *If R is a right self-injective ring, then for any $a \in R$, $(aR)^* \cong Ra$ as left R-modules.* (2) *If R is a self-injective ring, then principal 1-sided ideals of R are reflexive, and for $a, b \in R$, $aR \cong bR$ iff $Ra \cong Rb$.*

Proof. (1) Consider the exact sequence

$$0 \longrightarrow \mathrm{ann}_\ell(a) \longrightarrow R \stackrel{g}{\longrightarrow} Ra \longrightarrow 0 \qquad (g(x) = xa),$$

which yields

$$Ra \cong R/\mathrm{ann}_\ell(a) = R/\mathrm{ann}_\ell(aR) \cong (aR)^*,$$

where the last isomorphism follows from the second half of (15.4) (applicable as long as R_R is injective). Note that the isomorphism $\varphi : Ra \to (aR)^*$ obtained here is given by $\varphi(ya)(az) = yaz$ for $y, z \in R$.

(2) Assume now R is self-injective. Informally, we have

$$(aR)^{**} \cong (Ra)^* \cong aR \qquad (\forall a \in R),$$

by first applying (1) to $(aR)^*$ and then applying it again to $(Ra)^*$. The remark made at the end of the paragraph above enables us to check that this, indeed, means that the natural map $\varepsilon : aR \to (aR)^{**}$ is an isomorphism. By left-right symmetry, $\varepsilon' : Ra \to (Ra)^{**}$ is also an isomorphism. The last statement in (2) follows by taking the first duals. □

§16B. Definition of a Frobenius Ring

We are now ready to introduce the definition of a Frobenius ring (as a specialization of a quasi-Frobenius ring). We do this by stating several equivalent conditions. (The basic notations in (16.7) and (16.8) will remain in force.)

(16.14) Theorem. *For any artinian ring R, let $\bar{R} = R/J$, where $J = \mathrm{rad}(R)$. Then the following are equivalent:*

(1) *R is* QF *and* $\mathrm{soc}(R_R) \cong \bar{R}_R$.
(2) *R is* QF *and* $\mathrm{soc}({}_RR) \cong {}_R\bar{R}$.
(3) *R is* QF *and $n_i = n_{\pi(i)}$ $(1 \le i \le s)$ in the notation of* (16.7), *where π is the Nakayama permutation of R.*
(4) $\mathrm{soc}(R_R) \cong \bar{R}_R$ *and* $\mathrm{soc}({}_RR) \cong {}_R\bar{R}$.

If R satisfies these equivalent conditions, it is said to be a Frobenius ring.[90]

Proof. First assume R is QF. From (16.7), $R_R \cong \bigoplus_i n_i U_i$. Computing the socle using (16.9) and Exercise (6.12)(6), we have

$$\mathrm{soc}(R_R) \cong \bigoplus_i n_i \cdot S_{\pi(i)}. \tag{16.15}$$

Comparing this with $\bar{R}_R \cong \bigoplus_i n_i \cdot S_i$, we get (1)$\Longleftrightarrow$(3), and (2)$\Longleftrightarrow$(3) follows by symmetry. (Of course, it is crucial to note that the decomposition (16.7) is independent of side.) Clearly, (1),(2),(3) imply (4), so it only remains to show that (4) $\Longrightarrow$ R is QF. Assume (4). Since all S_i's appear in $\bar{R}_R$, they also appear in $\mathrm{soc}(R_R)$, so R is right Kasch. Similarly, R is left Kasch. Furthermore,

$$\bigoplus_i n_i \cdot S_i \cong \bar{R}_R \cong \mathrm{soc}(R_R) \cong \bigoplus_i n_i \cdot \mathrm{soc}(U_i).$$

The LHS has length $\sum_i n_i$, and the RHS has length $\sum_i n_i \cdot \mathrm{length}(\mathrm{soc}(U_i))$. Since each $\mathrm{length}(\mathrm{soc}(U_i)) > 0$, we must have $\mathrm{length}(\mathrm{soc}(U_i)) = 1$, so $\mathrm{soc}(U_i)$ is simple for all i. Similarly, $\mathrm{soc}(U_i')$ is simple for all i, so by Theorem (16.4), R is QF. □

(16.16) Corollary. *A* QF *ring R is a Frobenius ring iff $\bar{R}_R \cong ({}_R\bar{R})^*$ (as right R-modules), iff ${}_R\bar{R} \cong (\bar{R}_R)^*$ (as left R-modules).*

Proof. For the QF ring R, recall the following (R, R)-bimodule isomorphisms from (15.25):

$$\mathrm{soc}(R) \cong ({}_R\bar{R})^* \cong (\bar{R}_R)^*, \tag{16.17}$$

where $\mathrm{soc}(R)$ denotes the common socle $\mathrm{soc}(R_R) = \mathrm{soc}({}_RR)$. Specializing the above isomorphisms to one side, we see that the second condition in (16.14)(1)

[90] In §3B, we have defined separately the notion of *Frobenius k-algebras* over a field k. Fortunately, this double usage of the term "Frobenius" turns out to be harmless. In §16C, it will be shown that the Frobenius k-algebras in §3B are precisely the finite-dimensional k-algebras that are Frobenius rings.

translates into $\bar{R}_R \cong ({}_R\bar{R})^*$ (as right modules), and the second condition in (16.14)(2) translates into ${}_R\bar{R} \cong (\bar{R}_R)^*$ (as left modules). □

(16.18) Corollary. *Suppose the* QF *ring R is such that $\bar{R} \cong \mathbb{M}_n(D_1) \times \cdots \times \mathbb{M}_n(D_s)$ where the D_i's are division rings. Then R is a Frobenius ring. (In particular, this is the case if $\bar{R}$ is either a commutative ring or a simple ring.)*

Proof. The hypothesis on $\bar{R}$ implies that $n_1 = \cdots = n_s = n$ in the notation of (16.7) and (16.8′), so the condition (3) in (16.14) is automatic. Hence R is a Frobenius ring. If $\bar{R}$ is a simple ring, then $s = 1$ and we are certainly in the present case. If $\bar{R}$ is a commutative ring instead, then $n_1 = \cdots = n_s = 1$ (from (16.8′)) so we are again in the case of this corollary. □

(16.19) Examples.

(1) If $R = S/\mathfrak{B}$ where S is a Dedekind ring and $\mathfrak{B} \subseteq S$ is a nonzero ideal, then R is QF by (15.26)(2) and hence Frobenius by (16.18).

(2) *Any semisimple ring R is always Frobenius:* this follows readily from the criterion (16.14)(4) since $\bar{R} = R/\text{rad}\, R = R$ and $\text{soc}(R) = R$. Alternatively, in the notations of (16.7) and (16.8), we have $U_i = S_i$ and $U_i' = S_i'$. These imply that the Nakayama permutation of R is the identity; clearly, any QF ring with such a property is a Frobenius ring, by the criterion (16.14)(3).

(3) It is easy to check directly that a finite direct product $R = \prod_{i=1}^{s} R_i$ is a Frobenius ring iff each factor R_i is.

(4) For any division ring k, let R be the 4-dimensional k-ring consisting of matrices of the form

$$\gamma = \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{pmatrix}.$$

The Jacobson radical J consists of all matrices in R with a zero diagonal. The quotient $\bar{R} = R/J \cong k \times k$, with the isomorphism given by $\bar{\gamma} \mapsto (a, b)$. In the notation of (16.7), we can take

$$e_1 = \text{diag}(1, 0, 0, 1) \quad \text{and} \quad e_2 = \text{diag}(0, 1, 1, 0),$$

which give rise to the right principal indecomposables:

$$U_1 = e_1 R = \left\{ \begin{pmatrix} a & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \right\}, \quad \text{and} \quad U_2 = e_2 R = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

each 2-dimensional over k. We have

$$U_1J = \left\{\begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}, \quad \text{and} \quad U_2J = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}.$$

Let $S_1 = U_1/U_1J$, $S_2 = U_2/U_2J$ be the two simple right R-modules. Since $(uE_{12})\gamma = ubE_{12}$ and $(vE_{34})\gamma = vaE_{34}$ (E_{ij}'s being the matrix units), we see that

$$\text{soc}(U_1) = U_1J \cong S_2 \quad \text{and} \quad \text{soc}(U_2) = U_2J \cong S_1.$$

By (16.4), R is QF with Nakayama permutation given by the transposition (12). Since $R_R \cong U_1 \oplus U_2$, R is necessarily a Frobenius ring. Note that in this example, we have $J^2 = 0$ and $\text{soc}(R) = U_1J \oplus U_2J = J$.

One nice thing about this example is that it generalizes easily to an example of a Frobenius ring with an n-cycle as its Nakayama permutation. We need only take R to be the set of $2n \times 2n$ matrices with 2×2 diagonal blocks

$$\begin{pmatrix} a_1 & x_1 \\ 0 & a_2 \end{pmatrix}, \quad \begin{pmatrix} a_2 & x_2 \\ 0 & a_3 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} a_n & x_n \\ 0 & a_1 \end{pmatrix},$$

where all entries are chosen from k. This new ring R has dimension $2n$ over k, and is the direct sum of the distinct right principal indecomposable modules $U_i = e_iR$, where e_i is the primitive idempotent in R obtained by setting all $x_j = 0$ and $a_j = \delta_{ij}$. Exactly the same computation as given above shows that R is a Frobenius ring with a Nakayama permutation given by the n-cycle $(12\cdots n)$.

In the case when k is a field, one can also show directly that R is a Frobenius k-algebra in the sense of §3. A nonsingular bilinear pairing $B : R \times R \to k$ with the associativity property will be computed in (16.51) below.

It is also worth noting that, by coupling this example with a direct product construction, we can construct a Frobenius ring whose Nakayama permutation is a prescribed permutation in the symmetric group on any (finite) number of letters. See Exercise 3.

(5) Many books contained material on QF rings and Frobenius rings, but few of them offered any worked out examples of QF rings which are not Frobenius rings! We present such an example, from Nakayama's original paper [39]. Let k be any division ring, and let R be the subset of $\mathbb{M}_6(k)$ consisting of matrices of the form

(A)
$$\gamma = \begin{pmatrix} a & b & p & 0 & 0 & 0 \\ c & d & q & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & s & t \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{pmatrix}.$$

It is easy to check that R is a ring containing the scalar matrices $k \cdot I_6$. We can compute the U_i 's and S_i 's as follows (using the notations in (16.7) and (16.8)). The Jacobson radical rad(R) is given by the ideal J of R consisting of γ 's with

$a = b = c = d = r = 0$. This is clear since J is obviously nilpotent and $R/J \cong k \times \mathbb{M}_2(k)$ is semisimple. Let S_1, S_2 be the two simple right R/J-modules (viewed also as R-modules) labeled so that $\dim_k S_i = i$. We can think of S_1 as k with right action of γ above given by right multiplication by r, and S_2 as k^2 with right action of γ given by right multiplication by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have a decomposition $1 = e_1 + e_2 + e'_2$ where

(B) $$e_1 = E_{33} + E_{44}, \qquad e_2 = E_{11} + E_{55}, \qquad e'_2 = E_{22} + E_{66}$$

are orthogonal idempotents in R. (To save space, we shall henceforth express our matrices by using the matrix units E_{ij}.) By working in $\bar{R} = R/J$, we see that e_1, e_2, e'_2 are primitive idempotents in R with $S_1 \cong \bar{e}_1\bar{R}$ and $S_2 \cong \bar{e}_2\bar{R} \cong \bar{e}'_2\bar{R}$. (The fact that e_2, e'_2 are *isomorphic* idempotents in R can also be seen directly by noting that $e_2 = \alpha\beta$ and $e'_2 = \beta\alpha$ for $\alpha = E_{12} + E_{56}$ and $\beta = E_{21} + E_{65}$ in R.) A direct computation now yields the two (distinct) principal indecomposables $U_i = e_i R$:

(C)
$$U_1 = k(E_{33} + E_{44}) + kE_{45} + kE_{46}, \quad U_2 = k(E_{11} + E_{55}) + k(E_{12} + E_{56}) + kE_{13}.$$

For the record, we note that $e'_2R = k(E_{21} + E_{65}) + k(E_{22} + E_{66}) + kE_{23}$: we can "ignore" this one, since it is $\cong e_2R$. The maximal submodules in the U_i's are

(D) $$U_1J = kE_{45} + kE_{46} \quad \text{and} \quad U_2J = kE_{13},$$

with $U_i/U_iJ \cong S_i$. Now, for a general element $\gamma \in R$ as in (A), the actions of γ on U_iJ are as follows:

$$(xE_{45} + yE_{46})\gamma = (xa + yc)E_{45} + (xb + yd)E_{46}, \quad (zE_{13})\gamma = zrE_{13}.$$

Therefore, we have $U_1J \cong S_2$ and $U_2J \cong S_1$. In particular,

(E) $$\text{soc}(U_1) = U_1J \cong S_2 \quad \text{and} \quad \text{soc}(U_2) = U_2J \cong S_1,$$

so (16.4) *implies that R is* QF. The Nakayama permutation π is the transposition (12). Since $R_R \cong U_1 \oplus 2 \cdot U_2$, the multiplicity numbers in (16.7) are $n_1 = 1$ and $n_2 = 2$. This shows immediately that *R is not Frobenius*! We might also point out that, in this example,

(F) $$\text{soc}(R_R) \cong S_2 \oplus 2 \cdot S_1 \quad \text{and} \quad \bar{R}_R \cong S_1 \oplus 2 \cdot S_2.$$

These are indeed *not* isomorphic as right R-modules.

We mention in passing the notion of a Cartan matrix. For any right artinian ring, let $U_1, \ldots, U_s$ be a complete set of right principal indecomposables, and let S_i be the unique simple quotient of U_i, as in (16.8).

(16.20) Definition. The *(right) Cartan matrix* of R is the $s \times s$ matrix $(c_{ij}) \in \mathbb{M}_n(\mathbb{Z})$, where c_{ij} is the number of composition factors of U_i that are isomorphic to S_j. (The integers c_{ij} are called the *Cartan invariants* of R.) The left Cartan

matrix of R is defined similarly. (Of course, since we could have labeled the U_i 's in any way, these Cartan matrices are defined only up to a conjugation by a permutation matrix.)

For instance, the Cartan matrix of a *local* right artinian ring R is (c), where $c = \text{length}(R_R)$. The Cartan matrix of a right artinian ring R is the identity matrix iff R is semisimple. The Cartan matrix for the ring R in (16.19)(5) is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. And, for the (generalized) ring R in (16.19)(4), the Cartan matrix is

$$I_n + E_{12} + E_{23} + \cdots + E_{n-1,n} + E_{n1},$$

where the E_{ij} 's denote the matrix units. In general, the left and the Cartan matrices of an artinian ring may be different. For instance, the "Hint" for Exercise 2 below offers a local artinian with left Cartan matrix (2) and right Cartan matrix $(n + 1)$.

There are many interesting results on Cartan matrices. For instance, in the case of group algebras of finite groups over fields of characteristic p, the study of Cartan matrices is an important part of the modular representation theory of groups. Due to limitation of space, however, we shall not delve into this matter here.

§16C. Frobenius Algebras and QF Algebras

In §3, we have made an early introduction to a class of finite-dimensional algebras called *Frobenius algebras*, partly as a source of self-injective rings. We must now explain the relationship between these Frobenius algebras and the Frobenius rings defined in the last subsection.

Let R be a finite-dimensional algebra over a field k. Recall that the k-dual $\hat{R} := \text{Hom}_k(R, k)$ has a natural structure as an (R, R)-bimodule. For now we shall use the right R-structure on $\hat{R}$, which is defined via the equation $(fr)(r') = f(rr')$ for any $f \in \hat{R}$ and $r, r' \in R$. The two right R-modules R_R and $(\hat{R})_R$ have the same k-dimension so it is of interest to compare their isomorphism types. By definition, *R is a Frobenius k-algebra if $R_R \cong (\hat{R})_R$ as right R-modules.* Since $(\hat{R})_R$ is always an injective right R-module by (3.6C), R is a Frobenius algebra only if it is a right self-injective ring.

We shall now give the precise relationship between Frobenius rings and Frobenius algebras.

(16.21) Theorem. *Let R be a finite-dimensional algebra over a field k. Then R is a Frobenius k-algebra iff R is a Frobenius ring.*

In particular, this result shows that "R being a Frobenius algebra over k" is independent of the choice of the field k, as long as, of course, R is a finite-dimensional k-algebra. (This fact has been derived earlier by another argument: see the last paragraph of (3.41).) A similar statement applies to the notion of QF algebras to be introduced below, after (16.22).

Proof of (16.21). If R is a Frobenius k-algebra, then, as we have recalled above, R is a right self-injective ring. Since R is clearly right noetherian, R is a QF ring. Now let $\bar{R} = R/J$ where $J = \text{rad}(R)$. In (3.41), the socle $\text{soc}((\hat{R})_R)$ has been computed (for any finite-dimensional k-algebra R) to be isomorphic to $(\bar{R})_R$. In the present case, we have therefore

$$\bar{R}_R \cong \text{soc}((\hat{R})_R) \cong \text{soc}(R_R).$$

Thus, (16.14)(1) holds, so R is a Frobenius ring.

Conversely, suppose R is a Frobenius ring. Then R is QF, and we have $\bar{R}_R \cong \text{soc}(R_R)$. By (3.41), the injective hull $E(\bar{R}_R)$ is given by $(\hat{R})_R$. On the other hand, since R is right artinian, $\text{soc}(R_R)$ is essential in R_R. Since R_R is injective, the injective hull $E(\text{soc}(R_R))$ is given by R_R. Therefore, $\bar{R}_R \cong \text{soc}(R_R)$ leads to $(\hat{R})_R \cong R_R$ by taking injective hulls, so now R is a Frobenius k-algebra. □

(16.22) Corollary. *A commutative finite-dimensional algebra R over a field k is a Frobenius k-algebra iff R is a self-injective ring.*

Proof. This is now immediate from (16.21), since a commutative QF ring is always a Frobenius ring by (16.18). (Other proofs are possible too.) □

Classically, there is also a definition of QF algebras in generalization of that for Frobenius algebras, based on a comparison of the two right R-modules R_R and $(\hat{R})_R$. Instead of comparing their isomorphism types, we compare the isomorphism types of their (Krull-Schmidt) indecomposable components.

Definition. Let R be a finite-dimensional k-algebra. We say that R is a QF *algebra (over k)* if R_R and $(\hat{R})_R$ have the same distinct indecomposable components (occurring possibly with different multiplicities). Clearly, any Frobenius algebra is a QF algebra.

Fortuitously, we have the following complete analogue of Theorem (16.21).

(16.23) Theorem. *A finite-dimensional k-algebra R is a* QF *algebra iff it is a* QF *ring.*

Proof. First assume R is a QF algebra. Since $(\hat{R})_R$ is injective (by (3.6C)), so are its indecomposable components. By assumption, each indecomposable component of R_R is isomorphic to one of $(\hat{R})_R$, so R_R is also injective. Therefore, R is a QF ring. Conversely, assume R is a QF ring. Let $e_1, \ldots, e_s \in R$ be primitive idempotents as in the context of (16.8). We shall use the fact that the k-dual of an indecomposable left R-module is an indecomposable right R-module. Since R_R is injective, we have by (3.66):

$$R_R \cong m_1(Re_1)^\wedge \oplus \cdots \oplus m_s(Re_s)^\wedge \qquad (m_i \geq 0).$$

Here, the m_i 's must be all positive. (Every injective indecomposable $(Re_i)^\wedge$, being projective also, must show up in some free module R^n_R, and therefore must show up in R_R, by the Krull-Schmidt Theorem (*FC*–(19.23)).) On the other hand,

$$_RR \cong n_1(Re_1) \oplus \cdots \oplus n_s(Re_s) \quad (n_i > 0)$$

implies that

$$(\hat{R})_R \cong n_1(Re_1)^\wedge \oplus \cdots \oplus n_s(Re_s)^\wedge,$$

so $(\hat{R})_R$ has distinct indecomposable components $(Re_i)^\wedge$ $(1 \leq i \leq s)$, just as R_R does. Thus, R is a QF algebra. (Of course, by the Krull-Schmidt Theorem again, the $(Re_i)^\wedge$'s are just a permutation of the e_iR 's, up to isomorphism.) □

Next we shall give another interpretation for the Nakayama permutation π of a QF algebra R, in terms of the k-duals of R-modules. Recall that, upon forming R-duals, the principal indecomposables go to themselves in the same order ((16.12)), while the simple modules are permuted by π ((16.10)). We will show that, when we form k-duals, exactly the opposite happens; namely, the simple modules go to themselves in the same order, while the principal indecomposables are permuted by π^{-1}. We begin by checking the former, which is actually valid over any (finite-dimensional) algebra. Here again, the notations in (16.7) and (16.8) will be in force, and $\bar{R} := R/\text{rad}(R)$.

(16.24) Proposition. *For any primitive idempotent e in a finite-dimensional k-algebra R, we have $(\bar{R}\bar{e})^\wedge \cong \bar{e}\bar{R}$ as right R-modules, and $(\bar{e}\bar{R})^\wedge \cong \bar{R}\bar{e}$ as left R-modules.*

Proof. First observe that rad(R) acts as zero on all the modules concerned, and that the formation of the k-duals depends only on k and not on R. Therefore, we may replace R by $\bar{R}$ to assume that R is semisimple. Under this assumption, the principal indecomposables coincide with the simple modules. By the proof of (16.23), $(Re_1)^\wedge, \ldots, (Re_s)^\wedge$ are a permutation of $e_1R, \ldots, e_sR$, so $(Re_i)^\wedge \cong e_{i'}R$ for some i'. Now let $R = RE_1 \times \cdots \times RE_s$ be the decomposition of R into its simple components, where the E_i 's are central idempotents. For $j \neq i$, E_j acts trivially on Re_i, and hence also on $(Re_i)^\wedge$. This implies that the simple right R-module $e_{i'}R$ must be associated with the simple component RE_i, and so $(Re_i)^\wedge \cong e_{i'}R = e_iR$. Taking the k-duals, we have then also $(e_iR)^\wedge \cong Re_i$. □

(The fact that $(Re)^\wedge \cong eR$ for a primitive idempotent e in a semisimple algebra R is a special case of a much more general result to be obtained later. In fact, over any "symmetric algebra" R, it will be shown in (16.74) that $(Ra)^\wedge \cong aR$ and $(aR)^\wedge \cong Ra$ for *any* element $a \in R$.)

Returning to QF algebras, let us now proceed to study the way in which the formation of k-duals acts on the principal indecomposables.

(16.25) Proposition. *Let R be a* QF *algebra over a field k, and let π be the Nakayama permutation of R. (With respect to the notation in* (16.8), $(S_i')^* \cong S_{\pi(i)}$.*) Then, for the right and left principal indecomposables $\{U_i\}$ and $\{U_i'\}$, we have*

$$\hat{U}_i \cong U'_{\pi(i)}, \quad \text{and} \quad U_j'^{\wedge} \cong U_{\pi^{-1}(j)}.$$

Proof. By (16.9), $\mathrm{soc}(U_i) \cong S_{\pi(i)}$, so there exists an injection $S_{\pi(i)} \to U_i$. This induces a surjection from $\hat{U}_i$ to $\hat{S}_{\pi(i)}$, which is $S'_{\pi(i)}$ by (16.24). Since $\hat{U}_i$ is an indecomposable injective (and hence projective) left R-module, it must then be $U'_{\pi(i)}$. Taking k-duals again and replacing i by $\pi^{-1}(j)$, we get $U_j'^{\wedge} \cong (U_{\pi^{-1}(j)})^{\wedge\wedge} \cong U_{\pi^{-1}(j)}$. □

Throughout this subsection, we have assumed that R is an algebra over a field k. As it turns out, this assumption can be relaxed. In the 1950s, Eilenberg and Nakayama succeeded in extending the theory of Frobenius algebras to algebras over commutative rings. If k is a commutative ring (instead of a field), we consider k-algebras R such that R is f.g. projective as a k-module. Using the left R-module structure on R, we can, as before, view the k-dual $\hat{R} = \mathrm{Hom}_k(R, k)$ as a right R-module. We can then define R to be a Frobenius k-algebra if $R \cong \hat{R}$ as right R-modules. In the case when k itself is a Frobenius ring, it can be shown that any Frobenius k-algebra R is also a Frobenius ring. In this way, we can get new examples of Frobenius rings by taking, for instance, $(\mathbb{Z}/n\mathbb{Z})$-Frobenius algebras. This more general framework for studying Frobenius algebras is useful for coding theory; in fact, various results in classical coding theory over finite fields have been extended to finite commutative Frobenius rings (e.g., $\mathbb{Z}/4\mathbb{Z}$). For more details on this, see the article of J. A. Wood [97].

§16D. Dimension Characterizations of Frobenius Algebras

In this subsection, we shall obtain other classical characterizations of Frobenius algebras, in terms of certain formulas involving the dimensions of simple modules, their k-duals, and the annihilators of 1-sided ideals. To the extent possible, we shall formulate these dimension-theoretic results for QF algebras. The characterization theorems for Frobenius algebras will be deduced as easy consequences from them.

Throughout this subsection, R shall denote a finite-dimensional algebra over a field k. The notations $U_i = e_iR$, $U_i' = Re_i$, etc. in (16.7), (16.8) will be in force, and $\bar{R}$ shall always denote $R/\mathrm{rad}(R)$. Furthermore, we shall write $D_i = \bar{e}_i\bar{R}\bar{e}_i$; recall that these D_i's are division k-algebras and that we have a Wedderburn decomposition

$$\bar{R} \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_s}(D_s) \tag{16.26}$$

for the integers $n_1, \ldots, n_s$ in (16.7).

(16.27) Lemma. *Let R be a* QF *algebra over k, with Nakayama permutation π. Then* $\dim_k D_{\pi(i)} = \dim_k D_i$ *for all i.*

Proof. Let S be the ideal $\operatorname{soc}(R_R) = \operatorname{soc}({}_RR)$ in R. By (16.10) and Exercise (6.12)(7):

$$\bar{R}\bar{e}_i \cong \operatorname{soc}(Re_{\pi(i)}) = Se_{\pi(i)},$$

so $D_i = \bar{e}_i\bar{R}\bar{e}_i \cong e_iSe_{\pi(i)}$ as k-spaces. Similarly,

$$\bar{e}_{\pi(i)}\bar{R} \cong \operatorname{soc}(e_iR) = e_iS$$

gives $D_{\pi(i)} = \bar{e}_{\pi(i)}\bar{R}\bar{e}_{\pi(i)} \cong e_iSe_{\pi(i)}$ as k-spaces. It follows immediately that $\dim_k D_{\pi(i)} = \dim_k D_i$. □

Over a QF algebra R, $S_i = \bar{e}_i\bar{R}$ and $S_i' = \bar{R}\bar{e}_i$ are k-dual spaces (by (16.24)), so they have the same k-dimension. Let us denote this common dimension by d_i ($1 \le i \le s$). We have the following proportionality theorem between these d_i 's and the n_i 's in (16.26).

(16.28) Theorem. *For any* QF *algebra* R, $d_{\pi(i)}/d_i = n_{\pi(i)}/n_i$ *for all* i.

Proof. Note that $\bar{R}\bar{e}_i$ is a right vector space of dimension n_i over $\bar{e}_i\bar{R}\bar{e}_i = D_i$. Therefore,

(16.29) $$d_i = \dim_k \bar{R}\bar{e}_i = \dim_{D_i}(\bar{R}\bar{e}_i) \cdot \dim_k D_i = n_i \cdot \dim_k D_i\,.$$

Replacing i by $\pi(i)$ and using (16.27), we get by simple division $d_{\pi(i)}/d_i = n_{\pi(i)}/n_i$. □

(16.30) Remark. Recall from (16.10) that $(\bar{R}\bar{e}_i)^* \cong \bar{e}_{\pi(i)}\bar{R}$. Therefore, we can interpret the $d_{\pi(i)}$ above as $\dim_k(\bar{R}\bar{e}_i)^*$. This enables us to express (16.28) in the following equivalent form in terms of R-duals of simple left R-modules:

(16.31) $$\dim_k(\bar{R}\bar{e}_i)^*/\dim_k(\bar{R}\bar{e}_i) = n_{\pi(i)}/n_i.$$

For the record, we also state the right-side analogue of this formula:

(16.32) $$\dim_k(\bar{e}_i\bar{R})^*/\dim_k(\bar{e}_i\bar{R}) = n_{\pi^{-1}(i)}/n_i.$$

The above considerations lead us immediately to the following dimension-theoretic characterizations of Frobenius algebras:

(16.33) Theorem. *A* QF *algebra* R *is Frobenius iff* $d_{\pi(i)} = d_i$ *for all* i, *iff* $\dim_k M^* = \dim_k M$ *for all simple left (resp. right)* R*-modules.*

Remark. Since π is a permutation, it is easy to see that the above theorem remains to be true if, in the second and/or third condition, we replace the equal sign by an inequality sign (either "$\le$" or "$\ge$").

The fact that, over a Frobenius algebra R, a simple module has the same dimension as its R-dual admits the following generalization.

(16.34) Theorem. *Let R be any Frobenius k-algebra, and N be any f.g. left R-module. Then* $\dim_k N = \dim_k N^*$.

Proof. We induct on the integer length(N). If length$(N) = 1$, N is a simple module and the result follows from (16.33). Now assume length$(N) > 1$, and let $M \subseteq N$ be a simple submodule. Then M^* is a simple right R-module (of the same dimension as M^*), and by (15.14), we have

(16.35) $$M^\perp := \{f \in N^* : f(M) = 0\} \cong (N/M)^*, \quad \text{and} \quad N^*/M^\perp \cong M^*$$

as right R-modules. By the inductive hypothesis, $\dim_k (N/M) = \dim_k (N/M)^*$. Therefore,

$$\begin{aligned} \dim_k N &= \dim_k M + \dim_k (N/M) \\ &= \dim_k M^* + \dim_k (N/M)^* \\ &= \dim_k M^* + \dim_k M^\perp \\ &= \dim_k N^*, \end{aligned}$$

as desired. □

Since QF algebras need not be Frobenius algebras, (16.33) implies that $\dim_k M = \dim_k M^*$ need not hold over QF algebras. It behooves us to make this explicit by working with a concrete example.

(16.36) Example. Let k be a field, and R be the QF algebra (of dimension 9) defined in (16.19)(5). We shall use the notations in that example (and those in (16.8)). Since the Nakayama permutation of R is the transposition (12), we have $S_1^* \cong S_2'$ and $S_2^* \cong S_1'$. Thus, $\dim_k S_1 = 1$ while $\dim_k S_1^* = 2$, and $\dim_k S_2 = 2$ while $\dim_k S_2^* = 1$. The information on the dimensions of the S_i^* 's is also encoded in the socle equation: $\mathrm{soc}(R_R) \cong S_2 \oplus 2 \cdot S_1$. In fact, this equation tells us that there is essentially one way to embed S_2 into R, namely, sending

$$(x, y)(\in k^2 = S_2) \mapsto xE_{45} + yE_{46} \in \mathrm{soc}(U_1).$$

On the other hand, there are *two* essentially different ways of embedding S_1 into R: first by sending $z \in k = S_1$ into $zE_{13} \in \mathrm{soc}(U_2) = \mathrm{soc}(e_2R)$, and second, by sending the same $z \in k = S_1$ into $zE_{23} \in \mathrm{soc}(e_2'R)$ where $e_2'R = E_{22} + E_{66}$ is the "ignored" idempotent isomorphic to $e_2 = E_{11} + E_{55}$ in the decomposition $1 = e_1 + e_2 + e_2'$.

Although the dimension equation $\dim_k(N) = \dim_k(N^*)$ may not hold in the QF case, there is, nevertheless, a good analogue of (16.34) for QF algebras (in fact even for QF rings). For this analogue, we simply replace "dimension" by "length". We state the following result, the proof of which is left as an exercise.

(16.37) Theorem. *An artinian ring R is* QF *iff* length(N) = length(N^*) *for any f.g. module N_R and $_RN$.*

Recall that a Frobenius k-algebra R is characterized by the fact that it carries a nonsingular k-bilinear pairing $B : R \times R \to k$ with the associativity property $B(xy, z) = B(x, yz)$ (for all $x, y, z \in R$). For any set $T \subseteq R$, let

$$T^0 = \{x \in R : B(T, x) = 0\} \quad \text{and} \quad {}^0T = \{x \in R : B(x, T) = 0\}.$$

We note the following nice consequences of associativity on T^0 and ${}^0T$.

(16.38) Lemma. *Let (R, B) be as above.*

(1) *If $\mathfrak{A}$ is a left ideal, then $\mathfrak{A}^0$, ${}^0\mathfrak{A}$ are right ideals, with $\mathfrak{A}^0 = \operatorname{ann}_r(\mathfrak{A})$ and ${}^0\mathfrak{A} \supseteq \operatorname{ann}_\ell(\mathfrak{A})$. Moreover,*

$$\hat{\mathfrak{A}} \cong R/{}^0\mathfrak{A} \quad \textit{and} \quad (R/\mathfrak{A})\hat{} \cong {}^0\mathfrak{A} \quad \textit{as right } R\textit{-modules.}$$

(2) *If A is a right ideal, then ${}^0A$, A^0 are left ideals, with ${}^0A = \operatorname{ann}_\ell(A)$ and $A^0 \supseteq \operatorname{ann}_r(A)$. Moreover,*

$$\hat{A} \cong R/A^0 \quad \textit{and} \quad (R/A)\hat{} \cong A^0 \quad \textit{as left } R\textit{-modules.}$$

Proof. By symmetry, it is sufficient to prove (1). First let $x \in \operatorname{ann}_r(\mathfrak{A})$. Then $B(\mathfrak{A}, x) = B(\mathfrak{A}x, 1) = 0$, so $x \in \mathfrak{A}^0$. Conversely, if $x \in \mathfrak{A}^0$, then

$$B(R, \mathfrak{A}x) = B(R\mathfrak{A}, x) = B(\mathfrak{A}, x) = 0.$$

Since B is a nonsingular pairing, this implies that $\mathfrak{A}x = 0$; that is, $x \in \operatorname{ann}_r(\mathfrak{A})$. This gives $\mathfrak{A}^0 = \operatorname{ann}_r(\mathfrak{A})$, which is a right ideal. If $y \in {}^0\mathfrak{A}$, then $B(yR, \mathfrak{A}) = B(y, R\mathfrak{A}) = 0$; hence $yR \subseteq {}^0\mathfrak{A}$ so ${}^0\mathfrak{A}$ is also a right ideal. For any $z \in \operatorname{ann}_\ell(\mathfrak{A})$, we have $B(z, \mathfrak{A}) = B(1, z\mathfrak{A}) = 0$; this shows that $\operatorname{ann}_\ell(\mathfrak{A}) \subseteq {}^0\mathfrak{A}$.

Recall that, for the Frobenius algebra R, the pairing B above "corresponds" to a right R-module isomorphism $f : R \to \hat{R}$, with $f(x)(y) = B(x, y)$. (See the proof of (3.15).) Using the definition of ${}^0\mathfrak{A}$, we see that f induces the two isomorphisms asserted in (1). □

(16.39) Remark. In (1), the inclusion ${}^0\mathfrak{A} \supseteq \operatorname{ann}_\ell(\mathfrak{A})$ need not be an equality. In fact, ${}^0\mathfrak{A}$ is only a right ideal, but $\operatorname{ann}_\ell(\mathfrak{A})$ is always an ideal. For instance, if R is a simple algebra and $\mathfrak{A} \neq 0$, R, then we have necessarily $\operatorname{ann}_\ell(\mathfrak{A}) = 0$ but ${}^0\mathfrak{A} \neq 0$. A similar remark can be made about (2).

We are now in a position to prove the following remarkable characterization theorem of Frobenius algebras due to T. Nakayama.

(16.40) Theorem. *For any finite-dimensional k-algebra, the following are equivalent:*

(1) *R is a Frobenius algebra.*
(2) *For any right ideal $A \subseteq R$ and any left ideal $\mathfrak{A} \subseteq R$,*

$$\dim_k A + \dim_k \operatorname{ann}_\ell(A) = \dim_k R = \dim_k \mathfrak{A} + \dim_k \operatorname{ann}_r(\mathfrak{A}).$$

(3) *R is* QF, *and for any minimal right ideal $A \subseteq R$:*

$$\dim_k A + \dim_k \operatorname{ann}_\ell(A) = \dim_k R.$$

Proof. (1)$\Longrightarrow$(2). Fix a nonsingular (associative) k-bilinear pairing $B : R \times R \to k$ as in (16.38). Since B is nonsingular, we have

$$\dim_k R = \dim_k \mathfrak{A} + \dim_k \mathfrak{A}^0 = \dim_k \mathfrak{A} + \dim_k \operatorname{ann}_r(\mathfrak{A})$$

by (16.38)(1), and a similar formula for A follows.

(2)$\Longrightarrow$(3). For any left ideal $\mathfrak{A}$,

$$\begin{aligned}\dim_k \operatorname{ann}_\ell(\operatorname{ann}_r(\mathfrak{A})) &= \dim_k R - \dim_k(\operatorname{ann}_r(\mathfrak{A}))\\ &= \dim_k R - (\dim_k R - \dim_k \mathfrak{A})\\ &= \dim_k \mathfrak{A}.\end{aligned}$$

Thus, $\mathfrak{A} \subseteq \operatorname{ann}_\ell(\operatorname{ann}_r(\mathfrak{A}))$ must be an equality, and similarly, we have $A = \operatorname{ann}_r(\operatorname{ann}_\ell(A))$ for any right ideal $A \subseteq R$. Therefore, by (15.1), R is a QF ring.

(3)$\Longrightarrow$(1). According to (16.33), all we need is to show that, for any simple right R-module A, $\dim_k A^* = \dim_k A$. Since R is right Kasch, we may assume that A is a minimal right ideal of R. Then by (15.14), $A^* \cong R/\operatorname{ann}_\ell(A)$, so

$$\dim_k A^* = \dim_k R - \dim_k \operatorname{ann}_\ell(A) = \dim_k A,$$

as desired. □

Remark. In view of the Remark made after (16.33), we can strengthen the above result by adding another (ostensibly weaker) condition (3)′ where we replace the equality in (3) by an inequality (either "$\leq$" or "$\geq$"). For the proof, we just replace (3)$\Longrightarrow$(1) above by (3) $\Longrightarrow$ (3)′ $\Longrightarrow$ (1).

§16E. The Nakayama Automorphism

The idea here is that, for a Frobenius k-algebra R, there actually exists a k-algebra automorphism σ on R which "effects" the Nakayama permutation π of R. This is one of several special features which distinguish Frobenius algebras from QF algebras.

To define the automorphism σ of R, we first fix a nonsingular bilinear pairing $B : R \times R \to k$ with the associativity property. Let $a \in R$ be given. The map $x \mapsto B(a, x)$ is a k-linear functional on R, so it has the form $B(-, b)$ for a uniquely determined element b, which we then define to be $\sigma(a)$. In other words, $\sigma(a)$ is characterized by the equation

(16.41) $$B(a, x) = B(x, \sigma(a)) \qquad (\forall x \in R).$$

If we use the functional $\lambda : R \to k$ defined by $\lambda(x) = B(x, 1)$ (cf. proof of (3.15)), whose kernel contains no nonzero 1-sided ideals, then $B(x, y) = \lambda(xy)$,

so we can re-express the characterization (16.41) of σ in the form

(16.42) $$\lambda(ax) = \lambda(x\sigma(a)) \qquad (\forall x \in R).$$

It is easy to see that σ is a k-vector space automorphism. From

$$\lambda(x\sigma(a)\sigma(b)) = \lambda(bx\sigma(a)) = \lambda(abx) \qquad (\forall x \in R),$$

we see that $\sigma(ab) = \sigma(a)\sigma(b)$, so σ is a k-algebra automorphism. It is called the *Nakayama automorphism* of the Frobenius algebra R.

Of course, the definition of σ above depended on the choice of the pairing B. Fortuitously, changing the choice of B only changes σ by an inner automorphism, as we'll presently show.

(16.43) Proposition. *The Nakayama automorphism σ of a Frobenius algebra R is determined up to an inner automorphism.*

Proof. Suppose B, B' are both nonsingular, associative pairings. Then $g, g' : R \to \hat{R}$ defined by

$$(yg)(x) = B(x, y), \quad (yg')(x) = B'(x, y)$$

are isomorphisms of left R-modules (see the Remark after the proof of (3.15)), so there exists an automorphism h of R_R such that $g' = h \circ g$ (composition of right operators). Such h is given by right multiplication by some unit $u \in \mathrm{U}(R)$. Then,

(16.44) $$B'(x, y) = g'(y)(x) = ((yu)g)(x) = B(x, yu).$$

Bringing in the automorphisms σ and σ' defined by B and B', respectively, we can rewrite (16.44) as

$$B'(y, \sigma'(x)) = B(yu, \sigma(x)) = B(y, u\sigma(x)).$$

Using (16.44) to replace the LHS by $B(y, \sigma'(x)u)$, we see that $u\sigma(x) = \sigma'(x)u$, so $\sigma'(x) = u\sigma(x)u^{-1}$ for all $x \in R$, as desired. (The same proof shows that $x \mapsto u\sigma(x)u^{-1}$ is a Nakayama automorphism, for any unit $u \in \mathrm{U}(R)$.) □

(16.45) Proposition. *Let (R, B) be as above, and let σ be the Nakayama automorphism of R associated with B. Then for any right ideal A, we have the following left R-module isomorphisms:*

$$\hat{A} \cong R/\sigma(\mathrm{ann}_\ell A), \qquad (R/A)\hat{} \cong \sigma(\mathrm{ann}_\ell A).$$

Similarly for any left ideal $\mathfrak{A}$, we have right R-module isomorphisms:

$$\hat{\mathfrak{A}} \cong R/\sigma^{-1}(\mathrm{ann}_r \mathfrak{A}), \qquad (R/\mathfrak{A})\hat{} \cong \sigma^{-1}(\mathrm{ann}_r \mathfrak{A}).$$

Proof. It suffices to deal with the case of a right ideal A. First note that

(16.46) $$A^0 = \{x \in R : \ 0 = B(A, x) = B(x, \sigma A)\} = {}^0(\sigma A).$$

For the right ideal σA, we have by (16.38)(2):

$$^0(\sigma A) = \text{ann}_\ell(\sigma A) = \sigma(\text{ann}_\ell A). \tag{16.47}$$

Now the asserted isomorphisms in the Proposition follow from those in (16.38)(2) upon replacing A^0 there by $\sigma(\text{ann}_\ell A)$. □

(16.48) Proposition. *In the above notations, we have for any idempotent $e \in R$:*

$$(eR)^{\wedge} \cong R\sigma(e), \quad \textit{and} \quad (Re)^{\wedge} \cong \sigma^{-1}(e)R.$$

Proof. Let $A = eR$ in (16.45). Since

$$\sigma(\text{ann}_\ell\, eR) = \sigma(R(1-e)) = R(1-\sigma(e)),$$

we get $(eR)^{\wedge} \cong R/R(1-\sigma(e)) \cong R\sigma(e)$ as left R-modules. The other isomorphism follows similarly, or by taking "hat" (k-dual) of the isomorphism just obtained, and replacing e by $\sigma^{-1}(e)$. □

The Proposition bears out our earlier remark that the Nakayama automorphism σ "effects" the Nakayama permutation π (in the case of a Frobenius algebra). For, if $e_1R, \ldots, e_sR$ give all different types of right principal indecomposables, we will have from (16.48) and (16.25):

$$R\sigma(e_i) \cong Re_{\pi(i)} \quad \text{and} \quad \sigma^{-1}(e)R \cong e_{\pi^{-1}(i)}R. \tag{16.49}$$

In other words, *the idempotent $\sigma(e_i)$ is isomorphic to $e_{\pi(i)}$*. It follows that the R-duals of the simple R-modules can also be expressed in terms of σ, namely:

$$(\bar{R}\,\bar{e}_i)^* \cong \overline{\sigma(e_i)}\,\bar{R} \quad \text{and} \quad (\bar{e}_i\,\bar{R})^* \cong \bar{R}\,\overline{\sigma^{-1}(e_i)} \tag{16.50}$$

To conclude this subsection, let us compute an explicit example of a Frobenius automorphism σ. The case when σ is the identity is, of course, possible: this happens for instance when R is a "symmetric algebra". We shall treat this case in more detail in §16F below. Here, we would like to give an example of a nontrivial (noninner) σ.

(16.51) Example. Let k be a field and R be the $2n$-dimensional k-algebra defined at the end of (16.19)(4), consisting of the $2n \times 2n$ matrices γ with diagonal blocks

$$\begin{pmatrix} a_1 & x_1 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_2 & x_2 \\ 0 & a_3 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & x_n \\ 0 & a_1 \end{pmatrix},$$

where all entries are chosen from k. We have shown in (16.19)(4) that R is a Frobenius ring, so it is a Frobenius k-algebra by (16.21). A direct verification of this is also possible. We define the linear functional $\lambda : R \to k$ by $\lambda(\gamma) = x_1 + \cdots + x_n \in k$, where γ is as above. It suffices to show that $\ker(\lambda)$ contains no nonzero right ideal of R. Suppose, indeed, $\lambda(\bar{\gamma}R) = 0$, where $\bar{\gamma}$ has the

parameters $\bar{a}_i$'s and $\bar{x}_i$'s. By a direct calculation, this yields

$$0 = \lambda(\bar{\gamma}\,\gamma) = \sum_{i=1}^{n} \left(\bar{a}_i x_i + \bar{x}_i a_{i+1}\right) \qquad (\forall \gamma \in R),$$

where the subscripts are taken mod n (so that a_{n+1} means a_1). Clearly, this implies that $\bar{\gamma} = 0$. Therefore, $B(\gamma, \gamma') = \lambda(\gamma\gamma')$ defines a nonsingular bilinear pairing on R with the associativity property. This checks directly that R is a Frobenius k-algebra.

Now consider the map $\sigma : R \to R$ defined by cyclically permuting the diagonal blocks of a matrix γ in R. This is induced by the conjugation by a suitable permutation matrix in $\mathbb{M}_{2n}(k)$, so it is a k-algebra automorphism of R. For $\gamma,\ \bar{\gamma} \in R$ as above, an easy computation shows that

$$\lambda(\gamma \cdot \sigma(\bar{\gamma})) = \sum_{i=1}^{n} \left(a_i \bar{x}_{i-1} + x_i \bar{a}_i\right),$$

again with subscripts taken modulo n. An easy inspection shows that this is exactly the same sum as that computing $\lambda(\bar{\gamma}\,\gamma)$. Therefore, σ is precisely the Nakayama automorphism associated with B that we are looking for!

The n crucial primitive idempotents in this example are

$$e_i := E_{2i-2,2i-2} + E_{2i-1,2i-1} \quad (1 \le i \le n),$$

with subscripts taken now mod $2n$ (so that $e_1 = E_{2n,2n} + E_{11}$). Obviously, these are cyclically permuted by the automorphism σ. Therefore, as is predicted by the general theory, σ "effects" the Nakayama permutation π, which, according to (16.19)(4), is the n-cycle $(12\cdots n)$.

§16F. Symmetric Algebras

We shall conclude §16 by examining an important class of Frobenius algebras consisting of the so-called *symmetric algebras*. Throughout this subsection, R denotes a finite-dimensional algebra over a fixed field k.

Recall that, given R, we can form the k-dual $\hat{R}$ and give it the structure of an (R, R)-bimodule. The right and left R-actions of $\hat{R}$ are defined, respectively, by

(16.52) $$(\varphi \cdot x)(r) = \varphi(xr) \quad \text{and} \quad (y \cdot \varphi)(r) = \varphi(ry),$$

where $\varphi \in \hat{R}$ and $x, y, r \in R$. If $R \cong \hat{R}$ as right (or equivalently, left) R-modules, R is by definition a Frobenius k-algebra. It is, of course, equally natural to try to compare $\hat{R}$ to R in the category of (R, R)-bimodules. This leads us to the following:

(16.53) Definition. R is called a *symmetric algebra over k* (or a *symmetric k-algebra*) if $\hat{R} \cong R$ as (R, R)-bimodules. Of course, a symmetric k-algebra is always a Frobenius k-algebra.

The several initial characterizations of Frobenius algebras do have analogues for symmetric algebras. We collect them in the following result.

(16.54) Theorem. *For any (finite-dimensional) k-algebra R, the following are equivalent:*

(1) *R is a symmetric algebra.*
(2) *There exists a nonsingular, symmetric bilinear pairing $B : R \times R \to k$ with the associativity property $B(xy, z) = B(x, yz)$ for all $x, y, z \in R$.*
(3) *There exists a k-linear functional $\lambda : R \to k$ such that $\lambda(xy) = \lambda(yx)$ for all $x, y \in R$, such that* $\ker(\lambda)$ *contains no nonzero right ideals of R.*

Proof. The proof follows the same general pattern as that for (3.15). We simply "add on" the symmetric property.

(1)$\Longleftrightarrow$(2). First assume R is a symmetric algebra. We fix an (R, R)-bimodule isomorphism $f : R \to \hat{R}$ and define $B : R \times R \to k$ by $B(x, y) = f(x)(y)$. Then B is nonsingular and "associative" as in the proof of (3.15): this followed from the fact that f is a right R-module isomorphism. Let us now bring in the fact that f is also a *left* R-module homomorphism, which is expressed by the equation $f(zx) = zf(x)$ $(z, x \in R)$. This amounts to

$$f(zx)(y) = (zf(x))(y) = f(x)(yz) \qquad (\forall x, y, z \in R),$$

or equivalently, $B(zx, y) = B(x, yz)$. Setting $x = 1$, we get $B(z, y) = B(1, yz) = B(y, z)$ so B is symmetric. Conversely, if a nonsingular associative pairing B happens to be symmetric, then

$$B(zx, y) = B(y, zx) = B(yz, x) = B(x, yz),$$

so the associated map $f : R \to \hat{R}$ (defined by $f(x)(y) = B(x, y)$) will be an (R, R)-bimodule isomorphism.

(2)$\Longleftrightarrow$(3). Given B as in (2), we define λ by $\lambda(x) = B(x, 1)$. Then

$$\lambda(xy) = B(xy, 1) = B(x, y) = B(y, x) = B(yx, 1) = \lambda(yx),$$

and, as in the proof of (3.15), $\ker(\lambda)$ contains no nonzero right (or left) ideals of R. The converse is completely similar, since, if we define B via λ by $B(x, y) = \lambda(xy)$, the symmetry of B follows from that of λ. □

(16.55) Example. *Any commutative Frobenius k-algebra R is a symmetric algebra.* This is clear since, if λ is a nonzero k-linear functional on R such that $\ker(\lambda)$ contains no nonzero right ideals, the symmetry property $\lambda(xy) = \lambda(yx)$ is automatic because of the commutativity of R.

(16.56) Example. *For any finite group G, the group algebra $R = kG$ is always a symmetric algebra.* Recall from (3.15E) that R is a Frobenius algebra, with respect

to the functional $\lambda : R \to k$ given by $\lambda(\sum \alpha_g g) = \alpha_1$. Now for $\alpha = \sum \alpha_g g$ and $\beta = \sum \beta_h h$ in R, we have

$$\lambda(\alpha\beta) = \sum_{g \in G} \alpha_g \beta_{g^{-1}} = \sum_{h \in G} \beta_h \alpha_{h^{-1}} = \lambda(\beta\alpha),$$

so it follows from (16.54) that R is a symmetric algebra. (Historically, it is precisely this group ring example which prompted the definition of a symmetric algebra.)

(16.57) Example. *Any matrix algebra $R = \mathbb{M}_n(k)$ is a symmetric k-algebra.* We define $\lambda : R \to k$ to be the trace function. As is well-known, $\lambda(xy) = \lambda(yx)$ for all matrices $x, y \in R$. If $x \in R$ is such that $\lambda(xR) = 0$, then $\lambda(xE_{ij}) = 0$ for all matrix units E_{ij}. Since $\lambda(xE_{ij}) = x_{ji}$ (the (j, i)-entry of x), it follows that $x = 0$. Obviously, $\lambda \not\equiv 0$ also, so again by (16.54), R is a symmetric algebra.

(16.58) Example. If R, R' are symmetric algebras, then so are the algebras $R \times R'$ and $R \otimes_k R'$. The verification of this is left as an easy exercise.

(16.59) Example (Eilenberg-Nakayama). *Any semisimple k-algebra R is a symmetric algebra.* In view of Wedderburn's Theorem and (16.58) above, we need only consider the case $R = \mathbb{M}_n(D)$ where D is a division k-algebra. Since $\mathbb{M}_n(D) \cong D \otimes_k \mathbb{M}_n(k)$ and (by (16.57)) $\mathbb{M}_n(k)$ is a symmetric algebra, another application of (16.58) reduces us to the case when R is a division algebra. In this case, it suffices to show that $[R, R]$ (the additive group generated by $xy - yx$ for all $x, y \in R$) is not equal to R. (Note that $[R, R]$ is also a k-space. If $[R, R] \neq R$, any nonzero functional $\lambda : R \to k$ vanishing on $[R, R]$ will satisfy the properties stipulated in (16.54)(3), since the only nonzero right ideal in R is R itself.) To see that $[R, R] \neq R$, it is harmless to replace k by the center of R. If we "go up" to the algebraic closure K of k, we have

$$R^K := R \otimes_k K \cong \mathbb{M}_n(K)$$

for some n (by *FC*–(15.1)). Since $[R, R]^K \subseteq [R^K, R^K]$, it suffices to check that $[R^K, R^K] \neq R^K$. This is clear since the trace function $\text{tr}\colon \mathbb{M}_n(K) \to K$ is nonzero and $\text{tr}(xy) = \text{tr}(yx)$ for all matrices $x, y \in \mathbb{M}_n(K)$.

(16.60) Example (Tachikawa). Recall from (2.22)(A) that if M is any (A, A)-bimodule over any ring A, we can form the so-called "trivial extension" of M by A and get a new ring R containing M as an ideal such that $R/M \cong A$. By definition, $R = A \oplus M$, and multiplication on R is defined by

(16.61) $\quad (a, m) \cdot (a', m') = (aa', am' + ma') \qquad (a, a' \in A; \quad m, m' \in M).$

Here $M \subseteq R$ is an ideal with zero multiplication, and $(a, m) \mapsto a$ induces the ring isomorphism $R/M \cong A$. Now let A be any finite-dimensional k-algebra and take M to be the (A, A)-bimodule $\hat{A}$. We claim that

(16.62) *The trivial extension $R := A \oplus \hat{A}$ is always a symmetric k-algebra.*

To see this, we define $\lambda : R \to k$ by taking $\lambda(a, \varphi) = \varphi(1)$ for any $a \in A$ and $\varphi \in \hat{A}$. (Yes, the value of λ on (a, φ) is independent of a !) Clearly, $\lambda \not\equiv 0$, and

$$\lambda((a, \varphi)(a', \varphi')) = \lambda(aa',\, a\varphi' + \varphi a') = (a\varphi' + \varphi a')(1) = \varphi'(a) + \varphi(a')$$

shows that $\lambda(xy) = \lambda(yx)$ for all $x, y \in R$. Now suppose $\lambda((a, \varphi)R) = 0$. Then $\varphi'(a) + \varphi(a') = 0$ for all $a' \in A$ and $\varphi' \in \hat{A}$. Setting $a' = 0$ shows that $a = 0$, and $\varphi(a') = 0$ for all $a' \in A$ shows that $\varphi = 0$. Therefore, by (16.54), R is a symmetric k-algebra, as claimed.

Note that (16.62) gives a good supply of symmetric (in particular, Frobenius) algebras. This result also has the remarkable consequence that *any (finite-dimensional) k-algebra A is a quotient of a symmetric k-algebra.*

We have not yet given an example of a Frobenius algebra that is not a symmetric algebra. This will become easy once we observe the following basic result.

(16.63) Theorem. *A Frobenius k-algebra R is a symmetric algebra if and only if the Nakayama automorphism of R is an inner automorphism.*

Proof. To begin with, we first recall that the Nakayama automorphism σ of R is determined only up to an inner automorphism. So the result at hand is tantamount to: *R is a symmetric algebra iff σ can be taken to be the identity.* Let $B : R \times R \to k$ be the nonsingular associative pairing used to define σ, so that we have $B(x, y) = B(y, \sigma(x))$. If $\sigma = \mathrm{Id}_R$, then B is symmetric, so R is a symmetric algebra. Conversely, if R is a symmetric algebra, then we can take B to be symmetric. Then $B(x, y) = B(y, x)$ shows that the Nakayama automorphism defined via B is the identity automorphism. □

(16.64) Corollary. *The Nakayama permutation π of a symmetric algebra R is the identity. In other words, for any primitive idempotent $e \in R$, we have $\mathrm{soc}(eR) \cong \bar{e}\bar{R}$ and $(\bar{e}\bar{R})^* \cong \bar{R}\bar{e}$, where $\bar{R} = R/\mathrm{rad}(R)$.*

Proof. This follows by applying the theorem to (16.49), which says that π is "effected" by the Nakayama automorphism of R. □

The Corollary (16.64) suggests another viable notion: let us call a QF algebra R *weakly symmetric* if its Nakayama permutation π is the identity. (Note that, in this case, R is automatically a Frobenius algebra, by (16.14).) We have the following hierarchical relationships among the various classes of finite-dimensional k-algebras:

$$\begin{pmatrix}\text{semisimple}\\ \text{algebras}\end{pmatrix} \subset \begin{pmatrix}\text{symmetric}\\ \text{algebras}\end{pmatrix} \subset \begin{pmatrix}\text{weakly}\\ \text{symmetric}\\ \text{algebras}\end{pmatrix} \subset \begin{pmatrix}\text{Frobenius}\\ \text{algebras}\end{pmatrix} \subset \begin{pmatrix}\text{QF}\\ \text{algebras}\end{pmatrix}.$$

It can be shown that each "inclusion" is strict (as indicated). We have already seen an example of a QF algebra that is not a Frobenius algebra in (16.19)(5). The

algebra in (16.19)(4) (with k taken to be a field) is Frobenius with the Nakayama permutation $\pi = (12)$ (or, if we wish, $(12 \cdots n)$), so it is not weakly symmetric. A group algebra kG with $\text{char}(k) = p > 0$ and G a nontrivial finite p-group is a symmetric algebra and is not semisimple. Finally, to construct an example of a weakly symmetric algebra that is not symmetric, we make use of the following observation.

(16.65) Proposition. *For any ideal A in a symmetric k-algebra R, we have* $\text{ann}_r(A) = \text{ann}_\ell(A)$.

Proof. Let $B : R \times R \to k$ be a nonsingular symmetric and associative pairing. By (16.38), we know that $\text{ann}_r(A)$ is given by $A^0 := \{x \in R : B(A, x) = 0\}$, and that $\text{ann}_\ell(A)$ is given by ${}^0A := \{x \in R : B(x, A) = 0\}$. Since B is symmetric, we have $A^0 = {}^0A$, and therefore $\text{ann}_r(A) = \text{ann}_\ell(A)$. □

(16.66) Example (Nakayama-Nesbitt). To construct a weakly symmetric algebra that is not symmetric, the idea is to specialize to *local* algebras. If R is a local QF k-algebra, there is only one (left or right) principal indecomposable, so the Nakayama permutation of R is necessarily the identity. All we need then is an example of such an R that is not a symmetric algebra. Let k be a field with two nonzero elements u, v such that $u^2 \neq v^2$, and let R be the set of matrices

(16.67)
$$\gamma = \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & uc \\ 0 & 0 & a & vb \\ 0 & 0 & 0 & a \end{pmatrix}$$

over k. It is easy to see that R is a 4-dimensional local algebra over k, with

$$J := \text{rad } R = \{\gamma \in R : \ a = 0\},$$

and $R/J \cong k$. Moreover, a direct calculation shows that

$$\text{ann}_r(J) = \text{ann}_\ell(J) = \{\gamma \in R : \ a = b = c = 0\} \ \ (= J^2).$$

Since this is 1-dimensional over k, it is equal to $\text{soc}({}_RR)$ and $\text{soc}(R_R)$. Therefore, by (16.4) and (16.23), R is a QF algebra, and hence a weakly symmetric algebra. To show that R is not a symmetric algebra, consider the elements

(16.68)
$$\alpha = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & v & -u & 0 \\ 0 & 0 & 0 & -u^2 \\ 0 & 0 & 0 & v^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in R, for which we have $\beta\alpha = 0$, and $\alpha\beta = (v^2 - u^2)E_{14} \neq 0$. By matrix multiplication, we can check that

$$\alpha R = R\alpha = \left\{ \begin{pmatrix} 0 & a & a & d \\ 0 & 0 & 0 & ua \\ 0 & 0 & 0 & va \\ 0 & 0 & 0 & 0 \end{pmatrix} : \ a, d \in k \right\}.$$

Let A be this ideal. Then $\beta A = \beta\alpha R = 0$ so $\beta \in \text{ann}_\ell(A)$; but $A\beta \ni \alpha\beta \neq 0$ so $\beta \notin \text{ann}_r(A)$. This shows that $\text{ann}_\ell(A) \neq \text{ann}_r(A)$, so by (16.65), A is not a symmetric algebra! (The case when $u^2 = v^2$ can be treated too; for the details, see Exercise 29.)

We shall now conclude §16F by giving another remarkable characterization of symmetric algebras. This characterization is prompted by a comparison between the R-dual and the k-dual functors on the category of (say, left) R-modules over a finite-dimensional algebra R over a field k. In fact, this characterization theorem shows clearly why symmetric algebras are such an interesting (and desirable) class of k-algebras.

To facilitate our discussion, let us first introduce some notations. Given the (finite-dimensional) k-algebra R, let $F, G : {}_R\mathfrak{M} \to \mathfrak{M}_R$ be the contravariant functors from left R-modules to right R-modules, given by:

(16.69) $$F = \text{Hom}_k(-,\ k) \quad \text{and} \quad G = \text{Hom}_R(-,\ R).$$

This makes sense since, for any left R-module M, $F(M) = \text{Hom}_k(M, k)$ is a right R-module (the right R-module structure coming from the left R-structure on M), and $G(M) = \text{Hom}_R({}_RM,\ {}_RR_R)$ is also a right R-module (the right R-structure coming from that of R). We can introduce a third contravariant functor $F' : {}_R\mathfrak{M} \to \mathfrak{M}_R$, given by $F'(M) = \text{Hom}_R({}_RM,\ {}_R(\hat{R})_R)$ (the right R-structure coming from that of $\hat{R}$: recall that $\hat{R}$ is an $(R,\ R)$-bimodule). The following result is valid over any finite-dimensional k-algebra R.

(16.70) Brauer's Equivalence Theorem. *The two contravariant functors $F,\ F' : {}_R\mathfrak{M} \to \mathfrak{M}_R$ are naturally equivalent.*

Proof. For any left R-module M, we have the following isomorphism relations:

$$\begin{aligned}\text{Hom}_R(M,\ {}_R(\hat{R})_R) &= \text{Hom}_R(M,\ \text{Hom}_k({}_RR_R,\ k))\\ &\cong \text{Hom}_k(M \otimes_R R,\ k)\\ &\cong \text{Hom}_k(M,\ k) = \hat{M}.\end{aligned}$$

Clearly, all isomorphisms are functorial, so we have established a natural equivalence of functors: $F' \cong F$. □

The result above is attributed to Brauer since it was he who first observed it in a matrix form. Note that, in view of the natural equivalence $F' \cong F$, we can say that the k-dual functor F is "represented" by the bimodule ${}_R(\hat{R})_R$. This compares nicely with the fact that the R-dual functor G is (by its definition) "represented" by the bimodule ${}_RR_R$. With this observation, we are now in a position to state and prove the following nice characterization of a symmetric algebra.

(16.71) Theorem. *A finite-dimensional k-algebra R is a symmetric k-algebra iff the k-dual functor F is naturally equivalent to the R-dual functor G.*

Proof. First suppose R is a symmetric k-algebra. Then, by definition, the (R, R)-bimodules R and $\hat{R}$ are isomorphic. Since F is (up to natural equivalence) represented by ${}_R(\hat{R})_R$ (as we have observed above) and G is represented by ${}_R R_R$, it follows that $F \cong G$. Conversely, suppose $F \cong G$. Then, by (16.70), there is a natural equivalence of functors $\Phi : G \to F'$. For any left R-module homomorphism $h : N \to M$, we have therefore a commutative diagram

$$(16.72)\qquad \begin{array}{ccc} G(M) & \xrightarrow{\Phi(M)} & F'(M) \\ {\scriptstyle G(h)}\big\downarrow & & \big\downarrow{\scriptstyle F'(h)} \\ G(N) & \xrightarrow[\Phi(N)]{} & F'(N) \end{array}$$

where $\Phi(M)$, $\Phi(N)$ are right R-module isomorphisms. Let us apply this to $M = N = {}_R R$, taking h to be the endomorphism of ${}_R R$ given by *right* multiplication by an element $r \in R$. We have clearly

$$G(R) = \mathrm{Hom}_R({}_R R,\ {}_R R_R) \cong (R^*)_R \cong R_R\,, \quad \text{and}$$
$$F'(R) = \mathrm{Hom}_R({}_R R,\ {}_R(\hat{R})_R) \cong (\hat{R})_R\,.$$

By a direct calculation, we can check that $G(h)$ is left multiplication by r on R_R, and that $F'(h)$ is left multiplication by r on $(\hat{R})_R$. The isomorphism $\Phi(R) : G(R) \to F'(R)$ is a priori only an isomorphism of right R-modules, but the commutative diagram (16.72) (for $M = N = {}_R R$ and h as above) shows that $\Phi(R)$ is *also* an isomorphism of left R-modules. Therefore, $\Phi(R)$ is an (R, R)-bimodule isomorphism from R to $\hat{R}$, and so R is a symmetric k-algebra. □

The fact that $M^* \cong \hat{M}$ for any left module M over a symmetric k-algebra R is very nice indeed.[91] If R is only a Frobenius k-algebra and ${}_R M$ is f.g., we do know, from (16.34) that

$$\dim_k M^* = \dim_k M = \dim_k \hat{M}.$$

However, in general, M^* and $\hat{M}$ may not be isomorphic as right R-modules, even in the case when ${}_R M$ is a f.g. projective module to begin with. For instance, let M be the ith principal indecomposable left R-module U_i' (in the notation of (16.8)). By (16.12), we have $(U_i')^* \cong U_i$ (the ith principal indecomposable right R-module). However, by (16.25), we have $(U_i')\hat{}\cong U_{\pi^{-1}(i)}$ where π is the Nakayama permutation of R. Thus, if π is not the identity (as, for example, in (16.19)(4)), there exists a left principal indecomposable R-module M with $M^* \not\cong \hat{M}$ as right R-modules.

Returning now to a symmetric algebra R, let us explain a bit more explicitly the natural equivalence between the R-dual functor G and the k-dual functor F. Let $\lambda : R \to k$ be a k-linear functional with $\lambda(xy) = \lambda(yx)$, such that $\ker(\lambda)$

[91] For instance, a somewhat surprising consequence of this is the fact that, as a right R-module, $\hat{M}$ is actually independent of k.

contains no nonzero left ideal in R. Then an explicit (R, R)-bimodule isomorphism $f : R \to \hat{R}$ is given by $f(x)(y) = \lambda(xy)$. Using this to define the isomorphism $\Phi(M)$ in (16.72), we can then compute the composition $\Psi(M)$ of the following two isomorphisms:

$$G(M) \overset{\Phi(M)}{\longrightarrow} F'(M) \overset{\Phi'(M)}{\longrightarrow} F(M),$$

where $\Phi'(M)$ is the natural isomorphism worked out in the proof of (16.70). In fact, for any $g \in G(M) = \text{Hom}_R(M, R)$, $\Phi(M)(g)$ is just $f \circ g : M \to \hat{R}$, and we can see (by working through the isomorphisms in the proof of (16.70)) that $\Phi'(M)(f \circ g)$ is just $\lambda \circ g : M \to k$. Summarizing, we conclude that the natural isomorphism

$$\Psi(M) : G(M) = M^* = \text{Hom}_R(M, R) \longrightarrow \text{Hom}_k(M, k) = \hat{M} = F(M)$$

is given by $g \mapsto \lambda \circ g$ for any $g \in M^*$.

This unraveling of the map $\Phi'(M) \circ \Phi(M)$ actually leads to a more direct (and less abstract) view of the isomorphism between the R-dual and the k-dual of M. In fact, for a given ${}_RM$, define $\psi : M^* \to \hat{M}$ by $\psi(g) = \lambda \circ g$ for any $g \in M^*$. This is a right R-module homomorphism, since, for any $r \in R$ and $m \in M$:

$$\begin{aligned} \psi(g \cdot r)(m) &= \lambda((g \cdot r)(m)) \\ &= \lambda(g(m)r) \\ &= \lambda(r \cdot g(m)) && \text{(by the symmetry of } \lambda) \\ &= \lambda(g(r \cdot m)) && (g \text{ is an } R\text{-homomorphism}) \\ &= \psi(g)(r \cdot m) \\ &= (\psi(g) \cdot r)(m), \end{aligned}$$

which implies that $\psi(g \cdot r) = \psi(g) \cdot r$. Next, ψ is *injective*. In fact, if $\psi(g) = 0$, then $\lambda(g(M)) = 0$. Since $g(M)$ is a left ideal of R (g being a left R-module homomorphism), this implies that $g(M) = 0$; that is, $g \equiv 0$. In the case when ${}_RM$ is f.g., we know from (16.34) that $\dim_k M^* = \dim_k \hat{M}$, so it follows that $\psi : M^* \to \hat{M}$ is an isomorphism of right R-modules.

(16.73) Remark. Some special cases of $M^* \cong \hat{M}$ over a symmetric k-algebra R are already implicit in some of our earlier results. For instance, let $\mathfrak{A}$ be a left ideal in R. Since the Nakayama automorphism of R can be taken as the identity, (16.45) and (15.14) give isomorphisms

$$\hat{\mathfrak{A}} \cong R/\text{ann}_r\, \mathfrak{A} \cong \mathfrak{A}^* \quad \text{and} \quad (R/\mathfrak{A})\hat{} \cong \text{ann}_r\, \mathfrak{A} \cong (R/\mathfrak{A})^*$$

in the category of right R-modules. Combining, for instance, the first isomorphism with (16.13) in the case of a principal 1-sided ideal, we deduce the following:

(16.74) Corollary. *For a symmetric algebra R and any element $a \in R$:*

$$(Ra)\hat{} \cong (Ra)^* \cong aR \quad \textit{and} \quad (aR)\hat{} \cong (aR)^* \cong Ra.$$

There are other interesting special properties of symmetric algebras too which are not shared by general Frobenius algebras. Foremost among these is the symmetry property of the Cartan invariants of a symmetric algebra (over a sufficiently large ground field). We shall now conclude §16F by proving this result, in a somewhat more general setting. We start with the following observation.

(16.75) Lemma. *Let P, Q be f.g. right modules over a weakly symmetric k-algebra R. If P is projective, then* $\dim_k \mathrm{Hom}_R(P, Q) = \dim_k \mathrm{Hom}_R(Q, P)$.

Proof. We may assume that $P = eR$ where e is a primitive idempotent. Since eR is projective, $\mathrm{Hom}_R(eR, -)$ is an exact functor, so $\dim_k \mathrm{Hom}_R(eR, -)$ is additive over exact sequences of f.g. R-modules. Similarly, the injectivity of eR implies the same for $\dim_k \mathrm{Hom}_R(-, eR)$. Therefore, it suffices to prove the Lemma in the case when Q is a *simple* R-module. Let $J = \mathrm{rad}(R)$, and write $\bar{R}$ as usual for R/J. Since eJ is the unique maximal submodule of $P = eR$, we have $\mathrm{Hom}_R(P, Q) \cong \mathrm{Hom}_R(\bar{e}\bar{R}, Q)$. Invoking now the assumption that R is weakly symmetric, we also have $\mathrm{soc}(eR) \cong \bar{e}\bar{R}$, and this is the unique simple submodule of P. Thus, $\mathrm{Hom}_R(Q, P) \cong \mathrm{Hom}_R(Q, \bar{e}\bar{R})$. It follows that, if $Q \ncong \bar{e}\bar{R}$, both $\mathrm{Hom}_R(P, Q)$ and $\mathrm{Hom}_R(Q, P)$ are zero, and if $Q \cong \bar{e}\bar{R}$, both $\mathrm{Hom}_R(P, Q)$ and $\mathrm{Hom}_R(Q, P)$ are $\cong \mathrm{End}_R(\bar{e}\bar{R})$. □

For a simple R-module $\bar{e}\bar{R}$ as above, let us write $\langle \bar{e}\bar{R}, Q\rangle$ for the number of times $\bar{e}\bar{R}$ occurs as a composition factor in the f.g. R-module Q. The following result relating $\langle \bar{e}\bar{R}, Q\rangle$ to $\mathrm{Hom}_R(eR, Q)$ is valid over any finite-dimensional algebra.

(16.76) Lemma. *Let Q be any f.g. right module over a k-algebra R, and $e \in R$ be any primitive idempotent. Then*

$$\dim_k \mathrm{Hom}_R(eR, Q) = \langle \bar{e}\bar{R}, Q\rangle \cdot \dim_k \mathrm{End}_R(\bar{e}\bar{R}).$$

Proof. Again, both sides of the equation are additive over exact sequences of f.g. R-modules $0 \to Q' \to Q \to Q'' \to 0$. Therefore, it suffices to check this equation in the case Q is simple. If $Q \ncong \bar{e}\bar{R}$, both sides of the equation are zero. If $Q \cong \bar{e}\bar{R}$, both sides of the equation are equal to $\dim_k \mathrm{End}_R(\bar{e}\bar{R})$. □

(16.77) Theorem. *For any primitive idempotents e, f in a weakly symmetric k-algebra R, we have*

$$\langle \bar{e}\bar{R}, fR\rangle \cdot \dim_k \mathrm{End}_R(\bar{e}\bar{R}) = \langle \bar{f}\bar{R}, eR\rangle \cdot \dim_k \mathrm{End}_R(\bar{f}\bar{R}).$$

In particular, if $\dim_k \mathrm{End}_R(\bar{e}\bar{R}) = \dim_k \mathrm{End}_R(\bar{f}\bar{R})$, *we have* $\langle \bar{e}\bar{R}, fR\rangle = \langle \bar{f}\bar{R}, eR\rangle$.

Proof. By (16.76), the left-hand side is $\dim_k \mathrm{Hom}_R(eR, fR)$, and the right-hand side is $\dim_k \mathrm{Hom}_R(fR, eR)$. These two numbers are equal according to (16.75). □

From this, we immediately deduce the following classical result.

(16.78) Corollary. *Suppose a weakly symmetric k-algebra R splits over k (in the sense that every simple right R-module has endomorphism ring equal to k). Then the (right) Cartan matrix of R is a symmetric matrix.*

In particular, this applies to a group algebra $R = kG$ of a finite group G (over a splitting field k). In the case when char k does not divide $|G|$, the Cartan matrix (c_{ij}) is just the identity matrix, so the real interest in (16.78) lies in the case when char k divides $|G|$. In this case, (16.78) is a well-known result in modular representation theory. Here, the symmetry of the Cartan matrix takes on an even stronger form: (c_{ij}) has a factorization $D^T D$ where D is a certain (rectangular) matrix known as the *decomposition matrix* (and "T" denotes transposition). Moreover, if $p > 0$ is the characteristic of k, then the determinant of (c_{ij}) is a power of p; in particular, (c_{ij}) is always nonsingular. All of this belongs more properly to the modular representation theory of finite groups; for the details, we refer the reader to authoritative treatments of the subject, such as "Representation Theory of Finite Groups and Associative Algebras" by Curtis and Reiner [62].

§16G. Why Frobenius?

While Frobenius algebras have been covered in numerous textbooks, few authors have tried to explain why these algebras were called Frobenius algebras. Thus it behooves us to say a few things here on Frobenius' role in this study. (Besides, this author happens to be a great fan of Frobenius! See Part I of my article, [Lam: 98].) Throughout the following, R denotes a finite-dimensional algebra over a field k.

The idea of comparing the representations afforded by the two right modules R_R and $(\hat{R})_R$ originated with Frobenius' work on "hypercomplex systems" (an older term for algebras), ca. 1903. Of course modules were not in vogue then, so Frobenius was working solely with matrices. Let $\epsilon_1, \ldots, \epsilon_n$ be a fixed k-basis for R. For any element $r \in R$, let

(16.79) $$\epsilon_i r = \sum_j a_{ij}^{(r)} \epsilon_j \quad \text{and} \quad r\epsilon_i = \sum_j b_{ji}^{(r)} \epsilon_j ,$$

where $a_{ij}^{(r)}$, $b_{ji}^{(r)} \in k$. Define the matrices $A(r)$, $B(r)$ by $A(r)_{ij} = a_{ij}^{(r)}$ and $B(r)_{ji} = b_{ji}^{(r)}$. Note that the subscript notations are set up in such a way that we have $A(rr') = A(r)A(r')$ and $B(rr') = B(r)B(r')$ (along with the obvious additivity properties), so from the classical (in particular Frobenius') viewpoint, $r \mapsto A(r)$ and $r \mapsto B(r)$ give two representations of the algebra R. Let us call these the first and second regular representations of R. In modern terms, A is just the matrix representation afforded by the right module R_R with respect to the basis $\{\epsilon_i\}$, and a quick calculation involving dual spaces shows that B is the matrix representation arising from the right module $(\hat{R})_R$ given with the dual basis $\{\hat{\epsilon}_i\}$. (Clearly, A and B are both faithful representations.) Although Frobenius did not use the technique of modules, the idea of comparing the two matrix representations

$r \mapsto A(r)$ and $r \mapsto B(r)$ came to him naturally.[92] In 1903, Frobenius obtained the first criterion for the equivalence of these two representations, as follows.

Let $\{c_{\ell ij}\}$ be the structure constants of the algebra R with respect to the basis $\{\epsilon_i\}$, so that we have the equations

(16.80) $$\epsilon_i \epsilon_j = \sum_\ell c_{\ell ij} \epsilon_\ell \quad \text{for all } i,\ j.$$

For any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$, Frobenius defined a matrix $P_\alpha \in \mathbb{M}_n(k)$ by taking

(16.81) $$(P_\alpha)_{ij} = \sum_\ell \alpha_\ell \, c_{\ell ij}\,;$$

he called P_α a *paratrophic matrix*. Frobenius' basic result is the following:

(16.82) Frobenius' Criterion. *The first and second regular representations of R are equivalent iff there exists a nonsingular paratrophic matrix (or, in Frobenius' own terms, iff the paratrophic determinant* $\det(P_\alpha)$ *does not vanish identically for* $\alpha \in k^n$*).*

Proof. For the modern reader, Frobenius' Criterion can be proved easily as follows. We already know from (3.15) that the representations A and B are equivalent iff some hyperplane (passing the origin) in R contains no nonzero left ideals. Therefore, we need only show that this latter condition is equivalent to the existence of a nonsingular paratrophic matrix. For an arbitrary hyperplane

$$H_\alpha = \left\{ \sum x_\ell \, \epsilon_\ell : \sum \alpha_\ell \, x_\ell = 0 \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \neq 0$, we claim that

(16.83) $$R \cdot \beta \subseteq H_\alpha \iff P_\alpha \cdot \beta = 0$$

for any column vector $\beta = (\beta_1, ..., \beta_n)^T \in k^n$. Indeed, $R \cdot \beta \subseteq H_\alpha$ amounts to

$$H_\alpha \ni \epsilon_i \sum_j \beta_j \, \epsilon_j = \sum_j \beta_j \sum_\ell c_{\ell ij} \, \epsilon_\ell = \sum_\ell \left(\sum_j c_{\ell ij} \beta_j \right) \epsilon_\ell \quad \text{(for all } i\text{)},$$

which, in turn, amounts to

$$0 = \sum_\ell \alpha_\ell \sum_j c_{\ell ij} \, \beta_j = \sum_j (P_\alpha)_{ij} \, \beta_j = 0 \quad \text{(for all } i\text{)};$$

that is, $P_\alpha \cdot \beta = 0$. This proves (16.83). From the equivalence in (16.83), it is immediate that some H_α contains no nonzero left ideals in R iff a suitable paratrophic matrix P_α is nonsingular. □

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[92]Actually, Frobenius did not write down explicitly the basic equations (16.79) in his 1903 paper. Instead, he arrived at the two representations A and B by working with the structure constants of the algebra, which we shall introduce momentarily.

Some years later, Frobenius' result was generalized by Brauer, who determined the space of intertwining matrices for the (first and second) regular representations A and B defined above. In classical terminology, an $n \times n$ matrix P over k is said to *intertwine* the representations A, B if $A(r)P = PB(r)$ for all $r \in R$. Brauer's result (for an arbitrary n-dimensional k-algebra R) is as follows.

(16.84) Theorem. *A matrix $P \in \mathbb{M}_n(k)$ intertwines A, B iff P is a paratrophic matrix P_α (for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$).*

(Note that A and B are equivalent representations iff there exists a *nonsingular* matrix intertwining A and B. Therefore, the above theorem subsumes (16.82), and may be viewed as its generalization. From a logical standpoint, the proof for (16.82) above could have been omitted; we gave it only for the sake of illustrating the relationship between Frobenius' Criterion and our earlier result (3.15).)

Proof of (16.84). Again, we prove (16.84) by using the modern method of R-modules. From the module-theoretic viewpoint, an intertwining matrix P for the representations A, B amounts to an R-homomorphism from the module R_R (affording the representation A) to the module $(\hat{R})_R$ (affording the representation B). Therefore, the k-space S of matrices intertwining A and B corresponds to $\mathrm{Hom}_R(R_R, (\hat{R})_R)$, which we can identify with $(\hat{R})_R$ in the usual way. This shows that $\dim_k S = n$, and that a typical intertwining matrix is the matrix of a map $f : R_R \longrightarrow (\hat{R})_R$ given by

$$f(r) = (\alpha_1 \hat{\epsilon}_1 + \cdots + \alpha_n \hat{\epsilon}_n) \cdot r \qquad (r \in R),$$

where $\alpha_i \in k$ and $\{\hat{\epsilon}_i\}$ is the dual basis to $\{\epsilon_i\}$ in $\hat{R}$.

Now let us compute the right R-action on $\hat{R}$. Since

$$\Big(\hat{\epsilon}_\ell \cdot \epsilon_i\Big)(\epsilon_j) = \hat{\epsilon}_\ell(\epsilon_i \epsilon_j) = \hat{\epsilon}_\ell\big(\sum_p c_{pij}\epsilon_p\big) = c_{\ell ij},$$

we have $\hat{\epsilon}_\ell \cdot \epsilon_i = \sum_j c_{\ell ij} \hat{\epsilon}_j \in \hat{R}$. Therefore, for the R-homomorphism f described in the last paragraph, we have

$$f(\epsilon_i) = \Big(\sum_\ell \alpha_\ell \, \hat{\epsilon}_\ell\Big) \cdot \epsilon_i = \sum_j \Big(\sum_\ell \alpha_\ell \, c_{\ell ij}\Big) \hat{\epsilon}_j .$$

Consequently, the matrix of f (with respect to the bases $\{\epsilon_i\}$ on R and $\{\hat{\epsilon}_j\}$ on $\hat{R}$) is precisely the paratrophic matrix P_α. □

(16.85) Corollary. *A k-basis for the space S of matrices intertwining the representations A and B is given by $\{P_1, \ldots, P_n\}$, where $(P_\ell)_{ij} = c_{\ell ij}$.*

Proof. Note that P_ℓ is the paratrophic matrix P_α when we set $\alpha = (0, \ldots, 1, \ldots, 0)$ (the ℓth unit vector). Since

$$P_{(\alpha_1, \ldots, \alpha_n)} = \sum \alpha_\ell \, P_\ell,$$

$\{P_1, \ldots, P_n\}$ generate the space T of all paratrophic matrices. By the Theorem, we have $S = T$. Since $\dim_k S = n$, it follows that $\{P_1, \ldots, P_n\}$ form a k-basis for S. □

In the late 30s, with the result (16.84) as the starting point, R. Brauer and his student C. Nesbitt revived the study of the regular representations of an algebra. From this work, the idea of a Frobenius algebra came to the fore. Naming these algebras after Frobenius was, of course, natural and completely justified. But how *exactly* should one name them? Widely accepted terms such as "Abelian groups" and "Noetherian rings" would suggest that one uses Frobenius' name also in an adjectival form. In a footnote of his 1938 paper in the *Annals of Mathematics*, C. Nesbitt wrote: "The writer, in collaborating with T. Nakayama, adopted the term Frobeniusean algebra, but now, quailing before our critics, we return to simply Frobenius algebra." So apparently, people were not attracted by a six-syllable word. However, the "we" in Nesbitt's statement did not seem to apply to Nakayama, who continued to use the term "Frobeniusean algebra" in most of his subsequent papers. Today, there is no question that "Frobenius algebra" has won out. There are currently probably as many as 15 mathematical objects and results named after Frobenius; all of these are called "Frobenius X" for some "X". Frobenius even has the unique distinction that his name *is* a mathematical object without any "X" attached: on an affine variety V defined over a finite field $\mathbb{F}_q$, "the frobenius" is the endomorphism of V which maps a point $(x_1, \ldots, x_n) \in V$ to the point $(x_1^q, \ldots, x_n^q) \in V$.

This is my little spin on my favorite mathematician Ferdinand Georg Frobenius, 1849-1917.

Exercises for §16

0. Show that, for a QF ring R, two principal indecomposable right R-modules U, U' are isomorphic iff $\mathrm{soc}(U) \cong \mathrm{soc}(U')$.

1. Let $(R, \mathfrak{m})$ be a local artinian ring with $K = R/\mathfrak{m}$. Show that the following are equivalent:

   (1) R is QF;
   (1)′ R is Frobenius;
   (2) $\mathrm{soc}(R_R)$ is a simple right R-module and $\mathrm{soc}({}_RR)$ is a simple left R-module;
   (2)′ $\mathrm{ann}_\ell(\mathfrak{m})$ is a 1-dimensional right K-vector space and $\mathrm{ann}_r(\mathfrak{m})$ is a 1-dimensional left K-vector space;
   (3) R_R and ${}_RR$ are uniform R-modules;
   (4) $E((R/\mathfrak{m})_R) \cong R_R$;
   (4)′ $E({}_R(R/\mathfrak{m})) \cong {}_RR$.

   Show that these conditions imply each of the following:

   (5) $E((R/\mathfrak{m})_R)$ is a cyclic R-module;
   (5)′ $E({}_R(R/\mathfrak{m}))$ is a cyclic R-module;

(6) R is a subdirectly irreducible ring in the sense of FC–(12.2) (i.e. R has a smallest nonzero ideal).

If R is commutative, show that all ten conditions above are equivalent. (The next exercise will show that the condition (2) has to be imposed on both socles, and that (6) $\Longrightarrow$ (1) need not hold if R is not assumed to be commutative.)

2. Construct a local artinian (necessarily Kasch) ring $(R, \mathfrak{m})$ such that

   (1) R is subdirectly irreducible; and

   (2) $\text{soc}({}_RR)$ is simple but $\text{soc}(R_R)$ is not.

   Such a ring R is, in particular, *not* QF. (**Sketch.** Let σ be an endomorphism of a field K such that $[K : \sigma(K)]$ is an integer $n > 1$. Let $R = K[x;\sigma]/(x^2)$ where $K[x;\sigma] = \{\sum a_i x^i\}$ is the skew polynomial ring over K with $xa = \sigma(a)x$ for $a \in K$. Show that R is local with maximal ideal $\mathfrak{m} = K\bar{x}$, and that $\text{soc}({}_RR) = {}_R\mathfrak{m}$ is simple, but $\text{soc}(R_R) = \mathfrak{m}_R$ is semisimple with length $n > 1$.)

3. Let $R = R_1 \times \cdots \times R_r$, where each R_i is a QF ring. Describe the Nakayama permutation of R in terms of the Nakayama permutations π_i of R_i ($1 \le i \le r$). (Note that R is a QF ring by (15.26)(3).) Using this result and the computation in (16.19)(4), show that there exist Frobenius algebras over any given field whose Nakayama permutation π is any prescribed permutation on a finite number of letters.

4. Show that the Nakayama permutation of a commutative QF ring R is the identity.

5. For a QF ring R, it is shown in (15.25) that there is an (R, R)-bimodule isomorphism $\text{soc}(R_R) \cong ({}_R\bar{R})^*$, where $\bar{R} = R/\text{rad}(R)$. Confirm this as a right R-module isomorphism by using the Nakayama permutation π for R.

6. For any primitive idempotent f in a QF ring, it is shown in (16.5) that $\text{soc}(Rf) \cong (\bar{f}\bar{R})^*$ as left R-modules (where $\bar{R} = R/\text{rad}(R)$). Give a direct proof of this by using Exercise (15.8), assuming only that R is 1-sided artinian and that $\text{soc}(R_R) = \text{soc}({}_RR)$.

7. Prove (16.37).

8. Let $a \in R$ where R is a QF ring. Show that $\text{length}(Ra) = \text{length}(aR)$. If R is, in fact, a Frobenius algebra over a field k, show that $\dim_k Ra = \dim_k aR$. Does this equation hold over a QF algebra?

9. Let a, b, c be elements in a QF ring R such that $a = b+c$. If $aR = bR+cR$ and $bR \cap cR = 0$, show that $Ra = Rb + Rc$ and $Rb \cap Rc = 0$.

10. Let R be a QF ring with $J = \text{rad}(R)$ such that $J^n = 0 \neq J^{n-1}$. Show that if M_R is a f.g. indecomposable module such that $MJ^{n-1} \neq 0$, then M is isomorphic to a principal indecomposable module.

11. A ring R is said to have (right) *finite representation type* if R has only finitely many isomorphism types of f.g. indecomposable R-modules. Let R be a QF ring as in Exercise 10. (1) Show that R has (right) finite representation type iff R/J^{n-1} does. (2) If each R/J^i is QF, show that R has finite representation type.

12. Let M be a right module over a QF ring R. Show that M_R is faithful iff M has a direct summand isomorphic to $e_1R \oplus \cdots \oplus e_sR$, where $\{e_1R, \ldots, e_sR\}$ is a complete set of principal indecomposable right R-modules. (In view of (18.8) below, this means that a module M_R is faithful iff it is a generator. For a more general result, see (19.19).)

13. For the algebra $R = k[x, y]/(x, y)^{n+1}$ over a field k, show that the mth R-dual of the unique simple R-module V is isomorphic to $(n+1)^m \cdot V$.

14. Let R be the algebra $k[t]/(f(t))$ over a field k, where $f(t)$ is a nonconstant polynomial. Verify explicitly that R is a Frobenius k-algebra by applying Nakayama's "dimension characterizations" for such algebras (as in (16.40)(2)).

15. (D. Benson) For any division ring k, show that the set of matrices of the form

$$\gamma = \begin{pmatrix} a & 0 & b & 0 & 0 & 0 \\ 0 & a & 0 & b & p & 0 \\ c & 0 & d & 0 & 0 & 0 \\ 0 & c & 0 & d & q & 0 \\ 0 & 0 & 0 & 0 & r & 0 \\ s & 0 & t & 0 & 0 & r \end{pmatrix}$$

over k forms a QF ring that is not a Frobenius ring. (**Hint.** Compare this ring with the ring in (16.19)(5).)

16. Let $k \subseteq K$ be a field extension of degree $n > 1$. In (16.19)(4), it is shown that the ring S of matrices

$$\gamma = \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{pmatrix} \qquad (a, b, x, y \in K)$$

is a QF (in fact Frobenius) ring. For the subring $R = \{\gamma \in S : a \in k\}$ of S, show that R is Kasch but not QF, and compute the (right) Cartan matrix of R.

17. Let R be the ring of matrices

$$\gamma = \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & c \end{pmatrix}$$

over a division ring k. Show that the socle of any 1-sided principal indecomposable R-module is simple, and compute the (left, right) Cartan matrices of R. Is R a QF ring?

18. In a ring theory text, the following statement appeared: "If R is QF, then R is the injective hull of $(R/\text{rad } R)_R$." Find a counterexample; then suggest a remedy.

19. In the 4-dimensional Frobenius algebra R in (16.19)(4), find an ideal $A \subseteq R$ for which $\text{ann}_r(A) \neq \text{ann}_\ell(A)$.

20. Show that if R and S are symmetric algebras over a field k, then so are $R \times S$, $R \otimes_k S$, and $\mathbb{M}_n(R)$.

21. Let K/k be a field extension, and let R be a finite-dimensional k-algebra. Show that R is a symmetric algebra over k iff $R^K = R \otimes_k K$ is a symmetric algebra over K.

22. Let $K \supseteq k$ be a finite field extension with a non-identity k-automorphism τ on K. With the multiplication

$$(a, b)(c, d) = (ac,\ ad + b\tau(c)) \qquad (a, b, c, d \in K),$$

$R := K \oplus K$ is a k-algebra of dimension $2[K : k]$. Show that R is a weakly symmetric, but not symmetric, local k-algebra.

23. (Nakayama-Nesbitt) In the last exercise, assume that $[K : k] = 2$, char $k \neq 2$, and let R be the weakly symmetric k-algebra defined there. Show that the scalar extension $R^K := R \otimes_k K$ is a K-algebra isomorphic to

$$S = \left\{ \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, x, y \in K \right\}.$$

Deduce that R^K is not a weakly symmetric K-algebra. (This contrasts with the conclusions of Exercise 21.)

24. If R and S are both symmetric algebras over a field k, show that $R \otimes_k S$ is also a symmetric k-algebra.

25. If R is a symmetric k-algebra over a field k and $0 \neq e = e^2 \in R$, show that eRe is also a symmetric k-algebra, with $\text{soc}(eRe) = e(\text{soc}(R))e$. Using this, show that, for any nonzero f.g. projective right R-module P, $\text{End}_R(P)$ is also a symmetric k-algebra.

26. For any symmetric k-algebra R with center $Z(R)$, show that:

(1) $z \in (xRy)^0 \iff yzx = 0$;
(2) $xRy = 0 \iff yRx = 0$;
(3) $Z(R) = [R, R]^0$, where $[R, R]$ denotes the additive subgroup of R generated by $xy - yx$ $(x, y \in R)$.

27. For a field k, compute the paratrophic determinants of the commutative k-algebras $R = k[x, y]/(x^2, y^2)$,

$$S = k[x, y]/(xy,\ x^2 - y^2),\quad T = k[x, y]/(x^2, xy, y^2),$$

and apply Frobenius' Criterion (16.82) to determine which of these is a Frobenius algebra. (Of course, it helps to have known the answers from earlier work: R, S, but not T.)

28. (Nakayama-Nesbitt) Show that a finite-dimensional k-algebra (over a field k) is a symmetric k-algebra iff there exists a *symmetric* nonsingular paratrophic matrix.

29. Let k be a field with $u, v \in k$ (possibly zero), and let R be the 4-dimensional k-algebra defined in (16.66). Using Frobenius' Criterion (16.82) and the last exercise, show that:

(1) R is a Frobenius k-algebra iff $uv \neq 0$;
(2) R is a symmetric k-algebra iff $u = v \neq 0$.

30. For any field k and any finite group G, compute the paratrophic matrix of the group algebra $R = kG$. Using this computation and Exercise 28, give another proof for the fact that R is a symmetric k-algebra.

31. (Theorem on Structure Constants) Let R be an algebra over a field k with basis $\{\epsilon_1, \ldots, \epsilon_n\}$ and let $\epsilon_i\epsilon_j = \sum_\ell c_{\ell ij}\epsilon_\ell$. For $\alpha_1, \ldots, \alpha_n \in k$, show that:

(1) $\sum_\ell \alpha_\ell c_{\ell ij} = 0\ (\forall\, i, j) \Longrightarrow \alpha_\ell = 0\ (\forall\, \ell)$.
(2) $\sum_\ell \alpha_\ell c_{i\ell j} = 0\ (\forall\, i, j) \Longrightarrow \alpha_\ell = 0\ (\forall\, \ell)$.
(3) $\sum_\ell \alpha_\ell c_{ij\ell} = 0\ (\forall\, i, j) \Longrightarrow \alpha_\ell = 0\ (\forall\, \ell)$.

32. (Pascaud-Valette) For any field k of characteristic $\neq 2$, show that there exists a symmetric k-algebra R with a k-automorphism α of order 2 such that the fixed ring $R^\alpha = \{x \in R : \alpha(x) = x\}$ is any prescribed finite-dimensional k-algebra. In particular, R^α need not be QF. (**Hint.** Use a symmetric algebra R of the type (16.62).)

33. (K. Wang) Let R be a Frobenius algebra over a field k with a nonsingular associative k-bilinear form $B : R \times R \to k$. Let G be a finite group of k-automorphism of R such that B is G-invariant (i.e., $B(gr, gr') = B(r, r')$ for every $g \in G$). If $|G|$ is not divisible by the characteristic of k, show that the fixed ring R^G is also a Frobenius k-algebra. Prove the same result for symmetric algebras. (**Hint.** For $s \in R^G$, show that $B(s, R^G) = 0 \Longrightarrow B(s, R) = 0$.)

34. (Rim, Giorgiutti) Let R be a right artinian ring with (right) Cartan matrix (c_{ij}). If $\det(c_{ij}) \neq 0$, show that two f.g. projective right R-modules are isomorphic iff they have the same composition factors (counted with multiplicities).

Chapter 7

Matrix Rings, Categories of Modules, and Morita Theory

This last chapter offers an introduction to the basic categorical aspects of the theory of rings and modules. Since its introduction in the 1940s by Eilenberg and MacLane, the categorical viewpoint has been widely accepted by working mathematicians. For ring theorists especially, the convenient use of the categorical language in dealing with modules serves to provide a unifying force for the subject, and has subsequently become an indispensable tool in its modern study. In this chapter, we shall focus on two of the most important concepts in the application of category theory to rings and modules, namely, the *equivalence* and *duality* between two categories of modules. Both of these concepts come from the ground-breaking paper of K. Morita [58], which set in place the basic treatment of these topics pretty much as they are in use today.

In the spirit of concrete approach to ring theory used in this book, we preface our discussion of the equivalence of module categories by a section (§17) on matrix rings. The equivalence of the category of modules over a ring S and the category of modules over a matrix ring $R = \mathbb{M}_n(S)$ can be described in very simple terms; yet this simple description already exhibits most of the features of a general equivalence between the categories of modules over two *arbitrary* rings R and S. Thus, before going on to the general study of equivalences, the matrix ring example is definitely worthy of a good scrutiny. Besides presenting this material, §17 also features other related material on matrix rings, notably a study of the *recognition* of a ring R as an $n \times n$ matrix ring over some other ring S, and an investigation of the *uniqueness* in the choice of the base ring S. As far as I can tell, such material has not been made available in book form before.

In §18, we take up the formal study of equivalences between two module categories $\mathfrak{M}_R$ and $\mathfrak{M}_S$. Here, the role played by the free module of finite rank in the matrix ring example is simply replaced by that of a "progenerator", or a finitely generated projective generator. If ${}_SP$ is a progenerator over S with an endomorphism ring R, tensoring with ${}_SP$ gives a category equivalence from $\mathfrak{M}_S$ to $\mathfrak{M}_R$; and furthermore, *any* equivalence between two module categories essentially arises in this manner. (There is also an alternative description of equivalences by covariant "Hom" functors instead of tensor functors.) This main theorem

of Morita on module category equivalences was hailed as "probably one of the most frequently used results in modern algebra" in the article of Arhangel'skii, Goodearl, and Huisgen-Zimmermann [97]. Morita's theory leads directly to the important notion of Morita equivalence of rings, and the adjunct notion of Morita-invariant properties. All of these are presented in §18, where our treatment follows largely the influential lecture notes of H. Bass [62]. In particular, the terminology of "Morita Context" used in §18 is due to Bass.

Of course, a good mathematical theory is never created without historical precedent. In the case of the Morita theory of equivalences, the precursor was the famous Artin-Wedderburn classification of semisimple rings. From Morita's viewpoint, the Artin-Wedderburn Theorem can be simply retrieved in the form that any semisimple ring is Morita-equivalent to a unique finite direct product of division rings. More generally, a right artinian ring is Morita-equivalent to a unique right artinian "basic ring" (one which, modulo its Jacobson radical, is a finite direct product of division rings). The theory of basic rings was also fairly well understood by the time Morita wrote his paper, and was invented earlier by M. Osima and others working in the theory of finite-dimensional algebras in connection with the representation theory of groups.

The last section (§19) of our book is devoted to Morita's duality theory for categories of modules. Here Morita was motivated by the classical Pontryagin duality for locally compact Hausdorff topological groups. Simply put, Morita replaced the role of the circle group by a suitable injective module, nowadays called an "injective cogenerator". The case of self-duality for f.g. modules over quasi-Frobenius rings (where the injective cogenerator is taken to be the ring itself) provided a simple yet compelling example. Morita's main results on duality are parallel to those he had for equivalences, the only difference being that, here, the duality functors are restricted to suitable subcategories of $\mathfrak{M}_R$ and $\mathfrak{M}_S$. The same ideas were presented a little later (but independently) by G. Azumaya [59] in the case of right artinian rings. In this connection, we should mention that, again independently of Morita, E. Matlis arrived at his duality theory for modules over a commutative complete noetherian local ring in the same year, 1958. In retrospect, Matlis' theory is a most concrete and very lovely example of Morita (self)-duality. Our treatment of duality in §19 follows the classical line, though it does contain some of the later contributions to the subject (using the notion of linear compactness of modules) due to B. Müller, T. Onodera and others.

As a historical note, we might mention that the main objective of Morita's 1958 paper appeared to be the development of duality theory with applications to the case of artinian rings. The treatment of equivalences was almost an afterthought, occupying only a smaller part of this classic paper. Yet today, this paper is cited much more for equivalences than for dualities, and Morita's ideas on how to construct category equivalences have been repeated many times over in other branches of mathematics, usually with fruitful results. Morita died in August of 1995, but his basic contributions to the category theory of modules are likely to occupy a long-lasting place in algebra and ring theory. The afore-mentioned obituary article on Kiiti Morita written by Arhangel'skii, Goodearl, and Huisgen-Zimmermann

puts his work in excellent historical perspective, and is highly recommended as companion reading material for anyone studying this chapter.

§17. Matrix Rings

§17A. Characterizations and Examples

In this first subsection, we present the basic characterization theorems for matrix rings. Some of these, for instance (17.10), (17.17), and (17.18) are surprisingly recent results.

For any ring S, we write $R = \mathbb{M}_n(S)$ for the ring of $n \times n$ matrices over S. It will be convenient to identify S with the subring of R consisting of the "scalar matrices", $\{sI_n : s \in S\}$. Thus,

$$sI_n = \text{diag}(s, \ldots, s) = s(E_{11} + \cdots + E_{nn}), \tag{17.1}$$

where $\{E_{ij}\}$ are the matrix units in R. It is easy to see that R is a free left (resp. right) S-module on the basis $\{E_{ij}\}$ (cf. Exercise 1.12). Besides satisfying the additive relation

$$E_{11} + \cdots + E_{nn} = I_n\,, \tag{17.2}$$

the matrix units also satisfy the multiplicative relations:

$$E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell} \qquad (1 \le i, j, k, \ell \le n), \tag{17.3}$$

where δ_{jk} are the Kronecker deltas. Using these relations, we arrive at our first basic result in this subsection:

(17.4) Proposition. *The subring $S \subseteq R = \mathbb{M}_n(S)$ is the centralizer (in R) of the set of matrix units $\{E_{ij}\}$.*

Proof. First, we clearly have $sE_{ij} = E_{ij}s$ for any $s \in S$. Now consider any $x = \sum s_{ij}E_{ij} \in R$ commuting with all $E_{k\ell}$. From $xE_{k\ell} = E_{k\ell}x$ and the relations (17.3), we obtain $\sum_i s_{ik}E_{i\ell} = \sum_j s_{\ell j}E_{kj}$. For $i \neq k$, this shows that $s_{ik} = 0$, and the previous equation simplifies to $s_{kk}E_{k\ell} = s_{\ell\ell}E_{k\ell}$, so we also have $s_{kk} = s_{\ell\ell}$ for all k, ℓ. Thus, $x = s_{11}I_n \in S$. □

Axiomatizing the properties of the E_{ij}'s, let us say that a set of elements

$$\{e_{ij} :\ 1 \le i, j \le n\}$$

in a ring R is *a set of matrix units* if $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$ (for all i, j, k, ℓ). If, in addition, $e_{11} + \cdots + e_{nn} = 1 \in R$, we say that $\{e_{ij}\}$ is a *full set* of matrix units in R. The effect of fullness is to ensure some kind of nondegeneracy for the set $\{e_{ij}\}$. For instance, if $ab = 1$ in a ring R, then $e = ba$ is an idempotent, and (by *FC*–p. 328)

$$e_{ij} = b^i(1 - e)a^j \qquad (1 \le i, j \le n)$$

will be a set of matrix units for any fixed n. However, this may not be a full set. Indeed, if we also have $ba = 1$, then all $e_{ij} = 0$, and they are not of any interest. On the other hand, if e_{ij} $(1 \le i, j \le n)$ is a *full* set of matrix units in a ring R, the following theorem, together with Remark (17.6), will show that R can be represented as an $n \times n$ matrix ring, with the e_{ij} 's as matrix units in the usual sense.

(17.5) Theorem. *For any ring R and any fixed integer $n \ge 1$, the following are equivalent:*

(1) *$R \cong \mathbb{M}_n(S)$ for some ring S.*
(2) *R has a full set of matrix units $\{e_{ij} : 1 \le i, j \le n\}$.*
(3) *$R = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$ for suitable right ideals $\mathfrak{A}_i$ which are mutually isomorphic as right R-modules.*

Proof. It suffices to prove (2)$\Longrightarrow$(3)$\Longrightarrow$(1). Assume (2) holds, and let $\mathfrak{A}_i = e_{i1}R$. Then $\mathfrak{A}_i \cong \mathfrak{A}_1$ as right R-modules. In fact, if we define $f : \mathfrak{A}_1 \to \mathfrak{A}_i$ by left multiplication by e_{i1}, and $g : \mathfrak{A}_i \to \mathfrak{A}_1$ by left multiplication by e_{1i}, then f, g are mutually inverse isomorphisms of right R-modules. Next, note that $\sum_i \mathfrak{A}_i = R$, since the sum contains $\sum_i e_{i1}e_{1i} = \sum_i e_{ii} = 1$. To prove (3), it remains only to show that the sum $\sum_i \mathfrak{A}_i$ is *direct*. Say $\sum_i e_{i1}r_i = 0$, where $r_i \in R$. Then $0 = e_{jj}\sum_i e_{i1}r_i = e_{j1}r_j$ for all j, as desired.

To prove (3)$\Longrightarrow$(1), let $\mathfrak{A} = \mathfrak{A}_1$, where the $\mathfrak{A}_i$'s are as given in (3), and let $S = \text{End}(\mathfrak{A}_R)$. Then $R_R \cong \mathfrak{A} \oplus \cdots \oplus \mathfrak{A}$, and

$$R \cong \text{End}(R_R) \cong \text{End}(\mathfrak{A} \oplus \cdots \oplus \mathfrak{A})_R \cong \mathbb{M}_n(S).$$

□

(17.6) Remark. While we have proved (2)$\Longrightarrow$(3)$\Longrightarrow$(1) above, it would be desirable to know also how to get (2)$\Longrightarrow$(1) directly without using the right ideals $\mathfrak{A}_i$ in (3). To give such a direct proof, we use the idea in (17.4). Given the e_{ij} 's in (2), let S be the centralizer (in R) of the set $\{e_{ij}\}$. It suffices to show that ${}_SR$ is free on the basis $\{e_{ij}\}$, for then the multiplication rules for the e_{ij} 's will show that $R \cong \mathbb{M}_n(S)$. For any $x \in R$, define $a_{ij} = \sum_k e_{ki}xe_{jk} \in R$. By direct calculation, we see that

$$a_{ij}e_{uv} = e_{ui}xe_{jv} = e_{uv}a_{ij},$$

so $a_{ij} \in S$. Letting $u = i$ and $v = j$, we have $a_{ij}e_{ij} = e_{ii}xe_{jj}$. Summing over i, j, we get $x = \sum_{i,j} a_{ij}e_{ij}$. This shows that $R = \sum_{i,j} Se_{ij}$, and an easy calculation shows that the e_{ij} 's are (left) linearly independent over S (cf. Exercise 2).

(17.7) Corollary. *Let $f : R \to R'$ be a ring homomorphism, where $R = \mathbb{M}_n(S)$ for some ring S. Then R' can be expressed in the form $\mathbb{M}_n(S')$ for some ring S'*

such that f is "induced" by some ring homomorphism $f_0 : S \to S'$. (In particular, any ring containing $\mathbb{M}_n(S)$ has the form $\mathbb{M}_n(S')$ for some ring $S' \supseteq S$.)

Proof (Sketch). If $\{E_{ij}\}$ are the matrix units for $R = \mathbb{M}_n(S)$, then $\{f(E_{ij})\}$ give a *full* set of matrix units in R'. (It is important that $f(1) = 1$ here.) Let S' be the centralizer of these matrix units in R'. Then $R' = \mathbb{M}_n(S')$ and f is essentially $\mathbb{M}_n(f_0)$, where f_0 is the ring homomorphism $S \to S'$ induced (by f) from the centralizer of $\{E_{ij}\}$ to the centralizer of $\{f(E_{ij})\}$. □

The following basic result on ideals in matrix rings was proved by an ad hoc calculation in *FC*–(3.1). A new conceptual proof for it is given below, using (17.7).

(17.8) Corollary. *Let I be an ideal in $R = \mathbb{M}_n(S)$. Then $I = \mathbb{M}_n(\mathfrak{A})$ for some ideal $\mathfrak{A}$ in S.*

Proof. Let $f : R \to \overline{R} = R/I$ be the quotient map, and let $\mathfrak{A} = \ker(f|S)$. Clearly $\mathbb{M}_n(\mathfrak{A}) \subseteq I$, so we can finish by showing that $I \subseteq \mathbb{M}_n(\mathfrak{A})$. By (17.7), $\overline{R} = \mathbb{M}_n(A)$, where A is the centralizer (in $\overline{R}$) of $\{f(E_{ij})\}$. Clearly $f(S) \subseteq A$. For any $x \in \sum a_{ij}E_{ij} \in I$ ($a_{ij} \in S$), we have $\sum f(a_{ij})f(E_{ij}) = 0$, so $f(a_{ij}) = 0$ for all i, j. Therefore, $a_{ij} \in \mathfrak{A}$ and $x \in \mathbb{M}_n(\mathfrak{A})$. □

Another application of (17.5) is to the question of when an endomorphism ring of a module is a full $n \times n$ matrix ring. The following result, our second recognition theorem in this subsection, provides a natural answer.

(17.9) Theorem. *Let $S = \mathrm{End}_A(P)$, where P is a right A-module. Then $S \cong \mathbb{M}_n(T)$ for some ring T iff $P \cong n \cdot Q$ for some right A-module Q.*

Proof. If indeed $P \cong n \cdot Q$ (for some Q), then

$$S \cong \mathrm{End}_A(n \cdot Q) \cong \mathbb{M}_n(T), \quad \text{where} \quad T = \mathrm{End}_A(Q).$$

Conversely, assume that $S \cong \mathbb{M}_n(T)$ for some ring T. Then, by (17.5), $S_S = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$ where the $\mathfrak{A}_i$'s are mutually isomorphic right ideals in S. We can then write $\mathfrak{A}_i = e_iS$ where the e_i 's are mutually orthogonal isomorphic idempotents with sum 1. (For the notion of isomorphism between idempotents, see *FC*–(21.20).) Let $P_i = \mathfrak{A}_iP = e_iP$, which are right A-modules of P for all i. It is easy to see that $P = P_1 \oplus \cdots \oplus P_n$, so we are done if we can show that the P_i 's are mutually isomorphic. To simplify the notations, let e, f be isomorphic idempotents in S; it will be sufficient to show that $eP \cong fP$ (as A-modules). By *FC*–(21.20) we can write $e = ab$, $f = ba$ for suitable elements a, $b \in S$. Now define $\varphi : eP \to fP$ by $\varphi(ep) = fbp$, and $\psi : fP \to eP$ by $\psi(fp') = eap'$. A routine check shows that these maps are well-defined A-homomorphisms. Since

$$\psi\varphi(ep) = \psi(fbp) = eabp = ep, \quad \text{and} \quad \varphi\psi(p') = \varphi(eap') = fbap' = fp',$$

φ and ψ are mutually inverse isomorphisms, as desired. □

We come now to our third recognition theorem for matrix rings; this is a recent result due to Agnarsson, Amitsur, and Robson [95].

(17.10) Theorem. *Let R be a ring, and $p, q \geq 1$ be fixed integers. Then $R = \mathbb{M}_{p+q}(S)$ for some ring S iff there exist elements $a, b, f \in R$ such that*

$$f^{p+q} = 0, \quad \textit{and} \quad af^p + f^q b = 1. \tag{17.11}$$

In order to motivate the two equations in (17.11), we shall first prove the "only if" part of the theorem. Suppose $\mathbb{M}_{p+q}(S)$, where S is some ring. Along with the matrix

$$f = E_{21} + E_{32} + \cdots + E_{p+q,p+q-1} \in R \ \ (\text{with } f^{p+q} = 0),$$

consider the following block matrices

$$a = \begin{pmatrix} 0 & I_q \\ 0 & 0 \end{pmatrix} \in R, \qquad b = \begin{pmatrix} 0 & I_p \\ 0 & 0 \end{pmatrix} \in R.$$

Since $f^p = \begin{pmatrix} 0 & 0 \\ I_q & 0 \end{pmatrix}$ and $f^q = \begin{pmatrix} 0 & 0 \\ I_p & 0 \end{pmatrix}$, we have

$$\begin{aligned} af^p + f^q b &= \begin{pmatrix} 0 & I_q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I_q & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ I_p & 0 \end{pmatrix}\begin{pmatrix} 0 & I_p \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I_p \end{pmatrix} = I_{p+q}. \end{aligned}$$

This proves the "necessity" part of the theorem.

For the "sufficiency" part of (17.10), we need the following lemma.

(17.12) Lemma. *Let A, B be two additive subgroups of a ring R which are closed under squaring. Let $f \in R$ be an element such that fB and Bf are both contained in B. Then $1 \in Af + fB$ implies that $1 \in Af^2 + B$.*

Proof. Write $1 = xf + fy$, where $x \in A$ and $y \in B$. Then

$$\begin{aligned} 1 &= x(xf + fy)f + fy \\ &= x^2 f^2 + (xf)yf + fy \\ &= x^2 f^2 + (1 - fy)yf + fy \\ &= x^2 f^2 + yf + fy - fy^2 f. \end{aligned}$$

Since $x^2 \in A$ and $yf + fy - fy^2 f \in B$, we have $1 \in Af^2 + B$. □

(17.13) Corollary. *Let $f \in R$ be such that $1 \in Rf^p + f^q R$, where $p, q \geq 1$. Then $1 \in Rf^{p+q-1} + fR$.*

Proof. We may assume that $q \geq 2$ (for otherwise there is nothing to prove). Let $A = Rf^{p-1}$ and $B = f^{q-1}R$, which obviously have the properties stipulated in the above lemma. Since $1 \in Af + fB$, the lemma implies that

$$1 \in Rf^{p-1}f^2 + f^{q-1}R = Rf^{p+1} + f^{q-1}R.$$

Repeated use of this argument then gives $1 \in Rf^{p+q-1} + fR$. $\square$

We can now begin the proof of the "sufficiency" part of (17.10). Assume $a, b, f \in R$ exist as in (17.11). By (17.13), there also exist $c, d \in R$ such that $cf^{n-1} + fd = 1$, where $n = p + q$. Left multiplying this equation by f^{n-1} and using the fact that $f^n = 0$, we get $f^{n-1}cf^{n-1} = f^{n-1}$, so $e := cf^{n-1} \in R$ is an idempotent.

Let $\mathfrak{A}_i = f^{i-1}eR$, where $1 \leq i \leq n$. We claim that:

(A) *For each* $i \leq n$, $\mathfrak{A}_i \cong eR$ *as right ideals.*

(B) $R = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_n$.

Once these two facts are proved, we will have

(17.14) $$R \cong \text{End}(R_R) \cong \text{End}_R(n \cdot eR) \cong \mathbb{M}_n(S),$$

where $S = \text{End}_R(eR)$, as desired.

To prove (A), consider the R-epimorphism

$$\varphi_i : eR \longrightarrow \mathfrak{A}_i = f^{i-1}eR$$

defined by left multiplication by f^{i-1}. To show that φ_i is injective, suppose

$$0 = \varphi_i(er) = f^{i-1}er,$$

where $r \in R$. Left multiplying by cf^{n-i}, we get

$$0 = cf^{n-i}f^{i-1}er = cf^{n-1}er = e^2r = er,$$

so $\ker(\varphi_i) = 0$. For (B), note that

$$\begin{aligned} 1 &= e + fd \\ &= e + f(e + fd)d \\ &= e + fed + f^2d^2 \\ &= e + fed + f^2(e + fd)d^2 \\ &= e + fed + f^2ed^2 + f^3d^3 = \cdots \\ &= e + fed + f^2ed^2 + \cdots + f^{n-1}ed^{n-1} + f^nd^n. \end{aligned}$$

Since $f^n = 0$, this yields $1 \in \sum_i \mathfrak{A}_i$. Finally, to show that this sum is direct, consider an equation

$$er_1 + fer_2 + \cdots + f^{n-1}er_n = 0.$$

Since $ef = cf^{n-1}f = 0$, left multiplication by e shows that $0 = e^2r_1 = er_1$. Dropping the first term above and left multiplying by cf^{n-2}, we get

$$0 = cf^{n-2}fer_2 = e^2r_2 = er_2.$$

Repeating this argument, we see that $er_i = 0$ for all i.

This completes the proof of Theorem 17.10. However, there is much more to be said. To give the additional information, we continue the analysis started in the above proof. By "identifying" each $\mathfrak{A}_i$ with $\mathfrak{A}_1 \in eR$ via the isomorphism φ_i, we get an identification of R with the matrix ring $\mathbb{M}_n(S)$ as in (17.14) where $S = \mathrm{End}_R(eR)$. In particular, R has now a full set of $n \times n$ matrix units $\{E_{ij}\}$. The following result completes the information.

(17.15) Theorem. *With the above identifications, we have*

(1) $f = E_{21} + E_{32} + \cdots + E_{n,n-1} \in \mathbb{M}_n(S)$.
(2) $E_{ij} = f^{i-1}ed^{j-1}$ $(1 \le i, j \le n)$.
(3) $S \cong \mathrm{End}_R(f^{n-1}R) \cong \mathbb{I}_R(fR)/fR$, *where*

$$\mathbb{I}_R(fR) = \{r \in R : \ rf \in fR\}$$

is the "idealizer" of the right ideal fR in R.

Proof. Under the identification $\mathrm{End}(R_R) = R$, $f \in R$ acts as left multiplication on R_R, which takes $\mathfrak{A}_i$ "identically" to $\mathfrak{A}_{i+1}$ for $i < n$, and takes $\mathfrak{A}_n$ to (0). (Here, all $\mathfrak{A}_i$'s are "identified" with $\mathfrak{A}_1$.) Therefore, f corresponds to the matrix $E_{21} + E_{32} + \cdots + E_{n,n-1}$. Next, note that e corresponds to E_{11}, and so

$$\begin{aligned} f^{i-1}e &= (E_{21} + E_{32} + \cdots + E_{n,n-1})^{i-1}E_{11} \\ &= (E_{i,1} + E_{i+1,2} + \cdots + E_{n,n-i+1})E_{11} \\ &= E_{i1} \end{aligned}$$

for $i \le n$. To compute the other matrix units, think of $d \in R$ as an $n \times n$ matrix, and write it in the block form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where β is an $(n-1) \times (n-1)$ block. Since

$$fd = 1 - cf^{n-1} = 1 - e = E_{22} + \cdots + E_{nn},$$

we have

$$\begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix},$$

so $\alpha = 0$ and $\beta = I_{n-1}$. It follows now by easy matrix multiplications that $ed = E_{12}$, $ed^2 = E_{13}, \ldots,$ and $ed^{n-1} = E_{1n}$. Therefore, for all $i, j \le n$, we have

$$E_{ij} = E_{i1}E_{1j} = f^{i-1}e \cdot ed^{j-1} = f^{i-1}ed^{j-1},$$

as asserted in (2). We finish now by verifying the two descriptions of the base ring S in (3). For the first description, recall that $eR \cong f^{n-1}eR$. Left multiplying $e + fd = 1$ by f^{n-1}, we have $f^{n-1}e = f^{n-1}$, so

$$S = \mathrm{End}_R(eR) \cong \mathrm{End}_R(f^{n-1}R).$$

For the second description of S, note that

$$\begin{aligned} fR &= f(eR + feR + \cdots + f^{n-1}eR) \\ &= feR + f^2eR + \cdots + f^{n-1}eR. \end{aligned}$$

From this, we have $R = eR \oplus fR$, and hence $eR \cong R/fR$ as right R-modules. Taking endomorphism rings, we get $S \cong \mathrm{End}_R(R/fR)$, which is easily seen to be isomorphic to the quotient ring $\mathbb{I}_R(fR)/fR$. □

The good thing here is that we have been able to compute a base ring for the $n \times n$ matrix ring R by using only the element f, and not using the elements c, d (or the elements a, b in the original equations in (17.10)). The description (3) above for the base ring S appeared in Lam-Leroy [96].

In view of the results above, it is possible to define a "generic" $n \times n$ matrix ring, as follows. For any given commutative ring k, let R be the k-algebra with generators c, d, f and relations

(17.16) $$f^n = 0 \quad \text{and} \quad cf^{n-1} + fd = 1.$$

The ring R is a generic $n \times n$ matrix k-algebra in the sense that another k-algebra R' is an $n \times n$ matrix algebra iff there exists a k-algebra homomorphism from R to R'. The following result of Agnarsson, Amitsur, and Robson computes R explicitly and determines a base ring over which R is an $n \times n$ matrix ring.

(17.17) Theorem. *For the generic $n \times n$ matrix algebra R defined above, we have $R \cong \mathbb{M}_n(T)$, where $T = k\langle (x_{ij}) \rangle$ is the free k-algebra in the (noncommuting) variables $\{x_{ij} : 1 \leq i, j \leq n\}$.*

Proof. We shall use here the notations in the proof of (17.15). In particular, R is realized as an $n \times n$ matrix algebra $\mathbb{M}_n(S)$ over some k-algebra S, in such a way that $e, f \in R$ are realized, respectively, as E_{11} and $E_{21} + E_{32} + \cdots + E_{n,n-1}$, and $d \in R$ is realized as $\begin{pmatrix} 0 & I_{n-1} \\ \gamma & \delta \end{pmatrix}$. Since $E_{11} = e = cf^{n-1} = cE_{n,1}$, it follows similarly that $c \in R$ is realized as

$$\begin{pmatrix} s_{11} & \dots & s_{1,n-1} & 1 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \vdots \\ s_{n1} & \dots & s_{n,n-1} & 0 \end{pmatrix},$$

for suitable $s_{ij} \in S$. Now let the last row of the matrix d be

$$(\gamma, \delta) = (s_{1n}, s_{2n}, \dots, s_{nn}), \quad \text{where } s_{in} \in S.$$

Let $\varphi : T \to S$ be the unique k-algebra homomorphism with $\varphi(x_{ij}) = s_{ij}$ for all i, j. Then φ extends naturally to a k-algebra homomorphism

$$\Phi : \mathbb{M}_n(T) \longrightarrow \mathbb{M}_n(S) = R.$$

In $\mathbb{M}_n(T)$, we have the matrices

$$C := \begin{pmatrix} x_{11} & \dots & x_{1,n-1} & 1 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \vdots \\ x_{n1} & \dots & x_{n,n-1} & 0 \end{pmatrix}, \qquad D := \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{pmatrix}$$

and $F := E_{21} + \cdots + E_{n,n-1}$, which satisfy the relations

$$F^n = 0 \quad \text{and} \quad CF^{n-1} + FD = I_n.$$

Therefore, there exists a (unique) k-algebra homomorphism $\Psi : R \to \mathbb{M}_n(T)$ sending $c,\ d,\ f$, respectively, to $C,\ D,\ F$. Clearly, $\Phi\Psi = \mathrm{Id}_R$ (since it is the identity on $c,\ d,\ f$). If we can show that Ψ is a *surjection*, then Φ must be an isomorphism and we are done. To show that Ψ is a surjection, it suffices to check the following:

(1) $\Psi(E_{ij}) = E_{ij}$ *for all* $i,\ j$;
(2) $\mathrm{im}(\Psi) \supseteq T$ (*the subring of scalar matrices in* $\mathbb{M}_n(T)$).

Here, (1) follows readily since

$$E_{ij} = f^{i-1}(cf^{n-1})d^{j-1} \in \mathbb{M}_n(S),$$

and we have a similar equation in $\mathbb{M}_n(T)$. To prove (2), it suffices to show that $xI_n \in \mathrm{im}(\Psi)$ for *any* x in the set $\{x_{ij}\}$. Now x appears as an entry in either C or D. Since $\mathrm{im}(\Psi)$ contains $C,\ D$ and all matrix units, it follows easily that $\mathrm{im}(\Psi)$ contains xE_{pq} for all $p,\ q$. In particular, $\mathrm{im}(\Psi)$ contains $x\sum E_{pp} = xI_n$, as desired. □

Remark. Of course, the fact that Ψ is a surjection implies that $\Psi\Phi$ is also the identity map (on $\mathbb{M}_n(T)$), so Φ and Ψ are mutually inverse isomorphisms. From this, it follows that $\varphi : T \to S$ is an isomorphism, so we can conclude that $S \cong k\langle(x_{ij})\rangle$.

Using Theorem (17.10), one can come up with some rather surprising examples of matrix rings. We present below an example from Robson [91] and Chatters [92] on 2×2 matrix rings.

(17.18) Example. Let $H = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$ be the ring of integer quaternions, and let R be the subring $\begin{pmatrix} H & 3H \\ H & H \end{pmatrix}$ of $\mathbb{M}_2(H)$. *It turns out that* R *is a* 2×2 *matrix ring*! To see this, we apply the criterion (17.10) with $m = n = 1$. Let $\alpha = i + j + k \in H$, and

(17.19) $$f = \begin{pmatrix} \alpha & 3 \\ 1 & -\alpha \end{pmatrix}, \qquad a = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix} \ \text{in } R.$$

Since $\alpha^2 = -3$, we have $f^2 = (\alpha^2 + 3)I_2 = 0$, and a direct calculation shows that

$$af + fa = (3 + i\alpha + \alpha i)I_2 = I_2.$$

From (17.10) and (17.15)(3), it follows that $R \cong \mathbb{M}_n(S)$ where $S = \mathrm{End}_R(fR)$. To compute S more explicitly, consider the unit $u = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$ of $\mathbb{M}_n(H)$. We have $f' := uf = \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix} \in R$, and so

$$fR \cong ufR = f'R = \begin{pmatrix} 0 & 0 \\ H & \alpha H \end{pmatrix}.$$

We claim that $\mathrm{End}_R(f'R)$ *is isomorphic to* $T := \mathbb{I}_H(\alpha H)$ (the idealizer of the right ideal αH in H). To see this, define a ring homomorphism $\lambda : T \to \mathrm{End}_R(f'R)$ by taking $\lambda(\beta)$ ($\beta \in T$) to be left multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ on $\begin{pmatrix} 0 & 0 \\ H & \alpha H \end{pmatrix}$. It is easy to see that λ is injective. To see that λ is also surjective, consider any $g \in \mathrm{End}_R(f'R)$. Then $g\begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta & \alpha\gamma \end{pmatrix}$ for some $\beta, \gamma \in H$. Since

$$\begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} -\alpha & 0 \\ 0 & 0 \end{pmatrix},$$

we must have

$$\begin{pmatrix} 0 & 0 \\ \beta & \alpha\gamma \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta & \alpha\gamma \end{pmatrix}\begin{pmatrix} -\alpha & 0 \\ 0 & 0 \end{pmatrix};$$

that is, $\alpha\gamma = -\beta\alpha$. Thus $\beta \in T$, and $g = \lambda(\beta)$. Summing up, we have $R \cong \mathbb{M}_2(\mathbb{I}_H(\alpha H))$. (We leave it to the reader to show that the idealizer $\mathbb{I}_H(\alpha H)$ is just the subring $\mathbb{Z} + \alpha H$ in H.)

Using a little bit of number theory, it can be shown by a similar application of (17.10) that, for any *odd* integer n, the ring $R_n = \begin{pmatrix} H & nH \\ H & H \end{pmatrix}$ is also a 2×2 matrix ring. On the other hand, if n is *even*, then R_n is *not* a 2×2 matrix ring. To see this, note that

$$\begin{pmatrix} w & nx \\ y & z \end{pmatrix} \mapsto \overline{w} \in H/nH$$

defines a surjective ring homomorphism φ_n from R_n onto H/nH. Composing this with $H/nH \to H/2H$, we get a surjection $R_n \to H/2H$. Since $H/2H$ is commutative, it cannot be a 2×2 matrix ring. It follows from (17.8) that R_n also cannot be a 2×2 matrix ring (if n is even).

The fact that R_3 is a 2×2 matrix ring would appear less surprising in view of the existence of the ring homomorphism $\varphi_3 : R_3 \to H/3H$. The ring $H/3H$ is

just the Hamiltonian algebra of quaternions defined over the field $\mathbb{F}_3$. Since this is a central simple $\mathbb{F}_3$-algebra, it must be isomorphic to $\mathbb{M}_2(\mathbb{F}_3)$. The elements $\overline{\alpha}$ and $\overline{i}$ in $H/3H$ obviously satisfy

$$\overline{\alpha}^2 = -\overline{3} = 0 \quad \text{and} \quad \overline{i}\,\overline{\alpha} + \overline{\alpha}\,\overline{i} = 2\,\overline{i}^{\,2} = \overline{1},$$

thus reaffirming $H/3H \cong \mathbb{M}_2(\mathbb{F}_3)$ by (17.10). To apply (17.10) to R_3, it suffices to "lift" the above two equations via φ_3 to R_3. The two elements $a,\ f \in R_3$ constructed in (17.19) provide precisely such a lifting.

§17B. First Instance of Module Category Equivalences

This subsection is intended to be an introduction to a special case of the Morita Theory. As such, it provides important motivational material for the general Morita Theory which is to be developed in §§18–19.

In a nutshell, what we shall do in this subsection is to provide an explicit "equivalence" between the categories of right modules over a ring S and over the matrix ring $R = \mathbb{M}_n(S)$ (for a fixed n). The construction of this "equivalence" is entirely explicit, so we can see in a very concrete way how two different rings can have "equivalent" module categories. In our opinion, this construction is definitely worth knowing before the onslaught of all the tensor product formations necessary for developing the general Morita Theory. Furthermore, the particular construction of the equivalence between the module categories over S and $\mathbb{M}_n(S)$ turns out to contain most of the key features of an equivalence of module categories in general. Therefore, a good mastery of the material in this subsection will prove to be a valuable aid for understanding the general theory presented in §§18–19 below.

Continuing to write $\mathfrak{M}_R$ for the category of right R-modules, we state the main result in this subsection as follows.

(17.20) Theorem. *For $R = \mathbb{M}_n(S)$ (for any fixed $n \geq 1$), the module categories $\mathfrak{M}_R$ and $\mathfrak{M}_S$ are "equivalent".*

By this statement, we mean that there exist a functor[93] $F : \mathfrak{M}_R \to \mathfrak{M}_S$ and a functor $G : \mathfrak{M}_S \to \mathfrak{M}_R$ such that $G \circ F \cong$ identity functor on $\mathfrak{M}_R$, and $F \circ G \cong$ identity functor on $\mathfrak{M}_S$. (Here, we use the symbol "$\cong$" for the isomorphism, or more precisely, natural equivalence, of functors). If such functors F, G can be found, we shall say that F (or G) defines an *equivalence* between the two categories $\mathfrak{M}_R$ and $\mathfrak{M}_S$. At this point, we should note that, since we shall be dealing exclusively with additive categories, *all functors* (such as F and G above) *are assumed to be additive functors.*

Proof of (17.20). For $V \in \mathfrak{M}_S$ (shorthand for $V \in \text{Obj}\, \mathfrak{M}_S$), define $G(V)$ to be $V^{(n)}$, the space of row n-tuples $(v_1, \ldots, v_n)$ $(v_i \in V)$. Letting $R = \mathbb{M}_n(S)$ operate on $V^{(n)}$ by matrix multiplication from the right, we have $G(V) = V^{(n)} \in$

[93]Unless stated otherwise, the word "functor" shall always mean a *covariant* functor.

$\mathfrak{M}_R$. For $\alpha \in \text{Hom}_S(V, V')$ (written on the left, opposite the scalars), we define $G(\alpha) : G(V) \to G(V')$ by

$$(17.21) \qquad G(\alpha)(v_1, \ldots, v_n) = (\alpha v_1, \ldots, \alpha v_n).$$

We check easily that $G(\alpha)$ is an R-homomorphism, and that G defines a functor from $\mathfrak{M}_S$ to $\mathfrak{M}_R$. (Roughly speaking, G is given by "tensoring with a free left S-module of rank n". This is a useful remark to keep in mind when we try to generalize the construction of G later in §18.)

For any $U \in \mathfrak{M}_R$, define $F(U)$ to be UE_{11}. Since $UE_{11}s = UsE_{11} \subseteq UE_{11}$ for any $s \in S$, we have $F(U) \in \mathfrak{M}_S$. For any $\beta \in \text{Hom}_R(U, U')$, clearly

$$\beta(UE_{11}) = (\beta U)E_{11} \subseteq U'E_{11}.$$

Taking $F(\beta)$ to be the map $F(U) \to F(U')$ induced by β, we have

$$F(\beta) \in \text{Hom}_S(F(U), F(U')).$$

Again, it is routine to check that F defines a functor from $\mathfrak{M}_R$ to $\mathfrak{M}_S$.

For $V \in \mathfrak{M}_S$, $(F \circ G)(V) = F(V^{(n)}) = V^{(n)}E_{11}$ consists of rows of the form $(v, 0, \ldots, 0)$ $(v \in V)$, so $V^{(n)}E_{11}$ is "naturally isomorphic" to V. This shows that $F \circ G$ is naturally equivalent to the identity functor on $\mathfrak{M}_S$.

To compute $G \circ F$, consider $U \in \mathfrak{M}_R$ and write $V = UE_{11} = F(U)$. *Our job is to find a natural isomorphism* ε_U *from* U *to* $V^{(n)} = (G \circ F)(U)$. We define ε_U by:

$$(17.22) \qquad \varepsilon_U(u) = (uE_{11}, uE_{21}, \ldots, uE_{n1}) \in V^{(n)},$$

noting that $uE_{i1} = (uE_{i1})E_{11} \in V$ for all i. To check that ε_U is an R-homomorphism, it suffices to show that $\varepsilon_U(u \cdot sE_{ij}) = \varepsilon_U(u)sE_{ij}$ (for $s \in S$). By definition:

$$\begin{aligned} \varepsilon_U(u \cdot sE_{ij}) &= (u \cdot sE_{ij}E_{11}, \ldots, u \cdot sE_{ij}E_{n1}) \\ &= (0, \ldots, u \cdot sE_{i1}, \ldots, 0), \end{aligned}$$

where $u \cdot sE_{i1}$ occurs in the jth position. On the other hand, by direct matrix multiplication, $\varepsilon_U(u)sE_{ij}$ is seen to be the same thing. Hence $\varepsilon_U \in \text{Hom}_R(U, V^{(n)})$. Next, we check that ε_U is an *isomorphism*. First, if $\varepsilon_U(u) = 0$, then

$$uE_{ii} = (uE_{i1})E_{1i} = 0, \quad \text{so} \quad u = u\left(\sum E_{ii}\right) = 0.$$

To show that ε_U is surjective, it suffices to show that $(v, 0, \ldots, 0) \in \text{im}(\varepsilon_U)$ for any $v \in V$. Fix any element $u \in U$ such that $v = uE_{11}$. Then

$$\begin{aligned} \varepsilon_U(v) &= (vE_{11}, \ldots, vE_{i1}, \ldots, vE_{n1}) \\ &= (vE_{11}E_{11}, \ldots, uE_{11}E_{i1}, \ldots, uE_{11}E_{n1}) \\ &= (v, 0, \ldots, 0). \end{aligned}$$

Finally, it is easy to check that ε_U is a natural isomorphism. Therefore, $G \circ F \cong$ identity functor on $\mathfrak{M}_R$, and we have completed the proof of (17.20). □

Several remarks about the category equivalences F and G above are in order.

(17.23) Remarks.

(A) Suppose $\mathcal{P}$ is a certain "categorical" property; that is, a property on modules that can be defined by using only the module category, and not using the base ring. Then, the fact that $G : \mathfrak{M}_S \to \mathfrak{M}_R$ is a category equivalence implies that a right S-module V has the property $\mathcal{P}$ iff the right R-module $G(V)$ does (and a similar statement holds for F). Some examples of such properties are "projectivity", "injectivity", and "finite generation" of (right) modules. The first two cases are clear. For the third case, we have to give a characterization of "finite generation" in terms of submodules (rather than in terms of elements). This is easy to do: a module V_S is finitely generated iff, for any set of submodules $\{V_i : i \in I\}$ in V, $V = \sum_{i \in I} V_i$ implies that $V = \sum_{i \in J} V_i$ for some finite subset $J \subseteq I$.

(B) Being the right regular module is not a categorical property. In fact, neither F nor G preserves the right regular modules (if $n > 1$). For the right regular module $V = S_S$, $G(V) = S^{(n)}$ is only a projective R-module, with $n \cdot S^{(n)} \cong R_R$. Loosely speaking, $G(S_S)$ is "$1/n$" of R_R. On the other hand, for $U = R_R$,

$$F(U) = RE_{11} = E_{11}S \oplus \cdots \oplus E_{n1}S,$$

and this is "n copies" of S_S. Thus, F takes free modules to free modules, but (in general) G does not.

(C) The number of elements needed to generate a f.g. module is not a categorical quantity, and behaves in a rather interesting way with respect to the functors F, G. For instance, if $V \in \mathfrak{M}_S$ is generated by n elements, then $G(V)$ is a *cyclic* module in $\mathfrak{M}_R$. In fact, suppose $V = \sum_{i=1}^n a_i S$, and consider any $(v_1, \ldots, v_n) \in V^{(n)} = G(V)$. Writing $v_j = \sum_{i=1}^n a_i s_{ij}$, we have

$$(v_1, \ldots, v_n) = (a_1, \ldots, a_n)(s_{ij}),$$

so $(a_1, \ldots, a_n)$ is a cyclic generator for the R-module $G(V)$.

The last remark has, for instance, the following interesting application to the study of principal right ideal rings.

(17.24) Theorem. *Let S be a principal right ideal domain. Then for any $n \geq 1$, $R = \mathbb{M}_n(S)$ is a principal right ideal ring.*

Proof. As we have observed in (B) above, $F(R_R)$ is a free S-module $\cong S_S^n$. A right ideal $\mathfrak{A} \subseteq R_R$ corresponds to an S-submodule $V \subseteq S_S^n$. By (2.27), $V_S \cong S_S^m$ for some m, and since S is right noetherian, $m \leq n$ (cf. (1.35)). Thus, V_S can be generated by n elements. It follows from the discussion in (C) above that $\mathfrak{A}_R$ is cyclic, so $\mathfrak{A}$ is a principal right ideal in R. □

In general, a ring T is said to be *Morita equivalent to* S if there exists a category equivalence between $\mathfrak{M}_S$ and $\mathfrak{M}_T$. Obviously, Morita equivalence is an

equivalence relation among rings. For a given ring S, it is of great importance to describe the Morita equivalence class of S. By (17.20), all matrix rings $\mathbb{M}_n(S)$ ($n \geq 1$) belong to this class, but in general, there may be other rings in the class too. The following result is a sneak preview of Morita's main theorems in §18.

(17.25) Theorem. *A ring T is Morita equivalent to S iff $T \cong \text{End}_S(P)$, where P_S is a progenerator in $\mathfrak{M}_S$.*

The notion of a progenerator used here was briefly introduced at the end of §2: we say that P_S is a *progenerator* if P is f.g. projective, and $\text{tr}(P) = S_S$ (or equivalently, there exists an epimorphism $n \cdot P \to S_S$ for some $n \geq 1$). For instance, the free modules $n \cdot S$ (for any $n \geq 1$) are progenerators. Choosing P to be $n \cdot S$, we get the rings $\text{End}_S(P) \cong \mathbb{M}_n(S)$ in the Morita equivalence class of S. Other choices of P, in general, might lead to further rings in the class not isomorphic to any $\mathbb{M}_n(S)$.

The proof of Morita's Theorems will be presented in full in §18. At this point, the reader should assume (17.25) without proof, since we have an application of this result in mind for §17C.

§17C. *Uniqueness of the Coefficient Ring*

If R is an $n \times n$ matrix ring, say, $R \cong \mathbb{M}_n(S)$, it is natural to ask if S is uniquely determined by R, up to an isomorphism. In the first subsection (§17A), we were concerned with the problem of *recognition* of R as an $n \times n$ matrix ring, but the question of the *uniqueness* of the base ring was never raised. And then in §17B, we shifted our attention to category equivalences which do not seem to have any bearing on the uniqueness issue. In this subsection, we shall address the uniqueness question in earnest. As it turns out, the material developed in §17B is quite relevant to this discussion after all.

For convenience, let us say that a ring S is $\mathbb{M}_n$*-unique* if, for any ring S', $\mathbb{M}_n(S) \cong \mathbb{M}_n(S')$ implies that $S \cong S'$. The question is: *are there rings that are not $\mathbb{M}_n$-unique, and if so, which classes of rings are $\mathbb{M}_n$-unique*?

For the latter question, one class of rings comes to mind immediately, namely, the class of division rings. Using the Wedderburn-Artin Theorem (*FC*–(3.5)), it is easy to see that any division ring (and for that matter any semisimple ring) is $\mathbb{M}_n$-unique, for any $n \geq 1$. Another class of $\mathbb{M}_n$-unique rings is given by the following theorem.

(17.26) Theorem. *Let $\varphi : \mathbb{M}_n(S) \to \mathbb{M}_n(S')$ be a ring isomorphism, where S is a commutative ring. Then φ induces an isomorphism from S onto S' (where, as usual, S and S' are identified with the subrings of scalar matrices in $\mathbb{M}_n(S)$ and $\mathbb{M}_n(S')$.) In particular, any commutative ring S is $\mathbb{M}_n$-unique.*

Proof. We first make a useful observation without assuming commutativity on S. For any ring A, let $Z(A)$ denote the center of A. Consider any $M \in Z(\mathbb{M}_n(S))$.

Since M commutes with all E_{ij}'s, (17.4) implies that $M \in S$, and hence $M \in Z(S)$. This shows that $Z(\mathbb{M}_n(S)) = Z(S)$. Therefore, *any isomorphism* $\varphi : \mathbb{M}_n(S) \to \mathbb{M}_n(S')$ *must induce an isomorphism from* $Z(S)$ *to* $Z(S')$.

Now assume S is commutative. Then φ induces an injection of S into $Z(S')$. To simplify the notations, let us identify S with $\varphi(S)$, so that S "becomes" the center of S'. Our job is to show that, in fact, $S = S'$. Let us work in the category $\mathfrak{M}_S$. Since φ is an isomorphism, $\mathbb{M}_n(S)_S \cong \mathbb{M}_n(S')_S$. On the other hand,

$$\mathbb{M}_n(S)_S \cong S_S^{n^2} \quad \text{and} \quad \mathbb{M}_n(S')_S \cong (S'^{n^2})_S \cong (S'_S)^{n^2}.$$

Therefore, $(S'_S)^{n^2} \cong S_S^{n^2}$. This shows that S'_S is a f.g. projective right S-module of rank 1. By (2.50), $S'_S = S \oplus X$ for some (f.g. projective) right S-module X. Taking the rank, we see that $X = 0$, so $S = S'$, as desired. □

In order to get more definitive results on $\mathbb{M}_n$-uniqueness (or the lack of it), we must now bring to bear some module-theoretic techniques. We introduce the following two notions of "cancellation", where R is a ring, $\mathcal{C}$ is a class of right R-modules, and $n \geq 1$ is a given integer.

(17.27) Definition. (1) We say that $\mathcal{C}$ satisfies *n-cancellation* if, for any $P,\ P' \in \mathcal{C}$,

$$n \cdot P \cong n \cdot P' \Longrightarrow P \cong P'.$$

(2) We say that $\mathcal{C}$ satisfies *weak n-cancellation* if, for any $P,\ P' \in \mathcal{C}$,

$$n \cdot P \cong n \cdot P' \Longrightarrow \text{End}_R(P) \cong \text{End}_R(P').$$

Needless to say, if $\mathcal{C}$ satisfies n-cancellation, then it satisfies weak n-cancellation. The converse is, however, not true. For instance, if R is a commutative ring, the class of rank 1 projectives trivially satisfies weak n-cancellation for any n (in view of Exercise (2.27)), but may not satisfy n-cancellation.

The relevance of n-cancellation and weak n-cancellation to the problem of $\mathbb{M}_n$-uniqueness stems from the case where $\mathcal{C}$ is the class of progenerator modules. To explain this connection, we first make the following observation.

(17.28) Proposition. *Let A be any ring that is Morita-equivalent to S. Let $\mathcal{P}_S$ be the class of f.g. (resp. f.g. projective, progenerator) right S-modules, and let $\mathcal{P}_A$ be defined similarly. Then $\mathcal{P}_S$ satisfies n-cancellation iff $\mathcal{P}_A$ does. The same statement holds for weak n-cancellation.*

Proof. Let $G : \mathfrak{M}_S \to \mathfrak{M}_A$ be a category equivalence. Under G, it is easy to see that $\mathcal{P}_S$ corresponds to $\mathcal{P}_A$. (In case $\mathcal{P}_S$ is the class of progenerators, interpret a progenerator in $\mathfrak{M}_S$ as a f.g. projective P_S such that any f.g. projective in $\mathfrak{M}_S$ is an epimorphic image of some $n \cdot P$.) Since "direct sums" and "endomorphism rings" (of modules) are both categorical notions (therefore preserved by G), the desired conclusion follows. □

With the above proposition as a tool, we can now formulate more precisely the relationship between weak n-cancellation and $\mathbb{M}_n$-uniqueness.

(17.29) Theorem. *For any ring S, and any given integer $n \geq 1$, the following two statements are equivalent:*

(1) *Any ring T Morita-equivalent to S is $\mathbb{M}_n$-unique.*
(2) *The class $\mathcal{P}_S$ of progenerators in $\mathfrak{M}_S$ satisfies weak n-cancellation.*

Proof. (2)$\Longrightarrow$(1). Let T be as in (1), and let T' be another ring with $\mathbb{M}_n(T) \cong \mathbb{M}_n(T')$. To argue more symmetrically, let A be a ring isomorphic to these two matrix rings. Then A is Morita-equivalent to T, and hence to S. By (17.28), (2) implies that $\mathcal{P}_A$ satisfies weak n-cancellation. Identifying A with $\mathbb{M}_n(T)$ (resp. $\mathbb{M}_n(T')$), we get a decomposition of A_A into $n \cdot P$ (resp. $n \cdot P'$), where P is the right $\mathbb{M}_n(T)$-module given by $(T, \ldots, T)$ (and similarly for P'). Therefore, we have

$$n \cdot P \simeq n \cdot P' \cong A_A,$$

which implies that $P,\ P' \in \mathcal{P}_A$. By weak n-cancellation, we have then $\mathrm{End}_A(P) \cong \mathrm{End}_A(P')$. Now, by a straightforward calculation (cf. *FC*–p. 34), $\mathrm{End}_A(P)$ is isomorphic to T, and similarly $\mathrm{End}_A(P')$ is isomorphic to T'. Therefore, $T \cong T'$, as desired.

(1)$\Longrightarrow$(2). Suppose $n \cdot P \cong n \cdot P'$, where $P,\ P' \in \mathcal{P}_S$. Let $T = \mathrm{End}_S(P)$ and $T' = \mathrm{End}_S(P')$. By (17.25) (which we assumed without proof), T is Morita-equivalent to S. Taking the S-endomorphism rings of $n \cdot P \cong n \cdot P'$, we get an isomorphism $\mathbb{M}_n(T) \cong \mathbb{M}_n(T')$. But by (1), T is $\mathbb{M}_n$-unique, so we have $T \cong T'$; that is, $\mathrm{End}_S(P) \cong \mathrm{End}_S(P')$, as desired. □

Remark. Part of the subtlety of Theorem (17.29) lies in the fact that, in the condition (1) above, we must impose $\mathbb{M}_n$-uniqueness on *all* rings T that are Morita-equivalent to S. In general, this condition is not equivalent to the condition that S itself be $\mathbb{M}_n$-unique. For instance, by (17.26), a commutative ring S is $\mathbb{M}_n$-unique (for any n), but some matrix ring $T = \mathbb{M}_r(S)$ may fail to be $\mathbb{M}_n$-unique. Examples of this nature will be constructed later in this section.

A big payoff of Theorem (17.29) is the following.

(17.30) Corollary. *Let $\overline{R} = R/\mathrm{rad}\ R$, where R is any ring. If the class $\mathcal{P}_{\overline{R}}$ (of progenerators in $\mathfrak{M}_{\overline{R}}$) satisfies n-cancellation, then so does the class $\mathcal{P}_R$, and the rings R, $\overline{R}$ are both $\mathbb{M}_n$-unique.*

Proof. Let $J = \mathrm{rad}\ R$. If $P \in \mathcal{P}_R$, it is easy to see that $\overline{P} := P/PJ \in \mathcal{P}_{\overline{R}}$. Now assume $n \cdot P \cong n \cdot P'$, where $P,\ P' \in \mathcal{P}_R$. Then $n \cdot \overline{P} \cong n \cdot \overline{P'} \in \mathcal{P}_{\overline{R}}$, and our hypothesis implies that $\overline{P} \cong \overline{P'}$. By a standard lifting theorem (see *FC*–(19.27)),

we have $P \cong P'$. Since n-cancellation implies weak n-cancellation, it follows from (17.29) that R, $\overline{R}$ are both $\mathbb{M}_n$-unique. □

The most concrete case of (17.30) is the following.

(17.31) Corollary. *Any semilocal ring (a ring R for which $R/\text{rad}\ R$ is semisimple) is $\mathbb{M}_n$-unique for any n. In particular, any right artinian ring is $\mathbb{M}_n$-unique.*

Proof. It is easy to see that f.g. modules over the (semisimple) ring $R/\text{rad}\ R$ satisfy n-cancellation (for any n). Therefore, (17.30) applies. □

The point about the general formulation in (17.30) is, of course, that it has far wider applications than just to the case of semilocal rings. If $\overline{R} = R/\text{rad}\ R$ is a commutative PID, for instance, then by the fundamental structure theorem for modules over such rings, f.g. $\overline{R}$-modules satisfy n-cancellation. Therefore, (17.30) guarantees that R is $\mathbb{M}_n$-unique for any n. There are also many types of von Neumann regular rings over which f.g. projective right modules satisfy n-cancellation: see, for instance, Goodearl [91]. Thus, if $\overline{R} = R/\text{rad}\ R$ is one of these von Neumann regular rings, (17.30) will again guarantee that $\overline{R}$ and R are $\mathbb{M}_n$-unique for any n.

Another class of rings that are $\mathbb{M}_n$-unique for all n is the class of (2-sided) self-injective rings. This is a result of E. Gentile [67]; it is also proved by module-theoretic cancellation techniques.

Next we shall turn our attention to *non-uniqueness*. According to (17.29), the nonuniqueness of base rings for matrix rings can only be caused by the (possible) failure of weak n-cancellation for progenerators. Therefore, to produce examples of rings that are *not* $\mathbb{M}_n$-unique, we need only come up with rings whose progenerators fail to satisfy weak n-cancellation. We can proceed as follows.

(17.32) Example. Let A be a Dedekind domain, with a nonzero ideal I such that I^n is principal and I is not an rth power in the Picard group $\text{Pic}(A)$, where $n,\ r \geq 1$ are given integers. Let

$$P = (r-1)\cdot A \oplus I \quad \text{and} \quad P' = r \cdot A, \tag{17.33}$$

which are clearly progenerators in $\mathfrak{M}_A$. Let

$$S = \text{End}_A(P), \quad \text{and} \quad S' = \text{End}_A(P') \cong \mathbb{M}_r(A).$$

We claim that

$$n \cdot P \cong n \cdot P', \quad \text{but} \quad S \not\cong S', \tag{17.34}$$

so progenerators in $\mathfrak{M}_A$ *do not* satisfy weak n-cancellation. Also, since $n \cdot P \cong n \cdot P'$ implies that $\mathbb{M}_n(S) \cong \mathbb{M}_n(S')$, S and S' are *not* $\mathbb{M}_n$-unique. To prove

(17.34), note that, by repeated use of Steinitz' Isomorphism Theorem,[94]

$$\begin{aligned} n \cdot P &\cong n(r-1) \cdot A \oplus n \cdot I \\ &\cong n(r-1) \cdot A \oplus (n-1) \cdot A \oplus I^n \\ &\cong (nr-1) \cdot A \oplus A \\ &\cong n \cdot P'. \end{aligned}$$

For the second conclusion in (17.34), assume instead that

$$\operatorname{End}_A(P) = S \cong S' \cong \mathbb{M}_r(A).$$

By (17.9), we have $P \cong r \cdot J$ for some A-module J, which is necessarily f.g. projective of rank 1. We may assume that J is an ideal in A, in which case

$$r \cdot J \cong (r-1) \cdot A \oplus J^r \quad \text{(as above)}.$$

Comparing this with (17.33) and taking the rth exterior powers, we get $I \cong J^r$, in contradiction to the given properties of I. Thus, $S \not\cong S'$.

(17.35) Remark. In the above example, it should not go unnoticed that, although the (commutative) ring A is $\mathbb{M}_n$-unique (by (17.26)), the matrix ring $\mathbb{M}_r(A) \cong S'$ is *not*!

For an explicit instance of (17.32), let $A = \mathbb{Z}[\theta]$ where $\theta = \sqrt{-5}$, and let $I = (2,\ 1+\theta) \subseteq A$. As we saw in (2.19D), I is not principal, but $I^2 = 2A$. Since $\operatorname{Pic}(A) \cong \mathbb{Z}/2\mathbb{Z}$ (a well-known fact in number theory), (17.32) applies to I with $n = r = 2$. Thus, for $P = A \oplus I$ and $S = \operatorname{End}_A(P)$, we have

$$\mathbb{M}_2(S) \cong \mathbb{M}_2(\mathbb{M}_2(A)), \quad \text{but} \quad S \not\cong \mathbb{M}_2(A).$$

Note that S and $\mathbb{M}_2(A)$ are (2-sided) noetherian rings, so, although 1-sided artinian rings are $\mathbb{M}_n$-unique for all n (by (17.31)), 2-sided noetherian rings may not be $\mathbb{M}_2$-unique. The ring S above can be described explicitly: since $P = A \oplus I$, its endomorphism ring S is simply the subring of matrices $\begin{pmatrix} A & I^{-1} \\ I & A \end{pmatrix} \subseteq \mathbb{M}_2(K)$, where K is the quotient field of A, and

$$I^{-1} = \{a \in K : aI \subseteq A\} \cong \operatorname{Hom}_A(I, A).$$

To conclude this section, let us mention another well-known example on the lack of $\mathbb{M}_n$-uniqueness. Here, however, we can only describe the example in some detail, but we will not be able to give the complete verifications of our claims.

(17.36) Example. Let A be the coordinate ring of the real 2-sphere; that is, $A = \mathbb{R}[x, y, z]$ with the relation $x^2 + y^2 + z^2 = 1$. Let $\varphi : A^3 \to A$ be the A-homomorphism given by mapping the unit vectors $e_1, e_2, e_3 \in A^3$ to

[94] We use the following version of this theorem: *For any nonzero ideals J, J' in a Dedekind domain A, $J \oplus J' \cong A \oplus JJ'$ as A-modules.*

$x, y, z \in A$, and let $P := \ker(\varphi)$. Since $xA + yA + zA = A$, φ is a split epimorphism. This shows that $P \oplus A \cong A^3$; thus, P is a stably free A-module of rank 2. The following two interesting facts are known about P:

(1) $n \cdot P$ is free for any $n \geq 2$.
(2) P is an *indecomposable* A-module. (In particular, P is not free.)

Assuming these facts, we have then an instance of the failure of n-cancellation: for $F := A^2$, we have $n \cdot P \cong n \cdot F$ for all $n \geq 2$, but $P \ncong F$. In fact, even weak n-cancellation fails. To see this, let $S := \text{End}_A(P)$ and $S' := \text{End}_A(F)$. Taking endomorphism rings of $n \cdot P \cong n \cdot F$, we have $\mathbb{M}_n(S) \cong \mathbb{M}_n(S')$ for all $n \geq 2$. However, $S \ncong S'$, since $S' \cong \text{End}_A(A^2) \cong \mathbb{M}_2(A)$ has nontrivial idempotents, but according to (2), S does not. Thus, the ring S fails to be $\mathbb{M}_n$-unique (for all $n \geq 2$), and so does $S' \cong \mathbb{M}_2(A)$ (even though the commutative ring A is $\mathbb{M}_n$-unique for all n).

The fact (1) above can be proved in at least two different ways. First, it can be deduced from general stability results on f.g. projective modules over commutative rings of finite Krull dimension. Second, it can also be deduced, over *any* commutative ring A, from the isomorphism $P \oplus A \cong A^3$ (see [Lam: 76]). The fact (2) is, however, much harder to prove; most known proofs seem to involve topological considerations, using the fact that the projective A-module P "corresponds to" the tangent bundle of the 2-sphere. The fact that P is not free can be deduced from the "Hairy Ball Theorem" for the 2-sphere, but the indecomposability of P requires another topological step.

Exercises for §17

1. Let $R = \mathbb{M}_n(S)$, where $n \geq 1$. Show that R satisfies IBN (resp. the rank condition in §1) iff S does.

2. In the notation of Remark (17.6), show that the e_{ij}'s are left (and right) linearly independent over their centralizer S in R.

3. Suppose a ring R has three elements a, b, f such that $f^2 = 0$ and $af + fb = 1$. Show that R also has an element c such that $cf + fc = 1$.

   The next three exercises are due to G. Agnarsson.

4. Suppose a ring R has two elements a, f such that $f^{p+q} = 0$ and $af^p + f^q a = 1$, where $p, q \geq 1$ and $p \neq q$. Show that $R = 0$.

5. Let k be a commutative ring, and R be the k-algebra with generators a, f and relations $f^{2p} = 0$, $af^p + f^p a = 1$, where $p \geq 1$. Show that $R \cong \mathbb{M}_{2p}(S)$ for a suitable k-algebra $S \supseteq k$. (In particular, if $k \neq 0$, then $R \neq 0$.)

6. (1) Let k be a commutative ring, and R be the k-algebra with generators a, f and relations $a^n = f^n = 0$, $a^{n-1}f^{n-1} + fa = 1$. Show that $R \cong \mathbb{M}_n(k)$ as k-algebras.

(2) Show that a ring A is an $n \times n$ matrix ring iff there exist $a_0,\ f_0 \in A$ such that $a_0^n = f_0^n = 0$ and $a_0^{n-1} f_0^{n-1} + f_0 a_0 = 1$.

7. (Agnarsson-Amitsur-Robson) Show that, for $n \geq 3$, the existence of $c,\ d,\ f \in R$ such that $f^n = 0$ and $cf + fd = 1$ need not imply that R is an $n \times n$ matrix ring.

8. Let $R = \mathbb{M}_n(S)$ and $R' = \mathbb{M}_n(S')$, where $S,\ S'$ are rings.

(1) For any $(R,\ R')$-bimodule M, show that the triangular ring $T = \begin{pmatrix} R & M \\ 0 & R' \end{pmatrix}$ is an $n \times n$ matrix ring.

(2) Let N be an $(S,\ S')$-bimodule and let $M = \mathbb{M}_n(N)$, viewed as an $(R,\ R')$-bimodule in the obvious way. According to (1), $T = \begin{pmatrix} R & M \\ 0 & R' \end{pmatrix}$ is an $n \times n$ matrix ring. Determine a base ring for T.

9. For $s,\ t \in F^*$ where F is a field of characteristic $\neq 2$, let R be the F-quaternion algebra generated by two elements $i,\ j$ with the relations $i^2 = s,\ j^2 = t$, and $ij = -ji$. Assume that there exist $u,\ v \in F$ such that $su^2 + tv^2 = 1$. Using the Recognition Theorem (17.10) for $p = q = 1$, show that $R \cong \mathbb{M}_2(F)$ as F-algebras. (**Hint.** In the notation of (17.10), let $f = tvi + suj + ij$.)

10. (Fuchs-Maxson-Pilz) Show that a ring R is a 2×2 matrix ring iff there exist $f,\ g \in R$ such that $f^2 = g^2 = 0$ and $b := f + g \in \mathrm{U}(R)$. (**Hint.** Let $a = b^{-1}$, and show that $af = ga$, whence $1 = ga + fa = af + fa$.)

11. Let $R = \mathbb{M}_n(S)$, which may be viewed as a left S-module. Let $V,\ W$ be right S-modules.

(1) Show that $V \otimes_S R \cong n \cdot V^{(n)}$, where $V^{(n)} = (V, \ldots, V)$ is viewed as a right R-module in the obvious way.

(2) Show that $\mathrm{End}_R(V \otimes_S R) \cong \mathbb{M}_n(\mathrm{End}_S V)$.

(3) Show that $V \otimes_S R \cong W \otimes_S R$ as R-modules iff $n \cdot V \cong n \cdot W$ as S-modules.

12. For any S-module M_S, let $\mathrm{Lat}_S(M)$ denote the lattice of S-submodules of M. For any ring S and $R = \mathbb{M}_n(S)$, let $M = S_S^n$ be identified with RE_{11}. Define

$$\mathrm{Lat}_R(R_R) \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Lat}_S(M)$$

by $f(U) = UE_{11}$ for $U \in \mathrm{Lat}_R(R_R)$ and $g(V) = \sum_{j=1}^n VE_{1j}$ for $V \in \mathrm{Lat}_S(M)$. Show that f and g are mutually inverse lattice isomorphisms. (Note, in particular, that this "classifies" the right ideals in the matrix ring R.)

13. If $S,\ T$ are two rings such that $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$ as rings, show that $\mathrm{Lat}_S(S_S^n) \cong \mathrm{Lat}_T(T_T^m)$ as lattices.

14. Let S, T be nonzero commutative rings, and n, $m \geq 1$ be integers. If $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$, show that $n = m$ and $S \cong T$.

15. Let S, T be two rings such that $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$ as rings, where n, $m \geq 2$ are given integers. If S is commutative and n is prime, show that we must have $n = m$ and $S \cong T$.

16. Show that $Q^r_{\max}(\mathbb{M}_n(R)) \cong \mathbb{M}_n(Q^r_{\max}(R))$. If R is a semiprime ring, prove the same results for Martindale's right (resp. symmetric) ring of quotients.

§18. Morita Theory of Category Equivalences

§18A. Categorical Properties

Before going into the full discussion of how equivalences of module categories might arise, we shall devote this preliminary subsection to the notion of "categorical properties" of modules (and their morphisms).

A property $\mathcal{P}$ on objects (resp. morphisms) in a module category $\mathfrak{M}_R$ is said to be a *categorical property* if, for any category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, whenever $M \in \mathfrak{M}_R$ (resp. $g \in \text{Hom}_R(M, N)$) satisfies $\mathcal{P}$, so does $F(M)$ (resp. $F(g)$). Thus, if a property $\mathcal{P}$ is defined (or definable) purely in categorical terms (using only objects and morphisms, and without reference to elements of modules or to the underlying ring), then $\mathcal{P}$ is a categorical property, since the categorical equivalence F will "transport" $\mathcal{P}$ from M (resp. g) to $F(M)$ (resp. $F(g)$).

At the level of morphisms, $g : M \to N$ being an isomorphism is certainly a categorical property. Less obviously, g being a monomorphism (resp. epimorphism) are also categorical properties. Here, we must abandon the usual "elementwise" definition for "one-one" and "onto". Instead, we characterize g being a mono (resp. epi) by the property that, whenever $X \xrightarrow{h} M \xrightarrow{g} N$ has composition zero, then $h = 0$ (resp. whenever $M \xrightarrow{g} N \xrightarrow{k} Y$ has composition zero, then $k = 0$). From these observations, it follows readily that $0 \to M \to N \to M' \to 0$ being an exact sequence is a categorical property.

At the level of modules, $M \in \mathfrak{M}_R$ being zero, nonzero, simple, semisimple, indecomposable, uniform, noetherian, artinian, or having uniform dimension $= n$, composition length $= n$ $(n \leq \infty)$, etc., are certainly categorical properties. The module M_R being strongly indecomposable is also a categorical property, since this requires that the endomorphism ring $\text{End}_R M$ be local, which is preserved by a categorical equivalence. In a similar vein, a submodule $N \subseteq M$ being maximal, minimal, essential ("large"), superfluous ("small") in M, or being a direct summand or a complement in M, are also categorical properties (for the pair $N \subseteq M$). The property of N being *dense* in M is a bit trickier: the first definition for denseness given in (8.2) was certainly in terms of elements. However, if we resort to the

characterization of denseness:

(18.1) "$\text{Hom}_R(P/N, M) = 0$ whenever $N \subseteq P \subseteq M$"

in (8.6)(3), it becomes clear that N being dense in M is a categorical property as well.

For the study of module theory, "f.g." (finite generation) is an all-important property. It is very fortunate indeed that this property turns out to be categorical. To see this, we check (for instance) that the usual "elementwise" definition for $_RM$ being f.g. can be replaced by either one of the following categorical conditions:

(18.2) *For any family of submodules $\{N_i : i \in I\}$ in M, if $\sum_{i \in I} N_i = M$, then $\sum_{i \in J} N_i = M$ for some finite subset $J \subseteq I$.*

(18.3) *For any family of submodules $\{N_i : i \in I\}$ in M which form a chain, if each $N_i \neq M$, then $\bigcup_{i \in I} N_i \neq M$.*

The equivalence of (18.2) with "f.g." is clear; the equivalence of (18.3) with "f.g." is left as an exercise (*viz.* Exercise (18.0)). This example dealing with the notion "f.g." serves to show that sometimes there may be *several* interesting categorical formulations for a module-theoretic property first defined by using elements.

Coming now to some of the homological properties of modules, being *projective* or *injective* are certainly categorical properties, and therefore, so are the properties of being a projective cover or an injective hull. Using this and the above paragraph, we see that M_R being "f.p." (finitely presented) is also a categorical property (M_R is f.p. iff there exists $0 \to K \to P \to M \to 0$ with K, P f.g. and P projective). In view of this (plus (4.33) for instance), it follows in turn that M being *flat* is also a categorical property. Using resolutions, we conclude that $\text{pd}(M) = n$, $\text{id}(M) = n$, and $\text{fd}(M) = n$ $(n \leq \infty)$ are categorical properties as well.

Of course, not all module-theoretic properties are categorical. For instance, as we have seen near the end of §17C, the property of M_R being a *free* R-module is not categorical, nor is the property of M_R being a *cyclic* (or an *n-generated*) R-module.

Then there are properties for which we can make no immediate conclusions, due to an apparent lack of a categorical characterization. A good example is the property of M_R being a *faithful* R-module. The definition "$Mr = 0 \Longrightarrow r = 0 \in R$" certainly sheds no light. On the other hand, under the equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$ $(S = \mathbb{M}_n(R))$ constructed in §17C, if M is faithful, a quick matrix calculation shows that $F(M) = M^{(n)}$ does remain faithful over $S = \mathbb{M}_n(R)$. This gives us hope that faithfulness *may be* a categorical property. Indeed, we shall show that it is, in (18.37) below.

Next, we turn to ring-theoretic properties. We make the following formal definitions.

(18.4) Definition. Two rings R, S are said to be *Morita equivalent* ($R \approx S$ for short) if there exists a category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$. A ring-theoretic property $\mathcal{P}$ is said to be *Morita invariant* if, whenever R has the property $\mathcal{P}$, so does every $S \approx R$.

Thus, a property $\mathcal{P}$ is Morita invariant if it can be characterized purely in terms of the module category $\mathfrak{M}_R$ associated with R (without reference to elements of modules, or to the ring R itself).

(18.5) Remark. It will be shown later that if we have an equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, then we also have an equivalence $F' : {}_R\mathfrak{M} \to {}_S\mathfrak{M}$. In anticipation of this fact, we chose not to use the term "right Morita equivalent" in (18.4).

Some obviously Morita invariant properties for rings are: "semisimple", "right noetherian" (f.g. right modules have ACC), "right artinian" (f.g. right modules have DCC), "right (semi)hereditary" (see (2.26) and (2.30)), or "r.gl.dim $= n$", "weak dim $= n$". In the case $n = 0$, the last of these implies that R being von Neumann regular is a Morita invariant property (cf. (5.62)(a)). Using these facts, for instance, we can now deduce, in a purely conceptual way:

(18.6) Corollary. *If R is semisimple, right noetherian (resp. artinian), right hereditary (resp. semihereditary), or von Neumann regular, then so is $\mathbb{M}_n(R)$, and conversely.*

Granted the fact that a module being faithful is a categorical property, we see that R being right primitive (resp. semiprimitive) ["there exists a faithful simple (resp. semisimple) right R-module"] is a Morita invariant property. As we delve more deeply into the Morita Theory, we will be able to prove that other important properties such as "prime", "semiprime", "simple", "semilocal", "right nonsingular", "right self-injective", "QF", etc. are likewise Morita invariant. The finiteness of a ring turns out to be a Morita invariant property too.

On the other hand, *many* ring-theoretic properties are seen to be lost as we pass to matrix rings. Such properties are, therefore, *not* Morita invariant. These include, for instance, R being commutative, local, reduced, a domain, a division ring, right Goldie (remark following (11.18)), Dedekind-finite (Exercise (1.18)), ACC on right annihilators (remarks following (6.51)), "f.g. right projective R-modules are free", etc.

Checking whether a ring-theoretic property is Morita invariant can sometimes be downright confusing. For instance, my colleague G. Bergman once sent me email suggesting an example of a ring S with IBN such that some $R \approx S$ is without IBN. But later in the day, I found to my amazement an exercise on p. 10 of P. M. Cohn's book (Cohn [85]) asking the reader to show that IBN is a Morita invariant property! (For the construction of Bergman's example, see Exercise 11.)

§18B. Generators and Progenerators

In preparation for the development of the Morita Theory, we make the following formal definition.

(18.7) Definition. A right R-module P is said to be a *generator* (for $\mathfrak{M}_R$) if $\text{Hom}_R(P, -)$ is a faithful functor from $\mathfrak{M}_R$ to the category of abelian groups. This means that $\text{Hom}_R(P, -)$ *does not kill nonzero morphisms*; that is, if $f : M \to N$ is nonzero, then so is $\text{Hom}_R(P, f)$, or, more explicitly, there exists $g : P \to M$ such that the composition $P \xrightarrow{g} M \xrightarrow{f} N$ is nonzero.[95]

The most obvious example of a generator is the right regular module $P = R_R$. For this P, the functor $\text{Hom}_R(P, -)$ is the forgetful functor from $\mathfrak{M}_R$ to abelian groups, which is, of course, a faithful functor. The faithfulness property in this case amounts to the fact that a morphism $f : M \to N$ is nonzero iff it takes at least some element of M to a nonzero element of N. Other examples of generators can be produced by using the observation that, *if $P' \to P$ is a surjection and P is a generator, then so is P'*. In particular, if P is a generator, so is any direct sum $P \oplus Q$.

The notion of a generator bears a close relationship to the notion of the trace ideal of a module. Recall from §2 that, for any $P \in \mathfrak{M}_R$, $\text{tr}(P)$ (the trace ideal of P) is defined to be $\sum gP$ where g ranges over $P^* = \text{Hom}_R(P, R)$. We have the following alternative characterizations for P to be a generator.

(18.8) Theorem. *For any P_R the following are equivalent:*

(1) *P is a generator (for $\mathfrak{M}_R$).*
(2) $\text{tr}(P) = R$.
(3) *R is a direct summand of a finite direct sum $\bigoplus_i P$.*
(4) *R is a direct summand of a direct sum $\bigoplus_i P$.*
(5) *Every $M \in \mathfrak{M}_R$ is an epimorphic image of some direct sum $\bigoplus_i P$.*

Proof. (1)$\Longrightarrow$(2). Assume $\mathfrak{A} = \text{tr}(P) \neq R$. Then the projection map $R \to R/\mathfrak{A}$ is nonzero in $\mathfrak{M}_R$, so for some $g \in P^* = \text{Hom}_R(P, R)$, the composition $P \xrightarrow{g} R \longrightarrow R/\mathfrak{A}$ is nonzero. But this means that $gP \not\subseteq \mathfrak{A}$, a contradiction.

(2)$\Longrightarrow$(3). By (2), there exist $g_1, \ldots, g_n \in P^*$ with $g_1P + \cdots + g_nP = R$. Then $(g_1, \ldots, g_n) : P \oplus \cdots \oplus P \to R$ is a split epimorphism, hence (3).

(3)$\Longrightarrow$(4) is a tautology.

(4)$\Longrightarrow$(5). This is easy since M is an epimorphic image of a free module.

(5)$\Longrightarrow$(1). Let $f : M \to N$ be nonzero, and fix a surjection $\bigoplus_i P_i \to M$ where $P_i = P$. Clearly, the composition $\bigoplus_i P_i \to N$ is nonzero. This implies that, for some i, the composition $P_i (= P) \longrightarrow M \xrightarrow{f} N$ is nonzero, proving (1). □

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[95] In the older literature, a generator was also known as a "completely faithful" module.

(18.9) Remarks (on Generators).

(A) By (18.8)(4), over any commutative domain, a torsion module cannot be a generator.

(B) Also by (18.8)(4), any generator P_R is faithful.

(C) The converse of (B) is true in some interesting cases. For instance, it is true over any QF ring R, according to Exercise (16.12). (For more information on this, see §19B below.)

(D) The converse of (B) is certainly not true in general. For instance, over $R = \mathbb{Z}$, $P = \mathbb{Q}$ is faithful but is *not* a generator, since $P^* = 0$! The trouble here is that P_R is not f.g. If we consider only f.g. modules, the converse of (B) stands a better chance to hold. For instance, *if* R *is either* (1) *a commutative self-injective ring, or* (2) *a Prüfer domain, then any f.g. faithful module* P_R *is a generator.* In Case (1), write $P = x_1R + \cdots + x_nR$ and define $f \in \mathrm{Hom}_R(R, P^n)$ by $f(r) = (x_1r, \ldots, x_nr)$ for any $r \in R$. If $f(r) = 0$, then $Pr = (\sum x_iR)r = \sum x_irR = 0$, so $r = 0$. This shows that f embeds R into P^n. Since R is self-injective, R is isomorphic to a direct summand of P^n, so P is a generator. In case (2), let P_0 be the torsion submodule of P. Since $Q := P/P_0$ is f.g. and torsionfree, it is projective by (2.31). We have thus $P \cong P_0 \oplus Q$, and so P_0 is also f.g. Using this, we see that the faithfulness of P implies that of Q. By (18.11) below, Q is a generator, so P is also a generator.

(E) Let $\mathfrak{A} \subseteq R$ be a right ideal in any ring R. If $R\mathfrak{A} = R$, then $\mathfrak{A}_R$ is a generator. (If $\sum r_ia_i = 1$ where $a_i \in \mathfrak{A}$, $\mathrm{tr}(\mathfrak{A})$ contains $\sum r_i\mathfrak{A} = R$.) In particular, over a simple ring R, any nonzero right ideal is a generator for $\mathfrak{M}_R$.

Theorem (18.8) suggests an interesting "analogy" between generators and f.g. projective modules: P_R *is f.g. projective iff* P *is a direct summand of some* R^n, *and* P_R *is a generator iff* R *is a direct summand of some* P^n. By combining these two conditions, we obtain a powerful new notion: a module P_R is called a *progenerator* if it is a f.g. projective generator.

(18.10) Remarks (on Progenerators).

(A) From Def. (18.7), we see that P being a generator (progenerator) is a categorical property. Therefore, under a category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, P is a (pro)generator iff $F(P)$ is. In particular, $F(R_R)$ *is always a progenerator in* $\mathfrak{M}_S$. As it turns out, this observation holds the key to much of the Morita Theory to be developed later.

(B) Over any commutative ring R, (18.11) below implies that any invertible R-ideal is a progenerator.

(C) Let R be a right artinian ring (or more generally a semiperfect ring). Let $1 = e_1 + \cdots + e_n$ be a decomposition of 1 into a sum of primitive orthogonal idempotents, and suppose $e_1R, \ldots, e_rR$ is a complete set of principal indecomposable modules. *An arbitrary f.g. projective right module* $P = m_1(e_1R) \oplus \cdots \oplus m_r(e_rR)$ *is a progenerator iff* $m_i > 0$ *for* $1 \le i \le r$. This follows easily from (18.8).

(D) Let P_R be a f.g. projective module represented in the form $e \cdot R^n_R$ where e is an idempotent in $\mathbb{M}_n(R) = \text{End}(R^n_R)$. Let $e = (a_{ij})$. By Exercise 2.18,

$$\text{tr}(P) = \sum Ra_{ij}R, \quad \text{and} \quad \mathbb{M}_n(R)\, e\, \mathbb{M}_n(R) = \mathbb{M}_n(\text{tr}(P)).$$

Therefore, *P_R is a progenerator iff $e \in \mathbb{M}_n(R)$ is a full idempotent.* (Recall that an idempotent e in a ring A is *full* if $AeA = A$: see *FC*–(21.11).) In the special case $n = 1$, it follows that, for $e = e^2 \in R$, we have $\text{tr}(eR) = ReR$, and *eR is a progenerator iff e is a full idempotent.*

In the commutative case, we have the following easy (but important) criterion for the recognition of progenerators.

(18.11) Theorem (Azumaya). *Let P be a f.g. projective module over a commutative ring R. Then P is a progenerator iff it is faithful.*[96] *If R has no nontrivial idempotents, then any f.g. projective module $P \neq 0$ is a progenerator.*

Proof. This follows by combining (18.8) with (2.44). □

(18.12) Example. If R is not a commutative ring, a faithful f.g. projective module P may not be a progenerator. In fact, over the ring $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ (where $k \neq 0$ is any ring), let $e = E_{11}$ and $P = eR = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$. We check easily that the f.g. projective module P_R is faithful, but P is *not* a progenerator in $\mathfrak{M}_R$ since $\text{tr}(P) = ReR = eR \neq R$.

§18C. The Morita Context

We first introduce some general constructions and notations which will be useful for understanding category equivalences.

Let P be a right R-module, where R is any ring. We write $Q = P^* = \text{Hom}_R(P, R)$ and $S = \text{End}_R(P)$, both operating on the left of P. This makes P into an (S, R)-bimodule. Viewing P as ${}_SP_R$ and R as ${}_RR_R$, it follows that $Q = \text{Hom}_R(P, R)$ is an (R, S)-bimodule. The left R-action on Q is defined by $(rq)p = r(qp)$ ("RQP-associativity"), and the right S-action is defined by $(qs)p = q(sp)$ ("QSP-associativity"). Here and in the following, we write p, q, r, s (and p', q', r', s') for elements in P, Q, R, and S.

As far as P and Q are concerned, we have the following two basic pairings. For $p \in P$ and $q \in Q$, we have (as above) $qp \in R$ which is the result of applying q to p. We also have $pq \in S$ which can be defined by

(18.13) $$(pq)p' = p(qp') \qquad (\text{"}PQP\text{-associativity"}).$$

[96] For a criterion for P to be faithful in terms of localization, see Exercise (2.24).

We suppress the routine calculation for verifying that pq is an R-endomorphism of P. Instead, let us show that

$$(18.14) \qquad q(pq') = (qp)q' \qquad (\text{“}QPQ\text{-associativity”}).$$

It suffices to check that both sides have the same effect on $p' \in P$:

$$\begin{aligned}(q(pq'))p' &= q((pq')p') && (\text{“}QSP\text{-associativity”})\\ &= q(p(q'p')) && (\text{“}PQP\text{-associativity”})\\ &= (qp)(q'p') && (\text{“}QPR\text{-associativity”}: q \text{ is } R\text{-linear})\\ &= ((qp)q')p' && (\text{“}RQP\text{-associativity”}).\end{aligned}$$

(18.15) Lemma. *In the above notations:*

(1) $(q, p) \mapsto qp$ *defines an* (R, R)*-homomorphism* $\alpha : Q \otimes_S P \to R$*;*
(2) $(p, q) \mapsto pq$ *defines an* (S, S)*-homomorphism* $\beta : P \otimes_R Q \to S$*.*

Proof. (1) Since $P = {}_SP_R$ and $Q = {}_RQ_S$, $Q \otimes_S P$ makes sense and it is an (R, R)-bimodule. The well-definition of α results from QSP-associativity. The rest follows from RQP-associativity and QPR-associativity (already noted above). Similarly, the proof of (2) boils down to PRQ-, SPQ-, and PQS-associativities; the straightforward verification of these is left to the reader. □

The different kinds of associativities used above naturally suggest that there ought to be a larger ring at work whose associativity property subsumes all of the above. Indeed, if we let $M = \begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ with formal matrix multiplication

$$(18.16) \qquad \begin{pmatrix} r & q \\ p & s \end{pmatrix}\begin{pmatrix} r' & q' \\ p' & s' \end{pmatrix} = \begin{pmatrix} rr' + qp' & rq' + qs' \\ pr' + sp' & pq' + ss' \end{pmatrix},$$

then M is such a ring, called the *Morita ring* associated with P_R. (For more details, see Exercise 18.) The 6-tuple $(R, P, Q, S;\ \alpha, \beta)$ is called the *Morita Context* associated with P_R. (Note the order in which α, β are written: we first write down the pairing going into the ground ring R, then the pairing going into the endomorphism ring S of P.) *Fixing now the Morita Context* $(R, P, Q, S;\ \alpha, \beta)$, let us now prove a number of interesting propositions about it.

(18.17) Proposition. (1) *P_R is a generator iff α is onto.*
(2) *Assume P_R is a generator. Then*

(a) $\alpha : Q \otimes_S P \to R$ *is an* (R, R)*-isomorphism.*
(b) $Q \cong \mathrm{Hom}_S({}_SP, {}_SS)$ *as* (R, S)*-bimodules.*
(c) $P \cong \mathrm{Hom}_S(Q_S, S_S)$ *as* (S, R)*-bimodules.*
(d) $R \cong \mathrm{End}({}_SP) \cong \mathrm{End}(Q_S)$ *as rings.*

Proof. (1) is clear since $\operatorname{im}(\alpha) = \operatorname{tr}(P_R)$. For (2), assume P_R is a generator. Then there is an equation $1_R = \sum q_i p_i$. To prove (2a), suppose $0 = \alpha(\sum_j q_j' \otimes p_j') = \sum_j q_j' p_j'$. Then

$$\sum_j q_j' \otimes p_j' = \sum_{i,j} (q_i p_i) q_j' \otimes p_j' = \sum_{i,j} q_i (p_i q_j') \otimes_S p_j'$$
$$= \sum_{i,j} q_i \otimes_S (p_i q_j') p_j' = \sum_i q_i \otimes p_i \Big(\sum_j q_j' p_j'\Big) = 0.$$

To prove (2b), define $\lambda : Q \to \operatorname{Hom}_S({}_SP, {}_SS)$ by $p \cdot \lambda(q) = pq \in S$. The fact that $\lambda(q) \in \operatorname{Hom}_S(P, S)$ follows from SPQ-associativity. To show that λ is *injective*, suppose $pq = 0$ for all $p \in P$. Using the equation $1_R = \sum q_i p_i$ above, we have

(18.18) $$q = 1_R q = \Big(\sum q_i p_i\Big) q = \sum q_i (p_i q) = 0.$$

To show λ is *onto*, consider any $f \in \operatorname{Hom}_S(P, S)$. From
(18.18′)
$$pf = \Big(p \sum q_i p_i\Big) f = \sum ((pq_i)p_i) f = \sum (pq_i)(p_i f) = p \sum q_i (p_i f),$$

we see that $f = \lambda\big(\sum q_i (p_i f)\big)$. The fact that λ is an (R, S)-homomorphism follows from RPQ- and PQS-associativities. This completes the proof of (2b), and the proof for (2c) is similar. To prove (2d), define ring homomorphisms

$$\sigma : R \to \operatorname{End}({}_SP) \quad \text{and} \quad \tau : R \to \operatorname{End}(Q_S)$$

by $p \cdot \sigma(r) = pr$ and $\tau(r) q = rq$. The same calculations as in (18.18) and (18.18′) show that σ and τ are isomorphisms. □

The following is the "analogue" of (18.17) for f.g. projective modules.

(18.19) Proposition. (1) *P_R is f.g. projective iff β is onto.*
(2) *Assume P_R is f.g. projective. Then*

(a) *$\beta : P \otimes_R Q \to S$ is an (S, S)-isomorphism.*
(b) *$Q \cong \operatorname{Hom}_R(P_R, R_R)$ as (R, S)-bimodules.*
(c) *$P \cong \operatorname{Hom}_R({}_RQ, {}_RR)$ as (S, R)-bimodules.*
(d) *$S \cong \operatorname{End}(P_R) \cong \operatorname{End}({}_RQ)$ as rings.*

Proof. (1) β is onto iff there is an equation $1_S = \sum p_k'' q_k''$. This means that

$$p = \Big(\sum p_k'' q_k''\Big) p = \sum p_k'' (q_k'' p) \qquad \text{(for every } p \in P\text{)}.$$

By the Dual Basis Lemma, this amounts precisely to P_R being f.g. projective. The proof of (2) is completely similar to that of (18.17)(2), using the equation $1_S = \sum p_k'' q_k''$. □

(18.20) Remarks.

(A) Of course, the isomorphism in (2b) and the first isomorphism in (2d) are just the identity maps! They are included here for completeness (and symmetry with (18.17)).

(B) (2c) expresses the "reflexivity" of the f.g. projective module P_R, which already figured in Exercise 2.7.

(C) (2a) was also covered in an earlier exercise, namely, Exercise 2.20. In fact, if P_R is f.g. projective, this exercise gives a more general (abelian group) isomorphism: $U \otimes_R Q \cong \mathrm{Hom}_R(P, U)$ for any U_R. This fact will be used below in (18.25).

(18.21) Definition/Corollary. For rings A, B, an (A, B)-bimodule C is said to be *faithfully balanced* if the natural maps $A \to \mathrm{End}(C_B)$ and $B \to \mathrm{End}({}_AC)$ are both ring isomorphisms. With this terminology, (18.17)(d) and (18.19)(d) together imply that, *if P_R is a progenerator, ${}_SP_R$ and ${}_RQ_S$ are both faithfully balanced bimodules.*

(18.22) Proposition. *Suppose P_R is a progenerator. Then ${}_SP$, ${}_RQ$, Q_S are also progenerators (and α, β are isomorphisms).*

Proof. It suffices to prove that ${}_SQ$ is a progenerator, for the proofs of the other parts are similar. Starting with the right S-module Q, we have $\mathrm{Hom}_S(Q_S, S_S) \cong P$ and $\mathrm{End}(Q_S) \cong R$, by (18.17)(2). Since α, β are onto, it follows from (18.17)(1) and (18.19)(1) (applied to Q_S) that Q_S is a progenerator. □

(18.23) Remark. The proof above shows that, if P_R is a progenerator, then the Morita Context associated with the progenerator Q_S is $(S, Q, P, R;\ \beta, \alpha)$. Therefore, we have attained full symmetry between P and Q. Similarly, we have also attained full left-right symmetry. These phenomena will be made more explicit in the statement of "Morita I" below.

§18D. Morita I, II, III

With the preparatory material in the last subsection, we are now fully ready to state and prove Morita's main theorems for module category equivalences. We shall do this in three parts.

(18.24) Theorem ("Morita I"). *Let P_R be a progenerator, and $(R, P, Q, S; \alpha, \beta)$ be the Morita Context associated with P_R. Then*

(1) $-\otimes_R Q : \mathfrak{M}_R \to \mathfrak{M}_S$ *and* $-\otimes_S P : \mathfrak{M}_S \to \mathfrak{M}_R$ *are mutually inverse category equivalences.*

(2) $P \otimes_R - : {}_R\mathfrak{M} \to {}_S\mathfrak{M}$ *and* $Q \otimes_S - : {}_S\mathfrak{M} \to {}_R\mathfrak{M}$ *are mutually inverse category equivalences.*

Proof. For any U_R, we have (by (18.17)(a)) natural isomorphisms:

$$(U \otimes_R Q) \otimes_S P \cong U \otimes_R (Q \otimes_S P) \cong U \otimes_R R \cong U.$$

Similarly, for any V_S, $(V \otimes_S P) \otimes_R Q \cong V$. This proves (1), and (2) is proved similarly. □

(18.25) Remark. Let P_R be a progenerator. For any U_R, it is easy to check that $\beta_U : U \otimes_R Q \to \mathrm{Hom}_R(P_R, U_R)$ defined by $\beta_U(u \otimes q)(p) = u(qp)$ is an isomorphism of right S-modules (cf. (18.20)(C)).[97] Therefore, we can identify $- \otimes_R Q$ with the Hom-functor $\mathrm{Hom}_R(P_R, -)$ from $\mathfrak{M}_R$ to $\mathfrak{M}_S$. Similarly, we have functor isomorphisms $- \otimes_S P \cong \mathrm{Hom}_S(Q_S, -)$, and

$$P \otimes_R - \cong \mathrm{Hom}_R({}_RQ, -), \qquad Q \otimes_S - \cong \mathrm{Hom}_S({}_SP, -).$$

(18.26) Theorem ("Morita II"). *Let R, S be two rings, and*

$$F : \mathfrak{M}_R \to \mathfrak{M}_S, \quad G : \mathfrak{M}_S \to \mathfrak{M}_R$$

be mutually inverse category equivalences. Let $Q = F(R_R)$ and $P = G(S_S)$. Then we have natural bimodule structures: $P = {}_SP_R$, $Q = {}_RQ_S$. Using these, we have functor isomorphisms $F \cong - \otimes_R Q$ and $G \cong - \otimes_S P$.

Proof. Since $P \in \mathfrak{M}_R$ and $S \in \mathfrak{M}_S$ correspond under G, we have $\mathrm{End}(P_R) \cong \mathrm{End}(S_S) \cong S$. This enables us to view P as ${}_SP_R$. Similarly, we can view Q as ${}_RQ_S$. Also, since S_S is a progenerator for $\mathfrak{M}_S$, *P_R is a progenerator for $\mathfrak{M}_R$* (cf. (18.10)(A)). We can now compute the R-dual of P as follows:

$$\mathrm{Hom}_R(P, R) \cong \mathrm{Hom}_S(F(P), F(R)) \cong \mathrm{Hom}_S(S_S, Q_S) \cong Q.$$

Therefore, the Morita Context associated with P_R is $(R, P, Q, S;\ \alpha, \beta)$, where α, β are appropriate pairings. In particular, the various conclusions of Morita I apply. To "identify" F, note that, for any $M \in \mathfrak{M}_R$:

$$F(M) \cong \mathrm{Hom}_S(S_S, F(M)) \cong \mathrm{Hom}_R(P_R, M_R).$$

Therefore, $F \cong \mathrm{Hom}_R(P_R, -) \cong - \otimes_R Q$ by (18.25), and by a similar argument, $G \cong \mathrm{Hom}_S(Q_S, -) \cong - \otimes_S P$. □

To state the last part of the Morita Theorems, let us make the following convenient definition:

(18.27) Definition. Let R, S be rings. An (S, R)-bimodule P is called an (S, R)-*progenerator* if ${}_SP_R$ is faithfully balanced and P_R is a progenerator. (Note that by (18.22) this definition is left-right symmetric: we could have equally well required ${}_SP$ to be a progenerator.)

[97]In the special case $U = P_R$, we just get back the isomorphism $\beta : P \otimes_R Q \cong S$. Therefore, β_U is just a generalization of β.

With this terminology, we can state the following addendum to Morita I, II:

(18.28) Theorem ("Morita III"). *For two given rings R and S, the isomorphism classes of category equivalences $\mathfrak{M}_S \to \mathfrak{M}_R$ are in one-one correspondence with the isomorphism classes of (S, R)-progenerators. Composition of category equivalences corresponds to tensor products of such progenerators.*

Proof. Every (S, R)-progenerator ${}_SP_R$ gives rise to an equivalence $-\otimes_S P : \mathfrak{M}_S \to \mathfrak{M}_R$, the isomorphism type of which depends only on the isomorphism type of ${}_SP_R$. Conversely, if $G : \mathfrak{M}_S \to \mathfrak{M}_R$ is an equivalence, $P := G(S_S)$ is an (S, R)-progenerator (as we saw in the proof of Morita II), and its isomorphism type depends only on that of G. This proves the first conclusion. If ${}_RP'_T$ is an (R, T)-progenerator, the composition of the equivalences $\mathfrak{M}_S \to \mathfrak{M}_R \to \mathfrak{M}_T$ is given by $-\otimes_S (P \otimes_R P')$; hence the second conclusion. (It follows for free that $P \otimes_R P'$ is an (S, T)-progenerator.) □

For $S = R$, we obtain, in particular:

(18.29) Corollary. *The isomorphism classes of self-equivalences of $\mathfrak{M}_R$ under composition form a group isomorphic to the group of isomorphism classes of (R, R)-progenerators under tensor product.*

§18E. Consequences of the Morita Theorems

In this subsection, we would like to apply the Morita Theorems in §18D to study the finer aspects of a Morita equivalence of two rings. First we offer the following important example which provides what is perhaps the most explicit illustration of the situation in "Morita I".

(18.30) Example. Let e be a full idempotent of a ring R; that is, $e = e^2$ and $ReR = R$. We know from (18.10)(C) that $P = eR$ is a *progenerator* of $\mathfrak{M}_R$. Here we have $Q = P^* \cong Re$ and $S = \text{End}(P_R) = eRe$ (see *FC*–(21.6)). The Morita Context associated with P is $(R, eR, Re, eRe;\ \alpha, \beta)$, where the isomorphism

(18.30A) $$\alpha : Re \otimes_{eRe} eR \to R \quad \text{and} \quad \beta : eR \otimes_R Re \to eRe$$

are given by $\alpha(re \otimes er') = rer'$ and $\beta(er \otimes r'e) = err'e$, respectively. By Morita I, we get mutually inverse category equivalences:

(18.30B) $$F : \mathfrak{M}_R \longrightarrow \mathfrak{M}_{eRe} \quad \text{and} \quad G : \mathfrak{M}_{eRe} \longrightarrow \mathfrak{M}_R\,,$$

given explicitly (on the module level) by:

(18.30C) $$F(U) = U \otimes_R Re \cong \text{Hom}_R(eR,\ U) \cong (Ue)_{eRe}\,,$$

(18.30D) $$G(V) = V \otimes_{eRe} eR \cong \text{Hom}_{eRe}(Re,\ V),$$

for $U \in \mathfrak{M}_R$ and $V \in \mathfrak{M}_{eRe}$. Note that although the ring $S = eRe$ sits inside R, we should not call it a subring of R, since its identity element e may not be equal to 1_R.

In the special case when R is a matrix ring $\mathbb{M}_n(A)$ and e is the matrix unit E_{11}, we have clearly

$$(18.31) \qquad S := eRe = \{aE_{11} : a \in A\} \cong A \quad \text{(as rings)},$$

although A (always identified with the scalar matrices) is a subring of R and S is not (unless $n = 1$). It is easy to check that, upon identifying S with A as in (18.31), the two functors F and G between $\mathfrak{M}_R$ and $\mathfrak{M}_A$ in (18.30B) are essentially the ones constructed earlier in §17C. For F in (18.30C) this is clear. For G in (18.30D), just note that $eR = AE_{11} \oplus \cdots \oplus AE_{1n}$, so for $V \in \mathfrak{M}_A$, $G(V) = V \otimes_A eR \cong V^{(n)}$ (space of row n-tuples over V), with R acting on the right by matrix multiplications. (Here, eR happens to be a *free* left A-module of rank n, while in the general setting we just have to replace it with a left progenerator over A.)

Going back to general rings, we shall now record some consequences of the Morita Theorems in a sequence of propositions. Recall that two rings R and S are *Morita equivalent* ($R \approx S$) if there exists a category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$. The first consequence of Morita I, II is the left-right symmetry of this relation:

(18.32) Proposition. *If $\mathfrak{M}_R$ and $\mathfrak{M}_S$ are equivalent, then so are ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$.*

(18.33) Proposition. *For rings R and S, the following are equivalent:*

(1) $R \approx S$.
(2) $S \cong \mathrm{End}(P_R)$ *for some progenerator* P_R *of* $\mathfrak{M}_R$.
(3) $S \cong e\,\mathbb{M}_n(R)\,e$ *for some full idempotent* e *in a matrix ring* $\mathbb{M}_n(R)$.

Proof. (3)$\Longrightarrow$(1). If $S \cong e\,\mathbb{M}_n(R)\,e$ as in (3), then $S \approx \mathbb{M}_n(R)$ by (18.27).

(1)$\Longrightarrow$(2) follows from Morita II.

(2)$\Longrightarrow$(3). Identify P with a direct summand of some R^n, and write $R^n = P \oplus P'$. We also identify $\mathrm{End}(R^n_R)$ with $\mathbb{M}_n(R)$. Let $e \in \mathbb{M}_n(R)$ be the projection of R^n onto P (defined by $e|P' = 0$ and $e|P = \mathrm{Id}_P$). By (18.10)(D), the fact that $\mathrm{tr}(P) = R$ means that e is a *full* idempotent in $\mathbb{M}_n(R)$. The proof is now completed by noting that the ring homomorphism

$$(18.34) \qquad \lambda : \mathrm{End}(P_R) \to e \cdot \mathrm{End}(R^n) \cdot e = e\,\mathbb{M}_n(R)\,e$$

defined by $\lambda(f)|P = f$, $\lambda(f)|P' = 0$ for $f \in \mathrm{End}(P_R)$ is in fact an isomorphism. (The key observation here is that the endomorphisms in $e\,\mathbb{M}_n(R)\,e$ are exactly those sending P' to (0) and P to P.) □

The next two corollaries follow quickly from (18.33).

(18.35) Corollary. *A ring-theoretic property $\mathcal{P}$ is Morita invariant iff, whenever a ring R satisfies $\mathcal{P}$, so do eRe (for any full idempotent $e \in R$) and $\mathbb{M}_n(R)$ (for any $n \geq 2$).*[98]

From this, it follows, for instance, that finiteness of a ring is a Morita invariant property (cf. Exercise 5). Similarly, semiprimitivity, primeness, semiprimeness, etc. can be shown to be Morita invariant properties via (18.35). (Alternatively, see (18.45), (18.50) below.)

(18.36) Corollary. *Let R be a ring over which all f.g. projective right modules are free. Then $R \approx S$ iff $S \cong \mathbb{M}_n(R)$ for some n.*

The above result applies, in particular, to local rings, principal right ideal domains, and polynomial rings $k[x_1, \ldots, x_n]$ over a field k (the last case using the Quillen-Suslin solution of Serre's Conjecture: see (2.22)(F)). For any ring R satisfying the hypothesis of (18.36), the Corollary implies that R is the "smallest representative" of its Morita equivalence class.

It turns out that, for the class of semiperfect rings defined in *FC*–§23, there is also a smallest "canonical representative" for a Morita equivalence class. To see this, we need to use the notion of a *basic idempotent*, defined in *FC*–(25.5). Readers not familiar with this notion may, however, skip the following discussion without loss of continuity. Also, readers who have only dealt with basic idempotents in the context of right artinian rings rather than semiperfect rings may assume that the rings appearing below are right artinian rings.

Let R be any semiperfect ring. A basic idempotent in R is an idempotent of the form $e = e_1 + \cdots + e_r$ where the e_i 's are orthogonal primitive idempotents in R such that $e_1R, \ldots, e_rR$ represent a complete set of isomorphism classes of principal indecomposable right R-modules. For any such e, eRe is called a *basic ring* for R. From *FC*–(25.6), we know that eRe is also semiperfect, and e is necessarily a full idempotent.

(18.37) Proposition. *Let R, R' be semiperfect rings and $e \in R$, $e' \in R'$ be basic idempotents. Then $R \approx R'$ iff $eRe \cong e'R'e'$ as rings. In particular, the basic ring eRe is a "canonical representative" for the Morita equivalence class of the semiperfect ring R.*

Proof. The "if" part is clear, since $R \approx eRe$ and $R' \approx e'R'e'$. For the "only if" part, it will certainly be sufficient to show that the isomorphism type of the basic ring eRe can be determined from the category $\mathfrak{M}_R$. Using the notation set up in the paragraph preceding (18.37) we have

(18.38) $$eRe \cong \text{End}(eR)_R \cong \text{End}(e_1R \oplus \cdots \oplus e_rR)_R\,.$$

[98]In practice, many ring-theoretic properties $\mathcal{P}$ pass from R to eRe for any idempotent e, without the fullness assumption.

Since $P := e_1 R \oplus \cdots \oplus e_r R$ is the direct sum of a complete set of the f.g. indecomposable projective objects in $\mathfrak{M}_R$, P and hence $\text{End}(P_R)$ are uniquely determined by the category $\mathfrak{M}_R$. □

It is worth pointing out that the classical Wedderburn-Artin Theorem on the structure of semisimple rings can also be deduced from the Morita Theorems in this section. In fact, let R be a semisimple ring, and write $R_R = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_n$ where the $\mathfrak{A}_i$'s are minimal right ideals. Assume $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ give a complete set of isomorphism types of the $\mathfrak{A}_i$'s. Then, by (18.8), $P := \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ is a *progenerator* for $\mathfrak{M}_R$. Let $(R, P, Q, S; \alpha, \beta)$ be the Morita Context associated with P. By Schur's Lemma, $D_i = \text{End}(\mathfrak{A}_i)_R$ $(1 \leq i \leq r)$ are division rings, and

(18.39) $$S = \text{End}(P_R) = \text{End}(\mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r)_R \cong D_1 \times \cdots \times D_r .$$

Now, by (18.17)(2d), $R \cong \text{End}({}_S P)$. Let $n_i = \dim_{D_i}(\mathfrak{A}_i)$. (Note that $n_i < \infty$ since ${}_S P$ is f.g.) Then

(18.40) $$R \cong \text{End}({}_S P) \cong \prod \text{End}({}_{D_i}\mathfrak{A}_i) \cong \prod \mathbb{M}_{n_i}(D_i).$$

This is precisely the Wedderburn-Artin Theorem. By comparing (18.38) and (18.39) we see that the basic ring of R is $D_1 \times \cdots \times D_r$. This essentially gives the uniqueness of the D_i 's as well. It is true that, at the time when Artin proved the structure theorem for semisimple rings (1927), category theory was an unknown subject. In the light of the above discussion, however, it seems fair to say that, historically, the Wedderburn-Artin theory of semisimple rings was a harbinger for Morita's theory of category equivalences to come thirty years later.

Next, we shall explore a little bit the impact of the Morita Theorems on the structure of rings in general. The first proposition in this direction concerns $Z(R)$, the center of the ring R. We first make the following observation.

(18.41) Lemma. *Let ${}_A M_B$ be a faithfully balanced (A, B)-bimodule. Then $Z(A) \cong Z(B)$, and both rings are isomorphic to the ring E of bimodule endomorphisms of M (say, operating on the left of M).*

Proof. Define $f : Z(A) \to E$ by $f(z)(m) = zm$ (for $z \in Z(A)$ and $m \in M$). Note that left multiplication by z on M commutes with left multiplication by any $a \in A$, as well as with right multiplication by any $b \in B$: this shows that $f(z) \in E$. It is routine to check that f is a ring homomorphism. The fact that ${}_A M_B$ is faithfully balanced then implies that f is an isomorphism. By symmetry, there is also a natural isomorphism $g : Z(B) \to E$. (A direct isomorphism h from $Z(A)$ to $Z(B)$ is given by: $h(z) = z'$ where $zm = mz'$ for all $m \in M$.) □

Applying this lemma to the faithfully balanced (S, R)-bimodule P in Morita I, we obtain the following statement, which is certainly closely related to (17.26):

(18.42) Corollary. *$R \approx S \Longrightarrow Z(R) \cong Z(S)$ as rings. In particular, if R, S are commutative rings, $R \approx S \Longrightarrow R \cong S$.*

(18.43) Remark. The Corollary above shows that $Z(R)$ is an invariant of the Morita equivalence class of the ring R. Since this is the case, one is led to wonder if $Z(R)$ can be directly determined from the category $\mathfrak{M}_R$ (without reference to bimodules). Indeed, this turns out to be the case. Let I be the identity functor on $\mathfrak{M}_R$, and let C be the set of all natural transformations from I to I. These natural transformations can be added and multiplied (= composed), whereby C acquires the structure of a ring. This ring is known to category theorists as the *center* of the additive category $\mathfrak{M}_R$. *We claim that $C \cong Z(R)$ as rings*, which will certainly give (18.42). To prove our claim, construct $\lambda : Z(R) \to C$ as follows. For $r \in Z(R)$ and any $M \in \mathfrak{M}_R$, let $\lambda(r)_M : M \to M$ be given by right multiplication by r. Since $r \in Z(R)$, $\lambda(r)_M$ is a morphism in $\mathfrak{M}_R$. Also, for any morphism $f : M \to N$ in $\mathfrak{M}_R$, we have $\lambda(r)_N \circ f = f \circ \lambda(r)_M$. Thus, $\{\lambda(r)_M : M \in \mathfrak{M}_R\}$ defines a natural transformation $\lambda(r) \in C$. Clearly, $\lambda : Z(R) \to C$ is an injective ring homomorphism. To show that λ is onto, consider $c \in C$. Since $c_R : R_R \to R_R$ commutes with all endomorphisms of R_R, c_R is given by (left) multiplication by some $r \in Z(R)$. For any $M \in \mathfrak{M}_R$ and $m \in M$, the commutativity of the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & M \\ {\scriptstyle c_R}\downarrow & & \downarrow{\scriptstyle c_M} \\ R & \xrightarrow{f} & M \end{array} \qquad \text{(where } f(x) = mx\text{)}$$

shows that $c_M(m) = c_M f(1) = f(c_R(1)) = f(r) = mr$. Therefore, the natural transformation c is just $\lambda(r)$, which proves our claim.

(18.44) Proposition. *Let $(R, P, Q, S; \alpha, \beta)$ be as in Morita I, II. Then the lattice of right ideals in S (resp. R) is isomorphic to the lattice of submodules of P_R (resp. Q_S). Moreover, the following lattices are isomorphic:*

(1) *the lattice of ideals in R;*
(2) *the lattice of ideals in S;*
(3) *the lattice of (S, R)-submodules of ${}_SP_R$;*
(4) *the lattice of (R, S)-submodules of ${}_RQ_S$.*

Also, the isomorphism between the lattices in (1) *and* (2) *preserves products.*

Proof. The first conclusion is clear since $S \in \mathfrak{M}_S$ corresponds to $S \otimes_S P \cong P \in \mathfrak{M}_R$. Under this correspondence, a right ideal $\mathfrak{B} \subseteq S$ corresponds to $\mathfrak{B} \otimes_S P \cong \mathfrak{B}P \subseteq P$. We claim that $\mathfrak{B}$ is ideal iff $\mathfrak{B}P$ is an (S, R)-submodule of P. The "only if" part is clear. For the "if" part, assume $\mathfrak{B}P$ is an (S, R)-submodule of P. For $s \in S$, we have $s\mathfrak{B}P \subseteq \mathfrak{B}P$, so by the correspondence established earlier, $s\mathfrak{B} \subseteq \mathfrak{B}$. This shows that $\mathfrak{B}$ is an ideal, and we have proved the isomorphism

between (2) and (3). Considering the equivalence $P \otimes_R - : {}_R\mathfrak{M} \to {}_S\mathfrak{M}$, which sends ${}_RR$ to ${}_SP$, we have a correspondence between the left ideals $\mathfrak{A} \subseteq R$ and the S-submodules $P\mathfrak{A} \subseteq {}_SP$, with ideals corresponding to (S, R)-submodules. This shows the isomorphism between (1) and (3), and the isomorphism with (4) follows by symmetry. Note that under these isomorphisms, an ideal $\mathfrak{A} \subseteq R$ and an ideal $\mathfrak{B} \subseteq S$ correspond iff $\mathfrak{B}P = P\mathfrak{A}$ (iff $\mathfrak{A}Q = Q\mathfrak{B}$). Thus, if $\mathfrak{A} \leftrightarrow \mathfrak{B}$ and $\mathfrak{A}' \leftrightarrow \mathfrak{B}'$, we have $\mathfrak{B}\mathfrak{B}'P = \mathfrak{B}P\mathfrak{A}' = P\mathfrak{A}\mathfrak{A}'$, and therefore $\mathfrak{A}\mathfrak{A}' \leftrightarrow \mathfrak{B}\mathfrak{B}'$. □

(18.45) Corollary. *Under the above correspondence, nilpotent (resp. prime, semiprime) ideals correspond to nilpotent (resp. prime, semiprime) ideals, and the prime radical of R corresponds to the prime radical of S. In particular, R is prime (resp. semiprime) iff S is.*

(18.46) Remark. In view of (18.45), it seems natural to ask if, under the ideal correspondence in (18.44), nil ideals of R will correspond to nil ideals in S. Unfortunately, the answer to this question is not known. In fact, in the case when $S = \mathbb{M}_n(R)$, an affirmative answer to this question would amount to a positive solution of the very difficult "Köthe Conjecture" in noncommutative ring theory! (For more details on this, see *FC*–(10.25).)

(18.47) Proposition. *Keeping the notations in* (18.44), *suppose M_R and N_S correspond under the category equivalence $- \otimes_R Q : \mathfrak{M}_R \to \mathfrak{M}_S$. Then the ideal* $\text{ann}(M) \subseteq R$ *corresponds to the ideal* $\text{ann}(N) \subseteq S$. *In particular, M is faithful iff N is faithful.*

Proof. Let $\mathfrak{A} = \text{ann}(M) \subseteq R$, and suppose $\mathfrak{A}$ corresponds to an ideal $\mathfrak{B} \subseteq S$. Then

(18.48) $N\mathfrak{B} \cong (M \otimes_R Q)\mathfrak{B} = M \otimes_R (Q\mathfrak{B}) = M \otimes_R \mathfrak{A}Q = M\mathfrak{A} \otimes_R Q = 0.$

Thus, $\mathfrak{B} \subseteq \text{ann}(N)$, and symmetry gives $\mathfrak{B} = \text{ann}(N)$. □

(18.49) Corollary. *Keep the above notations, and suppose an ideal $\mathfrak{A} \subseteq R$ corresponds to an ideal $\mathfrak{B} \subseteq S$. Then $R/\mathfrak{A} \approx S/\mathfrak{B}$.*

Proof. For $M \in \mathfrak{M}_R$ and $N \in \mathfrak{M}_S$, we have seen that $M\mathfrak{A} = 0$ iff $N\mathfrak{B} = 0$. If we think of $\mathfrak{M}_{R/\mathfrak{A}}$ (resp. $\mathfrak{M}_{S/\mathfrak{B}}$) as the full subcategory of $\mathfrak{M}_R$ (resp. $\mathfrak{M}_S$) consisting of objects annihilated by $\mathfrak{A}$ (resp. $\mathfrak{B}$), then the given category equivalence between $\mathfrak{M}_R$ and $\mathfrak{M}_S$ induces an equivalence between $\mathfrak{M}_{R/\mathfrak{A}}$ and $\mathfrak{M}_{S/\mathfrak{B}}$. □

(18.50) Corollary. *In the above notations,* rad R *corresponds to* rad S, *and* $R/\text{rad } R \approx S/\text{rad } S$. *In particular, R is semiprimitive (resp. semilocal, semiprimary) iff S is.*

Proof. This first conclusion follows from the fact that rad R is the intersection of the annihilators of all simple right R-modules. The other conclusions follow from the first. (Recall that R is semiprimitive if rad $R = 0$, semilocal if $R/\text{rad } R$ is right artinian, and semiprimary if $R/\text{rad } R$ is right artinian and rad R is nilpotent.) □

§18F. The Category $\sigma[M]$

In this closing subsection of §18, we would like to present an important extension of Morita's theory of equivalences to a somewhat more general situation. In this new setting, we fix a right module M over a ring R, and introduce a certain subcategory $\sigma[M]$ of $\mathfrak{M}_R$ that is uniquely determined by M. Extending earlier definitions, we can set up the notions of $\sigma[M]$-projective modules and $\sigma[M]$-generators, etc. Then we generalize Morita's theory to the study of the equivalence of the category $\sigma[M]$ to another full module category $\mathfrak{M}_S$. By letting $M = R_R$, we retrieve the equivalence theory studied earlier in §18D.

The generalization of Morita's theory to the $\sigma[M]$ setting is worthwhile since it is very natural on the one hand, and also quite useful on the other. As the readers would expect, many of the steps needed in this generalization are simply repetitions of those used earlier in §18D. Therefore, in presenting the $\sigma[M]$ theory, we shall try to be a little brief, and just focus our discussion on that part of the generalization which might need special attention or clarification.

Although in retrospect the $\sigma[M]$ theory is a rather immediate generalization of Morita's theory, it had remained unnoticed for some time. In fact, the earliest instance of the use of $\sigma[M]$ I can find in the literature is in the work of A. Dress in the representation theory of finite groups, in the early 70s. Today, this generalization of Morita's theory is discussed in several textbooks. In fact, some authors even go to the length of developing module theory in the "relative" setting, working in the subcategory $\sigma[M]$ (for a fixed choice of M_R), rather than in the full category $\mathfrak{M}_R$; see, for instance, Wisbauer [91].

The beginning point of the relative theory is the formal introduction of the subcategory $\sigma[M]$, as follows.

(18.51) Definition. Let M be a fixed right module over a ring R. By $\sigma[M]$, we mean the full subcategory of $\mathfrak{M}_R$ whose objects are "subquotients" of direct sums of copies of M; that is, submodules of quotients of direct sums $M^{(I)}$ (for arbitrary indexing sets I). The reader need not feel unsure about the interpretation of the term "subquotient", since "submodules of quotients" of $M^{(I)}$ are, up to isomorphisms, the same as "quotients of submodules" of the same.

The following examples will give an idea of what the category $\sigma[M]$ may look like.

(18.52) Examples.

(1) Suppose M_R is a generator in $\mathfrak{M}_R$ (in the sense of (18.7)). Then clearly $\sigma[M]$ is the full module category $\mathfrak{M}_R$.

(2) The $\mathbb{Z}$-module $M = \mathbb{Q}$ is *not* a generator of $\mathfrak{M}_{\mathbb{Z}}$. But since $\mathbb{Z} \subseteq \mathbb{Q}$, we have nevertheless $\sigma[\mathbb{Q}] = \mathfrak{M}_{\mathbb{Z}}$. For a general statement on when $\sigma[M]$ equals $\mathfrak{M}_R$, see Exercise 32.

(3) Let $M = (R/A)_R$, where A is an ideal in R. It is easy to see that $\sigma[M]$ is just $\mathfrak{M}_{R/A}$, viewed (in the usual way) as a full subcategory of $\mathfrak{M}_R$. So, for instance, for $R = \mathbb{Z}$ and any prime p, $\sigma[\mathbb{Z}/p\mathbb{Z}]$ is the category of vector spaces over the field of p elements.

(4) Again let $R = \mathbb{Z}$. For the Prüfer p-group C_{p^∞} (p a prime), it is easy to see that $\sigma[C_{p^\infty}]$ is the category of all abelian p-groups. (Note that the injective hull of an abelian p-group is another abelian p-group, and use the classification of injective modules over $\mathbb{Z}$.) From this and the primary decomposition theorem, it can be seen that $\sigma[\mathbb{Q}/\mathbb{Z}]$ is the category of all torsion abelian groups.

(5) Let M be a f.g. module over a commutative ring R. Then for $A := \text{ann}^R(M)$, we have $\sigma[M] = \mathfrak{M}_{R/A}$: see Exercise 33.

Note that the class of modules $\sigma[M]$ (for any M) is closed with respect to the formation of submodules and quotient modules, so it yields a nice category in which the notions kernels, cokernels, and exact sequences, etc. are meaningful. Also, the class $\sigma[M]$ is closed under arbitrary direct sums; in particular, it is closed under the usual "pushout" and "pullback" constructions carried out in $\mathfrak{M}_R$. (It can be shown that direct products also exist in $\sigma[M]$, although it may not be the same as the direct product formed in $\mathfrak{M}_R$. In order to move the discussion along, however, we shall not dwell on this point here.)

Having introduced the category $\sigma[M]$, we can now define the *$\sigma[M]$-projective modules.* As the readers would expect, these are simply the modules P in $\sigma[M]$ for which the functor $\text{Hom}_R(P, -)$ is *exact on* $\sigma[M]$. We have the following natural characterization of these projective objects.

(18.53) Lemma. *A module P in $\sigma[M]$ is $\sigma[M]$-projective iff every short exact sequence $0 \to A \to B \to P \to 0$ in $\sigma[M]$ splits.*

Proof. (Sketch) This "only if" part follows as usual by "lifting" the identity map $P \to P$ to a map $P \to B$. For the "only if" part, assume the given splitting property, and consider an epimorphism $f : X \to Y$ in $\sigma[M]$ and another homomorphism $g : P \to Y$. We can form the pullback B of (f, g), which comes with a surjection $h : B \to P$, with B and $A := \ker(h)$ in $\sigma[M]$. By assumption,

$$0 \to A \to B \to P \to 0$$

splits. Composing a splitting of h with the natural map from B to X, we obtain a lifting of the given map $g : P \to Y$. $\square$

(18.54) Remark. Of course, if $M = R_R$, then the $\sigma[M]$-projectives are just the ordinary R-projectives. In general, a projective R-module in $\sigma[M]$ is always $\sigma[M]$-projective, though the converse is not true. Also, *M itself may not be $\sigma[M]$-projective*! For instance, for $M = \mathbb{Q}$ over the ring $R = \mathbb{Z}$, we know that $\sigma[M] = \mathfrak{M}_{\mathbb{Z}}$ (by (18.52)(2)), but M is certainly not a projective object in this category.

Next we define the *$\sigma[M]$-generators*: these are the modules P in $\sigma[M]$ for which the functor $\mathrm{Hom}_R(P, -)$ is *faithful on* $\sigma[M]$; in other words, for every nonzero $f : A \to B$ in $\sigma[M]$, there exists $h : P \to A$ such that $f \circ h \neq 0$. The proof of the following lemma can be safely left as an exercise, since it is very similar to part of the proof of (18.8).

(18.55) Lemma. *A module P in $\sigma[M]$ is a $\sigma[M]$-generator iff every module in $\sigma[M]$ is a quotient of some direct sum $P^{(I)}$.*

(18.56) Examples. Let $R = \mathbb{Z}$, and $M = \mathbb{Q}/\mathbb{Z}$. Then, as in (18.52)(4), $\sigma[M]$ is the category of all torsion abelian groups. In this case, a $\sigma[M]$-generator is given by the torsion group $P := \bigoplus_{i\geq 2} \mathbb{Z}/i\,\mathbb{Z}$. In fact, if $f : A \to B$ is a nonzero morphism in $\sigma[M]$, then $f(a) \neq 0$ for some $a \in A$. Letting j be the order of a, we can define $h : P \to A$ by mapping a generator of $\mathbb{Z}/i\,\mathbb{Z}$ to $\delta_{ij}a$, to ensure that $f \circ h \neq 0$. In a similar vein, we can check that, if C_{p^∞} is the Prüfer p-group (for a prime p), then a $\sigma[C_{p^\infty}]$-generator is given by $\bigoplus_{i\geq 1} \mathbb{Z}/p^i\,\mathbb{Z}$.

With the definitions of $\sigma[M]$-projectives and $\sigma[M]$-generators in place, we can now formulate the analogue of Morita I, which shows how a category equivalence can be constructed from $\sigma[M]$ to some full module category $\mathfrak{M}_S$, provided that a suitable object, called a "$\sigma[M]$-progenerator", exists in $\sigma[M]$.

(18.57) Theorem. *For a given module M_R, suppose there exists a $\sigma[M]$-progenerator, that is, a f.g. R-module P in $\sigma[M]$ which is both $\sigma[M]$-projective and a $\sigma[M]$-generator. Let $S := \mathrm{End}_R(P)$, and define (covariant) functors*

$$F : \sigma[M] \longrightarrow \mathfrak{M}_S \quad \textit{and} \quad G : \mathfrak{M}_S \longrightarrow \sigma[M] \quad \textit{by}$$
$$F := \mathrm{Hom}_R(P, -) \quad \textit{and} \quad G := - \otimes_S P.$$

Then F and G are mutually inverse category equivalences between $\sigma[M]$ and $\mathfrak{M}_S$.

This theorem has a converse also, which says that *any* equivalence between $\sigma[M]$ and another full module category $\mathfrak{M}_S$ must arise essentially in the above fashion (with $S = \mathrm{End}_R(P)$ for a suitable $\sigma[M]$-progenerator P). This converse is then the appropriate generalization of "Morita II" (i.e. (18.26)). To save space, however, we shall omit the proof of this converse, so the discussion below will be focused solely on (18.57), the generalization of "Morita I".

Let us first give a few details to elucidate the definition of the two functors F and G in (18.57). First, since P is a right R-module, we are assuming as usual

that S acts on the left of P, so that P becomes an (S, R)-bimodule. For any $X_R \in \sigma[M]$, $F(X) = \mathrm{Hom}_R(P, X)$ is then a right S-module since $P = {}_SP$. On the other hand, for any right S-module Y, we can form the tensor product $G(Y) = Y \otimes_S P$, which is a right R-module since $P = P_R$. *We need to check that* $G(Y) = Y \otimes_S P$ *is in* $\sigma[M]$. This is not hard. In fact, express P_R as a quotient of some $Q \subseteq M^{(I)}$, and express Y as a quotient of some $S^{(J)}$. Then $Y \otimes_S P$ is a quotient of

$$S^{(J)} \otimes_S Q \cong (S \otimes_S Q)^{(J)} \cong Q^{(J)},$$

which embeds in $M^{(I \times J)}$. Therefore, $Y \otimes_S P \in \sigma[M]$. (For this argument, we have used the right exactness of the tensor product functor, as well as the fact that tensor products commute with direct sums.)

Proof of (18.57). First note that F is an exact functor and G is a right exact functor. Therefore, their compositions $G \circ F$ and $F \circ G$ are both right exact. The functor G preserves (arbitrary) direct sums, and, using the hypothesis that P_R is f.g., it is easy to check that F also does. Therefore, the compositions $G \circ F$ and $F \circ G$ both preserve direct sums. Our job is to show that these are naturally equivalent to the identity functors on $\sigma[M]$ and $\mathfrak{M}_S$, respectively.

For any $X \in \sigma[M]$, we have a natural R-homomorphism

$$\alpha(X) : \ (G \circ F)(X) = \mathrm{Hom}_R(P, X) \otimes_S P \longrightarrow X$$

defined by $\alpha(X)(f \otimes p) = f(p)$, for any $p \in P$ and $f \in \mathrm{Hom}_R(P, X)$. (The well-definition of $\alpha(X)$ is checked by

$$\alpha(X)(fs \otimes p) = (fs)(p) = f(sp) = \alpha(X)(f \otimes sp)$$

for any $s \in S$.) We need to show that $\alpha(X)$ is an isomorphism. Since P is a $\sigma[M]$-generator, there exists a resolution $P^{(I)} \to P^{(J)} \to X \to 0$. The right exactness of $G \circ F$ leads to a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
(G \circ F)(P^{(I)}) & \longrightarrow & (G \circ F)(P^{(J)}) & \longrightarrow & (G \circ F)(X) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha(P^{(I)})} & & \downarrow{\scriptstyle \alpha(P^{(J)})} & & \downarrow{\scriptstyle \alpha(X)} & & \\
P^{(I)} & \longrightarrow & P^{(J)} & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Therefore, it suffices to show that $\alpha(P^{(I)})$ and $\alpha(P^{(J)})$ are isomorphisms. Since $G \circ F$ preserves arbitrary direct sums, we are reduced to showing that $\alpha(P)$ itself is an isomorphism. Now the domain of $\alpha(P)$ is

$$(G \circ F)(P) = \mathrm{Hom}_R(P, P) \otimes_S P \cong S \otimes_S P \cong P.$$

Upon viewing these isomorphisms as identifications, $\alpha(P)$ is just the identity map on P, so indeed $\alpha(P)$ is an isomorphism.

The fact that $F \circ G$ is naturally equivalent to the identity functor on $\mathfrak{M}_S$ can be checked similarly, this time using the natural maps

$$\beta(Y) : \ Y \longrightarrow \mathrm{Hom}_R(P, \ Y \otimes_S P) = (F \circ G)(Y) \quad (Y \text{ in } \mathfrak{M}_S)$$

defined by $\beta(Y)(y)(p) = y \otimes p$ for any $y \in Y$ and $p \in P$. □

(18.58) Examples.

(1) In the special case when $M = R_R$, we can choose P to be any progenerator in $\mathfrak{M}_R$. Here, the two functors F, G in (18.55) are essentially those in (18.24) for the equivalence of $\mathfrak{M}_R$ and $\mathfrak{M}_S$ (where $S = \text{End}_R(P)$); see Remark (18.25). Therefore, in this case, we simply recapture the original "Morita I".

(2) Let $M = (R/A)_R$, where A is a given ideal in a ring R. As in (18.52)(3), we may identify $\sigma[M]$ with $\mathfrak{M}_{R/A}$ (the latter thought of as a full subcategory of $\mathfrak{M}_R$). A rather "trivial" illustration for (18.57) is provided by choosing P to be M, which is indeed a $\sigma[M]$-progenerator. For the functor F constructed in (18.57), we have

$$F(X) = \text{Hom}_R(P, X) \cong \text{Hom}_{R/A}(R/A, X) \cong X$$

for any X in $\sigma[M]$. In particular, $S := \text{End}_R(P)$ is just R/A, and F is naturally equivalent to the identity functor on $\sigma[M] = \mathfrak{M}_{R/A}$.

(3) Of course, the equivalence of categories in (18.57) exists only if we have a $\sigma[M]$-progenerator P. In general, such an object may not exist. An example is given by $\sigma[\mathbb{Q}/\mathbb{Z}]$ over the ring of integers. Since this is the category of all torsion abelian groups, a f.g. $\mathbb{Z}$-module therein must be finite, and therefore killed by some integer n. By (18.55), it cannot be a $\sigma[\mathbb{Q}/\mathbb{Z}]$-generator. Hence there does not exist a $\sigma[\mathbb{Q}/\mathbb{Z}]$-progenerator. (It turns out that $\sigma[\mathbb{Q}/\mathbb{Z}]$ has, in fact, no projective objects other than (0): see Exercise 34.)

In general, even if a $\sigma[M]$-progenerator does not exist, all is not lost, since we can still use the module M itself in lieu of P to get some useful information. We proceed as follows. Let $\mathcal{D}(M)$ (resp. $\mathcal{D}_0(M)$) be the full subcategory of $\sigma[M]$ whose objects are *direct summands* of direct sums (resp. finite direct sums) of copies of M. For $S := \text{End}_R(M)$, let $\mathcal{P}(S)$ (resp. $\mathcal{P}_0(S)$) be the full subcategory of $\mathfrak{M}_S$ whose objects are the projective (resp. f.g. projective) right S-modules. Then we have the following useful result which is due to A. Dress.

(18.59) Theorem. *For a given module M_R, define the two functors F, G as in (18.57) using M in the place of P, and let $S = \text{End}_R(M)$. Then:*

(1) *F and G define mutually inverse category equivalences between $\mathcal{D}_0(M)$ and $\mathcal{P}_0(S)$;*

(2) *In case M is a f.g. R-module, F and G define mutually inverse category equivalences between $\mathcal{D}(M)$ and $\mathcal{P}(S)$.*

Proof. (1) Note that both F and G preserve finite direct sums, and

$$F(M) = \text{Hom}_R(M, M) = S, \quad G(S) = S \otimes_S M \cong M.$$

For any $X \in \mathcal{D}_0(M)$, we have the natural map $\alpha(X) : X \to (G \circ F)(X)$ defined in the proof of (18.57). Pick $X' \in \mathcal{D}_0(M)$ such that $X \oplus X' \cong M^n$ for some

integer n. From what we said above, $\alpha(M)$ is an isomorphism. Therefore, $\alpha(M)^n$ is also an isomorphism, and it follows that $\alpha(X)$ is an isomorphism. Similarly, we can show that, for any $Y \in \mathcal{P}_0(S)$, the map $\beta(Y) : Y \to (F \circ G)(Y)$ defined in the proof of (18.57) is an isomorphism.

The proof of (2) is similar, upon noting again that, in case M is f.g., the functor F preserves arbitrary direct sums (as G does). □

The point about (18.59) is that, in proving theorems about modules associated with a given module M, it is sometimes possible to use (18.59) to make a "transfer" from the category $\mathcal{D}_0(M)$ to the category $\mathcal{P}_0(S)$, so that the consideration is reduced to that of finitely generated projective modules over the ring S.

Exercises for §18

0. Show that a module M_R is f.g. iff it satisfies the property for chains of submodules in (18.3). (**Hint** ("Sufficiency"). Assume M is not f.g. and apply Zorn's Lemma to $\mathcal{B} = \{B \subseteq M : M/B \text{ is not f.g.}\}$.)

1A. In an additive category $\mathfrak{M}$ with arbitrary direct sums, an object M is said to be *small* if every morphism $f : M \to \bigoplus_{i \in I} A_i$ factors through a *finite* direct sum of the A_i 's. Show that

(1) every quotient object of a small object is small;
(2) $M = N \oplus N'$ is small iff N, N' are small.

1B. For any module category $\mathfrak{M}_R$, prove the following:

(1) A module M_R is small iff there does not exist an infinite family of submodules $M_j \subsetneq M$ $(j \in J)$ such that every $m \in M$ lies in almost all M_j.
(2) Every f.g. R-module is small in the category $\mathfrak{M}_R$.
(3) M_R is noetherian iff every submodule of M is small in $\mathfrak{M}_R$.
(4) If R is right noetherian, M_R is small iff M is f.g.
(5) If M_R is projective, M_R is small iff M is f.g.
(6) Give an example of a module that is small but not f.g.

2. Show that M_R being a singular (resp. nonsingular) R-module is a categorical property.

3. Characterize right nonsingular rings R by the property that every right projective R-module is nonsingular, and deduce that R being right nonsingular is a Morita invariant property.

4. Show that R being semiperfect (resp. right perfect) is a Morita invariant property (cf. *FC*–(24.16) and *FC*–(24.18)).

5. Show that *finiteness* of a ring is a Morita invariant property without using (18.35). (**Hint.** Consider endomorphism rings of f.g. modules.)

6. Show that "u.dim $R_R < \infty$" is a Morita invariant property.

7A. Show that the property of R being right self-injective or quasi-Frobenius can be characterized by suitable categorical properties of $\mathfrak{M}_R$. Deduce that "right self-injective" and "QF" are Morita invariant properties of rings.

7B. (1) Show that the basic ring of a QF ring is always a Frobenius ring.

(2) Compute the basic ring of the QF ring in Example (16.19)(5).

(3) Show that being a Frobenius ring is not a Morita invariant property.

7C. Show that being a symmetric algebra over a field k is a Morita invariant property.

8. (A slight variation of (18.35).) Show that a necessary and sufficient condition for a ring-theoretic property $\mathcal{P}$ to be Morita invariant is that, for any full idempotent e in a ring R, R satisfies $\mathcal{P}$ iff eRe does.

9. Use (18.35) (instead of (18.45), (18.50)) to show that semiprimitivity, primeness, and semiprimeness are Morita invariant properties. (**Hint.** Refer to *FC*–(10.20) and *FC*–(21.13).)

10. Let $S = \text{End}(P_R)$ where P_R is a progenerator over the ring R. Show that the ring S has IBN iff $P^n \cong P^m$ (in $\mathfrak{M}_R$) implies $n = m$.

11. (Bergman) For any ring R, let $\mathcal{P}(R)$ be the monoid of isomorphism classes of f.g. projective right R-modules (under the direct sum operation). Using the technique of coproducts from Bergman [74], it can be shown that there exists a ring R for which $\mathcal{P}(R)$ is generated as a monoid by $[R]$ together with $[M]$, $[N]$, with the defining relations $[M]+[N] = [R] = [R]+[R]$. Show that $S = \text{End}_R(M \oplus R)$ is Morita equivalent to R and has IBN, but R does not have IBN. (This shows that IBN is *not* a Morita invariant property.) (**Hint.** Apply Exercise 10 with $P := M \oplus R$.)

12. Suppose a ring-theoretic property $\mathcal{P}$ is such that, for any $n \geq 1$ and any ring R, R satisfies $\mathcal{P}$ iff $\mathbb{M}_n(R)$ does. Is $\mathcal{P}$ necessarily a Morita invariant property?

13. Show that a projective module P_R is a generator iff every simple module M_R is an epimorphic image of P.

14. Find the flaw in the following argument: "Let R, S be division rings. Construct functors $F : \mathfrak{M}_R \to \mathfrak{M}_S$ and $G : \mathfrak{M}_S \to \mathfrak{M}_R$ by: $F(U_R) = \bigoplus_{i \in I} S_S$ where $|I| = \dim_R U$, and $G(V_S) = \bigoplus_{j \in J} R_R$ where $|J| = \dim_S V$. Then F, G are inverse category equivalences and hence $R \approx S$."

15. Let $e = e^2 \in R$, and let α, β be as defined in (18.30A).

(1) Give a direct proof for the fact that β is an isomorphism by explicitly constructing an inverse for β.

(2) Show that $eR \cdot \ker(\alpha) = 0$. Using this, give another proof for the fact that α is an isomorphism if e is a full idempotent.

16. (Partial refinement of (18.22).) Let P_R be a right R-module, with associated Morita Context $(R, P, Q, S;\ \alpha, \beta)$.

    (1) If P_R is a generator, show that ${}_SP$ is f.g. projective.

    (2) If P_R is f.g. projective, show that ${}_SP$ is a generator.

    (**Hint.** For both cases, apply the functor $\mathrm{Hom}_R(-,\ {}_SP_R)$ from $\mathfrak{M}_R$ to ${}_S\mathfrak{M}$.)

17. (Morita) Show that P_R is a generator iff, for $S = \mathrm{End}(P_R)$, ${}_SP$ is f.g. projective and the natural map $R \to \mathrm{End}({}_SP)$ is an isomorphism. (**Note.** The "only if" part is already covered by (18.17)(2d) and (1) of the above exercise.)

18. Let R, S be rings, and ${}_SP_R$, ${}_RQ_S$ be bimodules. Let $\alpha : Q \otimes_S P \to R$ be an (R, R)-homomorphism, and $\beta : P \otimes_R Q \to S$ be an (S, S)-homomorphism. Define $pq = \beta(p \otimes q) \in S$ and $qp = \alpha(q \otimes p) \in R$, and let $M = \begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ (formally). Show that M is a ring under formal matrix multiplication (as in (18.16)) iff $q'(pq) = (q'p)q$ and $p(qp') = (pq)p'$ hold for all $p, p' \in P$ and $q, q' \in Q$. (Note that, in the special case when $P = 0$, the additional conditions are vacuous, and we get back the "triangular ring" construction $M = \begin{pmatrix} R & Q \\ 0 & S \end{pmatrix}$ in *FC*–(1.14).)

19. Suppose the "associativity" conditions in the above exercise are satisfied, so M is a ring. Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $R = eMe$, $S = fMf$, $P = fMe$, and $Q = eMf$.

20. Let M be a ring with idempotents e, f such that $e + f = 1$. Let $R = eMe$, $S = fMf$, $P = fMe$, and $Q = eMf$. Show that the natural maps $\alpha : Q \otimes_S P \to R$, $\beta : P \otimes_R Q \to S$ satisfy the "associativity" conditions in Exercise 18, and that the formal Morita ring $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ constructed there is isomorphic to the original ring M.

21. Suppose, in Exercise 18, the associativity conditions are satisfied, and that α, β are both *onto*. Show that

    (1) α, β are isomorphisms;
    (2) P_R is a progenerator; and
    (3) the Morita Context for P_R is $(R, P, Q, S;\ \alpha, \beta)$.

22. Using the category equivalence between $\mathfrak{M}_S$ and $\mathfrak{M}_R$ where $R = \mathbb{M}_n(S)$, show that there is an isomorphism from the lattice of right ideals of R to the lattice of S-submodules of the free module S_S^n. From this, deduce that, if two rings S, T are such that $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$, then the free modules S_S^n and T_T^m have isomorphic lattices of submodules.

23. Show that, under the ideal correspondence between S and $\mathbb{M}_n(S)$ established in (18.44), an ideal $\mathfrak{B} \subseteq S$ corresponds to the ideal $\mathbb{M}_n(\mathfrak{B}) \subseteq \mathbb{M}_n(S)$. Deduce from (18.50) that $\operatorname{rad} \mathbb{M}_n(S) = \mathbb{M}_n(\operatorname{rad} S)$ (cf. *FC*–p. 61).

24. Let e be a full idempotent in a ring R, so that $R \approx eRe$. Show that the ideal correspondence between R and eRe is as given in *FC*–(21.11); that is, $\mathfrak{A} \subseteq R$ goes to $e\mathfrak{A}e \subseteq eRe$, and $\mathfrak{B} \subseteq eRe$ goes to $R\mathfrak{B}R \subseteq R$.

25. Under a category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, show that the ideal correspondence between R and S sends an ideal $\mathfrak{A} \subseteq R$ to the annihilator of the right S-module $F(R/\mathfrak{A})$.

26. *True or False*: "If $R \approx S$, then there is a one-one correspondence between the subrings of R and those of S".

27. *True or False*: "If $R \approx S$, then there exist natural numbers n and m such that $\mathbb{M}_n(R) \cong \mathbb{M}_m(S)$"?

28. For any (right) R-module P, let $P^{(\mathbb{N})}$ denote the direct sum $P \oplus P \oplus \dots$, and let $S = \operatorname{End}_R(P)$. Show that, if P_R is f.g., then $\operatorname{End}_R(P^{(\mathbb{N})}) \cong \operatorname{CFM}(S)$, the ring of column-finite $\mathbb{N} \times \mathbb{N}$ matrices over S.

29. Show that a f.g. right R-module P is a progenerator iff $P^{(\mathbb{N})} \cong R^{(\mathbb{N})}$ as right R-modules. Is this still true if P is not assumed f.g.? (**Hint.** If P is a progenerator, show that $P^{(\mathbb{N})}$ and $R^{(\mathbb{N})}$ are both isomorphic to $P^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$.)

30. (Camillo) Show that two rings R and S are Morita equivalent iff $\operatorname{CF}\mathbb{M}(R) \cong \operatorname{CF}\mathbb{M}(S)$. (**Sketch.** For the "only if" part, take $S = \operatorname{End}(P_R)$, where P_R is a progenerator. Then $P^{(\mathbb{N})} \cong R^{(\mathbb{N})}$ by Exer. 29. Now take their endomorphism rings. For the converse, assume there is an isomorphism $\sigma : \operatorname{CFM}(S) \to \operatorname{CFM}(R)$. Let $\{e_{ij}\}$ be the matrix units in $\operatorname{CFM}(S)$, and identify $\operatorname{CFM}(R)$ with $\operatorname{End}_R(R^{(\mathbb{N})})$. Show that $R^{(\mathbb{N})} = \bigoplus_i \sigma(e_{ii}) R^{(\mathbb{N})}$, and that $P := \sigma(e_{11}) R^{(\mathbb{N})}$ is f.g. as an R-module. Deduce from Exercise 29 that P_R is a progenerator, and show that $\operatorname{End}(P_R) \cong S$. One final note: Camillo's result also holds with "CFM" replaced by "RCFM", where, for a ring A, $\operatorname{RCFM}(A)$ denotes the ring of *row and column finite* $\mathbb{N} \times \mathbb{N}$ matrices over A. This is a recent (1997) result of J. Haefner, A. del Río, and J. J. Simón.)

31. (Hattori-Stallings Trace) For any ring R, let $\overline{R}$ be the quotient group $R/[R, R]$, where $[R, R]$ denotes the additive subgroup $\{\sum (ab - ba) : a, b \in R\}$ of R.

    (1) Show that the projection map "bar": $R \to \overline{R}$ is a universal group homomorphism with respect to the "trace property" $\overline{ab} = \overline{ba}$ (for all $a, b \in R$).

    (2) Show that the group $\overline{R}$ is uniquely determined by the Morita equivalence class of R. (**Hint.** Say $R \approx S$ and use the notations in "Morita I". Define a map $f : Q \otimes_S P \to \overline{S}$ by $f(q \otimes_S p) = \overline{pq} \in \overline{S}$.)

32. For any right R-module M, let $\sigma[M]$ be the full subcategory of $\mathfrak{M}_R$ defined in §18F. Show that $\sigma[M] = \mathfrak{M}_R$ iff R_R can be embedded in M^n for some integer n.

33. Let M be a right R-module with $A = \text{ann}^R(M)$. Let $S = \text{End}_R(M)$ (operating on the left of M). If M is f.g. as a left S-module, show that $\sigma[M] = \mathfrak{M}_{R/A}$. Deduce that, if M is a f.g. (right) module over a commutative ring R, then $\sigma[M] = \mathfrak{M}_{R/A}$ (for $A = \text{ann}^R(M)$).

34. For $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$, show that the only $\sigma[M]$-projective module is (0).

35. (Blackadar) In this exercise, we write $r \cdot P$ for the direct sum of r copies of a module P. Let P, Q, X be right modules over a ring R such that $P \oplus X \cong Q \oplus X$. If X embeds as a direct summand of $r \cdot P$ and of $r \cdot Q$ (for some integer r), show that $n \cdot P \cong n \cdot Q$ for all $n \geq 2r$. Deduce that if P, Q are generators over R and X is a f.g. projective R-module such that $P \oplus X \cong Q \oplus X$, then $n \cdot P \cong n \cdot Q$ for all sufficiently large n.

§19. Morita Duality Theory

§19A. Finite Cogeneration and Cogenerators

The goal of §19 will be to give an exposition of Morita's theory of equivalences of (subcategories of) module categories under *contravariant* functors. Some features of this so-called duality theory are similar to those in the theory developed in §18. However, there are various new problems to be addressed and solved, and it is not just a matter of reversing the arrows of the morphisms. (The pitfalls of blindly "dualizing" a statement have already been well illustrated by the example in Remark (6.9)(c).) The main motivation for this duality theory will be given in §19C. In this preliminary subsection, we first discuss the notion of finitely cogenerated modules, and the notion of cogenerator modules.

The notion of a finitely cogenerated module was first introduced in Vámos [68]. In this paper, the definition for such a module was given in terms of its injective hull. For us, it is more natural to arrive at the definition by "dualizing" the conditions (18.2) and (18.3) for f.g. modules. The dualized conditions are given as (1) and (2) in the proposition below. It is, however, a bit tricky to see directly that these two conditions are equivalent. The proof becomes easier if we add another condition, namely (3), in the following. It turns out, fortunately, that (3) itself is a very useful condition to work with. For good measure, we shall also add (4), which is just a reformulation of (1).

(19.1) Proposition. *For any right module M over a ring R, the following are equivalent:*

(1) *For any family of submodules $\{N_i : i \in I\}$ in M, if $\bigcap_{i\in I} N_i = 0$, then $\bigcap_{i\in J} N_i = 0$ for some finite subset $J \subseteq I$.*

(2) *For any family of submodules* $\{N_i : i \in I\}$ *in* M *which form a chain, if each* $N_i \neq 0$, *then* $\bigcap_{i\in I} N_i \neq 0$.
(3) *The socle* $S := \mathrm{soc}(M)$ *is f.g., and* $S \subseteq_e M$.
(4) *If* $M \to \prod_{i\in I} M_i$ *is an embedding (where the* M_i*'s are any right* R*-modules), then* $M \to \prod_{i\in J} M_i$ *is already an embedding for some finite subset* $J \subseteq I$.

Proof. (4) $\Longleftrightarrow$ (1) $\Longrightarrow$ (2) are clear.

(2) $\Longrightarrow$ (3). If the (semisimple) module S is not f.g., it would contain an infinite direct sum $\bigoplus_{j=1}^{\infty} S_j$ where each $S_j \neq 0$. Then

$$S_1 \oplus S_2 \oplus \cdots \supseteq S_2 \oplus S_3 \oplus \cdots \supseteq S_3 \oplus S_4 \oplus \cdots$$

is a chain of nonzero modules with a zero intersection, in contradiction to (2). Thus, S must be f.g. To see that $S \subseteq_e M$, consider any nonzero submodule $N \subseteq M$, and the family of nonzero submodules of N, ordered by (reverse) inclusion. In view of (2), we can apply Zorn's Lemma to this family, to conclude that it has a minimal member N_0. This is then a simple submodule of N, and so it lies in $\mathrm{soc}(M) = S$. Thus, $N \cap S \supseteq N_0 \neq 0$, and we have checked that $S \subseteq_e M$. □

(3) $\Longrightarrow$ (1). Suppose that, in (1), $\bigcap_{i\in J} N_i \neq 0$ for any finite $J \subseteq I$. We may assume that the family $\{N_i\}$ is closed under finite intersections. The first assumption in (3) implies that S is artinian, and the second implies that $N_i' := N_i \cap S \neq 0$. Therefore, the family $\{N_i'\}$ has a minimal member, say, $N_{i_0}' \neq 0$. Then, for any $i \in I$, $N_i' \cap N_{i_0}' = N_{i_0}'$ by the minimality of N_{i_0}'. This shows that $\bigcap_{i\in I} N_i \supseteq N_{i_0}' \neq 0$. □

(19.2) Definition. A module M_R is said to be *finitely cogenerated*[99] (f.cog. for short) if it satisfies the equivalent conditions in (19.1). (For Vámos' characterizations of f.cog. modules in terms of their injective hulls, see Exercise 7.)

Let us make two remarks concerning the various conditions in (19.1). First, since (1) and (2) are conditions of a similar nature, it would be of interest to find a direct proof for their equivalence which does not involve an extrinsic notion such as that of a socle. We challenge the reader to provide such a proof: see Exercise 1. Second, since $S := \mathrm{rad}(M)$ in (19.1)(3) is a *semisimple* module, to say that S is f.cog. is the same as saying that it is f.cog., according to Exercise 3 below. If we replace "f.g." by "f.cog." in (19.1)(3), we see that the resulting characterization of a f.cog. module becomes the dual of one of the known characterizations for a f.g. module: "a module M is f.g. iff $\mathrm{rad}(M)$ is small in M and $M/\mathrm{rad}(M)$ is f.g." (Recall that a submodule $X \subseteq M$ is small if, for any submodule Y, $X + Y = M$

[99]Other names are sometimes used in the literature, e.g. "co-finitely generated", "finitely embedded", or "essentially artinian".

implies $Y = M$; rad(M) is the intersection of all maximal submodules of M, and is M itself if there are no maximal submodules.)

Working with a f.cog. module M requires different intuition from working with a f.g. module. Note, for instance, that while R_R is always f.g. (it is "cyclic"), it is *not* always f.cog., as the example $R = \mathbb{Z}$ shows. However, a number of facts about f.g. modules can be "dualized" to give suggestive statements about f.cog. modules. These "suggestive statements", of course, would require proofs (and in general there is no guarantee that they are always true!). For instance, the fact that "any quotient module of a f.g. module is f.g." dualizes into:

(19.3A) *Any submodule of a f.cog. module is f.cog.*

This happens to be true, and can be checked easily by using (for instance) the condition (1) in (19.1). The fact that "any proper submodule of a f.g. module is contained in a maximal submodule" dualizes into:

(19.3B) *Any nonzero submodule of a f.cog. module contains a minimal submodule.*

This is also true, and was in fact proved in the argument for (2) $\Longrightarrow$ (3) in (19.1). Finally, the well known fact that "a module M_R is noetherian iff every submodule of M is f.g." also admits a valid dual, as follows.

(19.4) Proposition. *A module M_R is artinian iff every quotient of M is f.cog. (In particular, R is a right artinian ring iff every cyclic (resp. f.g.) right R-module is f.cog.)*

The easy proof of this is left as Exercise 0. Other relevant facts about f.cog. modules are collected at the beginning of the set of exercises for this section. Their proofs are mostly routine.

Our next goal is to study cogenerator modules, which will play a very substantial role in the rest of this section. The definition of cogenerator modules is obtained by dualizing that of generator modules.

(19.5) Definition. A module U_R is called a *cogenerator* if $\text{Hom}_R(-, U)$ is a faithful (contravariant!) functor from $\mathfrak{M}_R$ to the category of abelian groups (in other words, if for every nonzero $f : M \to N$ in $\mathfrak{M}_R$, there exists $g : N \to U$ such that $gf \neq 0$.) Clearly if U is a cogenerator, so is any module $U' \supseteq U$.

(19.6) Proposition. *For $U_R \in \mathfrak{M}_R$, the following are equivalent:*

(1) *U is a cogenerator.*
(2) *For any $N \in \mathfrak{M}_R$ and $0 \neq x \in N$, there exists $g : N \to U$ such that $g(x) \neq 0$.*
(3) *Any $N \in \mathfrak{M}_R$ can be embedded into some direct product $\prod_i U$.*

Proof. (1)$\Longrightarrow$(2). Let $f : R \to N$ be defined by $f(1) = x \neq 0$. Take $g : N \to U$ such that $gf \neq 0$. Then $g(x) = (gf)(1) \neq 0$.

(2)$\Longrightarrow$(3). For any $0 \neq x \in N$, fix $\pi_x : N \to U$ with $\pi_x(x) \neq 0$. Then $\pi = (\pi_x)$ gives an embedding of N into $\prod_{x \neq 0} U$.

(3)$\Longrightarrow$(1). Say $f : M \to N$ is nonzero, and take an embedding $\pi = (\pi_i) : N \to \prod_i U$. Then $\pi f \neq 0$, so we must have $\pi_i f \neq 0$ for some i. □

(19.7) Corollary. *(1) Any cogenerator U_R is faithful. (2) If some direct product $\prod_i U$ is a cogenerator, so is U.*

Proof. Apply Criterion (3) in (19.6) to $N = R_R$. □

(19.8) Theorem. *U_R is a cogenerator iff, for any simple module V_R, U contains a copy of $E(V)$ (the injective hull of V).*

Proof. First suppose U is a cogenerator. For any simple module V, there exists $g : E(V) \to U$ with $g|V \neq 0$. This implies that $(\ker g) \cap V = 0$, and hence $\ker g = 0$. Thus, $E(V)$ embeds in U. Conversely, assume U contains a copy of $E(V)$ for every simple V. To see that U is a cogenerator, we check Criterion (2) in (19.6). Let $0 \neq x \in N$, where $N \in \mathfrak{M}_R$. Take a maximal submodule M of xR, and let $V = xR/M$. We have a map $g_0 : xR \to E(V)$ with kernel M and image V. By the injectivity of $E(V)$, g_0 extends to some $g_1 : N \to E(V)$. Composing this with an embedding $E(V) \hookrightarrow U$, we get a homomorphism $g : N \to U$ with $g(x) \neq 0$. □

Since projective generators played a major role in §18, we can expect that injective cogenerators will be important for this section. By specializing (19.8), we get the following useful characterization of injective cogenerators.

(19.9) Corollary. *An injective module U_R is a cogenerator iff every simple module V_R embeds in U.*

Proof. The "only if" part is clear from (19.8). (This part does not require injectivity of U.) For the "if" part, simply note that if V embeds in U, so does $E(V)$, since U is injective. □

It is worth noting that there does exist a "projective analogue" of (19.9): see Exercise (18.13). However, there is no projective analogue of (19.8), since not every module has a "projective cover" in the sense of *FC*–(24.9).

It is not a priori clear from (19.5) that cogenerator modules exist. However, the following explicit construction of a cogenerator solves this problem, and more.

(19.10) Theorem. *Let $\{V_i\}$ be a complete set of simple right R-modules. Then $U_0 = \bigoplus_i E(V_i)$ is a cogenerator, called the canonical cogenerator of $\mathfrak{M}_R$. A module U_R is a cogenerator iff U_0 can be embedded in U.*

Proof. The first conclusion is clear from (19.8). For the second conclusion, we need only prove the "only if" part. Let U be a cogenerator. By (19.8), we may assume that $E(V_i) \subseteq U$ for all i. Since the V_i 's are nonisomorphic simple modules, the sum $\sum_i V_i$ in U must be direct. By Exercise 3.8, the sum $\sum_i E(V_i) \subseteq U$ must also be direct, and so U contains a copy of $U_0 = \bigoplus_i E(V_i)$. □

Let us call a cogenerator M_R *minimal* if every cogenerator contains a copy of M. According to (19.10), *the canonical cogenerator* U_0 *is minimal.* However, as is recently shown by Osofsky [91] (and contrary to what was stated in earlier literature), a minimal cogenerator need not be unique. (If M_1, M_2 are minimal cogenerators, they can be embedded in each other; but this need not imply that $M_1 \cong M_2$.)

If the number of V_i 's in (19.10) is finite, or if the ring R is noetherian, the canonical cogenerator U_0 will be an *injective* module. The following are some standard examples drawn from earlier material on injective hulls.

(19.11) Examples.

(1) Let $R = \mathbb{Z}$. The simple modules are $V_p = \mathbb{Z}_p$ for $p = 2, 3, 5 \ldots$, and $E(V_p)$ is the Prüfer p-group C_{p^∞} (see (3.37)). Therefore, the canonical cogenerator over $\mathbb{Z}$ is

(19.12) $$U_0 = \bigoplus_p E(V_p) = \bigoplus_p C_{p^\infty} \cong \mathbb{Q}/\mathbb{Z}.$$

The fact that this is an (injective) cogenerator for $\mathfrak{M}_\mathbb{Z}$ was also proved directly in (4.7). With a little additional work, one can show that, over a Dedekind domain R with quotient field K, the canonical cogenerator is $U_0 = K/R$.

(2) Let R be a right artinian ring. Then we have only a finite set of simple right R-modules $\{V_1, \ldots, V_n\}$. By (3.61),

$$U_0 = E(V_1) \oplus \cdots \oplus E(V_n)$$

is just the direct sum of a complete set of indecomposable injectives over R. However, U_0 need not be f.g.: see Exercise (3.34).

(3) In the case when R is a finite-dimensional algebra over a field k, the description of U_0 can be made even more explicit. Using the notations in (3.65), the indecomposable injective right R-modules are exactly the k-duals $(Re_1)\hat{}, \ldots, (Re_n)\hat{}$. In particular, the canonical cogenerator

$$U_0 \cong (Re_1)\hat{} \oplus \cdots \oplus (Re_n)\hat{}$$

remains finite-dimensional over k. Also, if we consider the k-dual

$$R\hat{} = \mathrm{Hom}_k({}_R R, k)$$

(viewed as a right R-module), then by (3.41), $R\hat{} \cong E((R/\mathrm{rad}\ R)_R)$. Since $(R/\mathrm{rad})_R$ contains a copy of every simple right R-module, $R\hat{}$ *is an injective cogenerator of* $\mathfrak{M}_R$. (Alternatively, we can also deduce this from the fact that ${}_R R$ is a projective generator in ${}_R\mathfrak{M}$!)

Let us now come back to the general set-up and notations in the proof of (19.10). Over a general ring R, U_0 need not be an injective module, since a direct sum of injectives may fail to be injective. However, this can be easily remedied by considering instead $U^0 := E(\bigoplus_i V_i)$. Clearly U^0 contains a copy of $E(V_i)$ for every i. As before, we can show that $\sum_i E(V_i) \subseteq U^0$ is direct, so we have $U_0 \subseteq U^0$. We conclude easily that:

(19.13) Proposition. *$U^0 = E(U_0)$, and it is an injective cogenerator of $\mathfrak{M}_R$. An injective module U_R is a cogenerator iff it contains U^0 (necessarily as a direct summand).*

We shall call U^0 the *minimal injective cogenerator* of $\mathfrak{M}_R$. Note that if U is another injective cogenerator with the property that any injective cogenerator contains U, then U and U^0 can be embedded in each other, and therefore $U \cong U^0$ by Exercise (3.31). This is why we call U^0 "the" minimal injective cogenerator of $\mathfrak{M}_R$. Note that both U_0 and U^0 are solely determined by the module category $\mathfrak{M}_R$.

The "easiest" (but nevertheless quite important) case of a minimal injective cogenerator is that over a local ring $(R, \mathfrak{m})$. In this case, $U_0 = U^0 = E((R/\mathfrak{m})_R)$: we have dealt with this module before in §3I, where we have referred to it informally as the "standard module" over R. Being a cogenerator, this standard module is faithful: this fact was independently proved before in (3.76).

§19B. Cogenerator Rings

For any ring R, the regular modules R_R and ${}_RR$ are always generators. However, in general, they may not be cogenerators. For instance, according to (19.8), a *necessary* condition for R_R to be a cogenerator is that R be right Kasch; i.e., that every simple module V_R embeds in R_R. Thus, for instance, $\mathbb{Z}$ is not a cogenerator for $\mathfrak{M}_{\mathbb{Z}}$.

(19.14) Definition. A ring R is said to be a *cogenerator ring* if R_R and ${}_RR$ are both cogenerators. (In the literature, a cogenerator ring is also known occasionally as a *Morita ring.*)

Recall that a (say right) module N is *torsionless* if the natural map from N to its double dual is injective; i.e., if for every $x \in N$ there exists $g \in \mathrm{Hom}_R(N, R)$ with $g(x) \neq 0$. In view of (19.6), *R_R is a cogenerator iff every N_R is torsionless.* Therefore, to say that R is a cogenerator ring simply means that all (left and right) R-modules are torsionless. From (15.11)(1), it follows that *any* QF *ring is a cogenerator ring.* (Alternatively, see Exercise 12.)

Our first goal is to obtain several characterizations of cogenerator rings. We begin with the following observation, which is part of Exercise (15.6). Since we shall need this result, a proof of it is included here for the sake of completeness.

(19.15) Proposition. *For a right ideal A in any ring R, R/A is torsionless iff A is a right annihilator (i.e., iff $\text{ann}_r(\text{ann}_\ell A) = A$).*

Proof. First suppose R/A is torsionless. Let $x \notin A$. There exists $g : R_R \to R_R$ with $g(A) = 0$ and $g(x) \neq 0$. Let $g(1) = r$. Then $rA = 0$ but $rx \neq 0$, whence $x \notin \text{ann}_r(\text{ann}_\ell A)$. This proves the "only if" part, and the "if" part can be proved by reversing the argument. □

Let us assemble a few key properties of a cogenerator ring below.

(19.16) Proposition. *Let R be a cogenerator ring. Then:*

(1) *R is a Kasch ring.*
(2) *R satisfies both of the double-annihilator conditions* (15.1)(3a,b).
(3) *Both R_R and ${}_RR$ are f.cog.*

Proof. (1) has already been pointed out, and (2) follows from (19.15) (and its left analogue), since all modules are torsionless. For (3), suppose $\bigcap_i A_i = 0$, where $\{A_i : i \in I\}$ are right ideals. Let $B_i = \text{ann}_\ell A_i$. Then $A_i = \text{ann}_r B_i$ by (2), and

$$0 = \bigcap A_i = \bigcap \text{ann}_r B_i = \text{ann}_r \sum B_i .$$

By (2) again, $\sum B_i = R$, so $\sum_{i \in J} B_i = R$ for a finite subset $J \subseteq I$. Taking right annihilator now gives $\bigcap_{i \in J} A_i = 0$. This shows that R_R is f.cog., and so is ${}_RR$ by symmetry. □

Using (19.16) we can now clarify the exact relationship between cogenerator rings and QF rings.

(19.17) Corollary. *A ring R is* QF *iff it is a cogenerator ring satisfying right* ACC *(resp.* DCC*).*

Proof. We have already observed that a QF ring is a cogenerator ring. Conversely, if R is a right noetherian cogenerator ring, then, by (19.16)(2), R satisfies (15.1)(3). Therefore, R is QF. □

We come now to the following important characterization of a cogenerator ring.

(19.18) Theorem. *A ring R is a cogenerator ring iff it is a self-injective Kasch ring.*

Proof. First assume R is right self-injective right Kasch. The latter property means that any simple module V_R embeds into R_R. Hence R_R is a cogenerator by (19.9). Since the argument also applies to ${}_R\mathfrak{M}$, we have the "if" part of (19.18).

For the converse, assume now R is a cogenerator ring. Then R is a Kasch ring by (19.16)(1). Let $E = E(R_R)$. By (19.6), there exists an embedding

$$\pi = (\pi_i) : E \longrightarrow \prod_i R_R .$$

Let $r_i = \pi_i(1)$, and consider the left ideal $B = \sum_i Rr_i$. If $Br = 0$, then $r_i r = 0$ for all i and hence $\pi(r) = 0$, which implies that $r = 0$. Thus, $\text{ann}_r(B) = 0$ and so (by (19.16)(2)) $B = \text{ann}_\ell(\text{ann}_r B) = R$. This gives an equation $\sum_i a_i r_i = 1$ where the $a_i \in R$ are almost all zero. Now construct a homomorphism

$$g : \prod_i R_R \longrightarrow R_R \quad \text{by} \quad g((x_i)) = \sum_i a_i x_i .$$

This obviously splits the embedding $\pi | R : R \to \prod_i R_R$. In particular, R_R splits in E and hence $R_R = E(R_R)$. By symmetry, the same holds for ${}_R R$, so R is a self-injective ring. □

Remark. A somewhat stronger version of the "if" part of the theorem is true, namely: *if R is self-injective and 1-sided Kasch, then R is already a cogenerator ring.* We will not prove this sharper version here.

In general, "right Kasch" and "right self-injective" are independent properties. We can see this easily already for commutative rings. For instance, an infinite direct product $\mathbb{Q} \times \mathbb{Q} \times \cdots$ is self-injective but not Kasch (see Exercise 17). On the other hand, the commutative local ring $\mathbb{Q}[u, v]$ with relations $u^2 = v^2 = uv = 0$ is Kasch but not self-injective (see (3.69)). Neither ring is a cogenerator ring.

(19.19) Proposition. *Let R be a cogenerator ring. Then a right R-module M is faithful iff it is a generator, iff it is a cogenerator.*

Proof. Let M_R be faithful. We are done if we can show that M is both a generator and a cogenerator. The faithfulness of M can be expressed by the equation $\bigcap_{m \in M} \text{ann}(m) = 0$. Since R_R is f.cog. by (19.16)(3), $\bigcap_{i=1}^n \text{ann}(m_i) = 0$ for suitable $m_1, \ldots, m_n \in M$. We have, therefore, an embedding $f : R_R \to \bigoplus_{i=1}^n M$ given by $f(1) = (m_1, \ldots, m_n)$. Since R_R is a cogenerator, so is $\bigoplus_i M$. Therefore, by (19.7)(2), M is a cogenerator. Since R_R is injective, there also exists a surjection $g : \bigoplus_i M \to R_R$ splitting f. Therefore, M is also a generator. □

The Proposition yields immediately the following additional characterization of a cogenerator ring.

(19.20) Corollary. *A ring R is a cogenerator ring iff any (left and right) generator R-module is a cogenerator, iff any (left and right) faithful R-module is a cogenerator. In particular, being a cogenerator ring is a Morita invariant property.*

From (19.17), we see that the class of cogenerator rings may be thought of as a generalization of QF rings to the class of rings without artinian or noetherian conditions. To make this generalization convincing, *we need to produce some*

cogenerator rings that are not QF. To this end, we introduce first a useful procedure for constructing (right, left, or 2-sided) self-injective rings, due to C. Faith.

Recall that, if S is a ring and M is an (S, S)-bimodule, we can form the "trivial extension" $R := S \oplus M$, which is a ring with the multiplication

$$(19.21) \qquad (s+m)(s'+m') = ss' + (sm' + ms').$$

The ring R contains M as an ideal of square zero, and $R/M \cong S$ as rings: see (2.22)(A). We have used this construction earlier in (3.15C) and (8.30) in the case when S is commutative, and in (16.60) (and Exer. (16.22)) in the case when S need not be commutative. To exploit trivial extensions as a means of constructing self-injective rings, we start with a lemma in the possibly noncommutative setting, and then specialize in the Proposition to the commutative setting.

(19.22) Lemma. *Assume, in the above, that ${}_SM$ is faithful. Then R_R is injective iff M_S is injective and the natural map $S \to \operatorname{End}(M_S)$ (giving the left S-action on M) is onto (hence an isomorphism).*

Proof. For the "if" part, we apply Baer's Test. Let $A \subseteq R$ be a right ideal, and $f \in \operatorname{Hom}_R(A, R)$. Since

$$f(A \cap M) \cdot M = f((A \cap M) \cdot M) = 0,$$

the faithfulness of ${}_SM$ implies that $f(A \cap M) \subseteq M$. Since M_S is injective, $f : A \cap M \to M$ extends to some $g \in \operatorname{End}(M_S)$, which is, by assumption, left multiplication by some $s \in S$. After subtracting from f the left multiplication map by s, we may thus assume that $f(A \cap M) = 0$. Then

$$f(A)M = f(AM) \subseteq f(A \cap M) = 0$$

implies that $f(A) \subseteq M$ (as before), and we have an induced S-homomorphism

$$\overline{f} : A/(A \cap M) \longrightarrow M.$$

Since $A/(A \cap M)$ is isomorphic to a right ideal in $R/M \cong S$, and M_S is injective, $\overline{f}$ is given by left multiplication by some $m \in M$. This completes Baer's Test, and proves the injectivity of R_R. The "only if" part of the lemma (which is not needed below) is left as an exercise. □

(19.23) Proposition. *Let S be a commutative ring, and M_S be the minimal injective cogenerator, viewed as an (S, S)-bimodule in the obvious way. Assume that the map $S \to \operatorname{End}(M_S)$ is onto. Then the trivial extension $R = S \oplus M$ is a (commutative) cogenerator ring.*

Proof. By (19.7)(1), M_S is faithful, so R is a self-injective ring according to (19.22). As in (8.30), R is also a Kasch ring. Therefore, R is a cogenerator ring by (19.18). □

Using (19.23), it is now very easy to produce some non-noetherian (commutative) cogenerator rings.

(19.24) Example (Osofsky). Let S be a complete discrete valuation ring with quotient field K and maximal ideal πS ($\pi \neq 0$). Let $V = S/\pi S$. By (19.11)(1), the minimal injective cogenerator for S is

$$M := E(V_S) \cong K/S \cong \varinjlim \pi^{-n} S/S \cong \varinjlim S/\pi^n S.$$

Since $\mathrm{End}_S(S/\pi^n S) \cong S/\pi^n S$, it is easy to see (cf. (8.14)) that[100]

$$\mathrm{End}(M_S) = \varprojlim S/\pi^n S.$$

Since S is complete, we have $\mathrm{End}(M_S) \cong S$. By (19.23), the trivial extension $R = S \oplus M$ is a (commutative) cogenerator ring. But of course R is not noetherian. As an addendum, note that R is a local ring with maximal ideal $\pi S + M = \pi R$. Since R has no nontrivial idempotents, it follows further that $R = E(V_R)$. Finally by taking the matrix rings $\mathbb{M}_n(R)$ and applying the last part of (19.20), we arrive at examples of *noncommutative* nonnoetherian cogenerator rings.

In closing this subsection, we should point out that there are natural 1-sided generalizations of the notion of a cogenerator ring, due to Azumaya, Osofsky, and others. One defines a ring R to be *right* PF (*pseudo-Frobenius*) if R_R is an injective cogenerator in $\mathfrak{M}_R$. (As far as the right side goes, R_R being injective and R_R being a cogenerator are independent conditions: see Exercises 17 and 20.) Left PF rings are defined similarly. In view of (19.18), *cogenerator rings are simply the 2-sided PF rings.* Quite a few characterizations of right PF rings are found in the following result.

(19.25) Theorem. *For any ring R, the following are equivalent:*

(1) *R is right PF;*
(2) *R is right self-injective and right Kasch;*
(3) *R is right self-injective and* $\mathrm{ann}_r(\mathrm{ann}_\ell(A)) = A$ *for any right ideal $A \subseteq R$;*
(4) *R is right self-injective and R_R is f.cog.;*
(5) *R is right self-injective, semiperfect (or just semilocal), and* $\mathrm{soc}(R_R) \subseteq_e R_R$*;*
(6) *Any faithful right R-module is a generator;*
(7) *R_R is a cogenerator and R is left Kasch;*
(8) *R_R is a cogenerator and there are only finitely many isomorphism classes of simple right R-modules.*

In particular, a right PF ring is always a semiperfect Kasch ring.

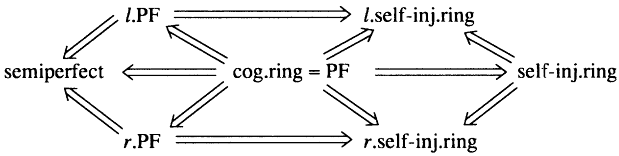
Since we will not be particularly concerned with right PF-rings in the following, we do not feel justified to present the full proof of the equivalences here. Some

[100] So far, we have not used the *completeness* of S. Thus, for instance, the endomorphism ring of the Prüfer p-group C_{p^∞} is the complete local ring of the p-adic integers. Of course, all of this is a special case of Matlis' Theorem (3.84).

of the equivalences are easy: for instance (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), which we offer as Exercise 16C below. The other equivalences require more work, for which we just refer the reader to two good sources: Osofsky [66] and Kasch [82: p. 322].

Azumaya, one of the originators of the notion of a 1-sided PF ring, posed the subtle question whether a left PF ring need to be right PF. This has been answered negatively: in Dischinger-Müller [86], where a *local* ring is constructed that is left PF but not right PF.

In summary, we have the following implication relations, where "ℓ" means left and "r" means right:



And, for rings with any one of: left/right ACC, left/right DCC, all of the above coincide with "QF ring".

§19C. Classical Examples of Dualities

The notion of duality between categories is essentially the contravariant version of the notion of equivalence.

(19.26) Definition. Let $\mathcal{A}$, $\mathcal{B}$ be categories. A *duality* between $\mathcal{A}$ and $\mathcal{B}$ means a pair of contravariant functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that $G \circ F \cong 1_{\mathcal{A}}$ and $F \circ G \cong 1_{\mathcal{B}}$, where $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ are the identity functors.

Under such a duality, the objects of $\mathcal{A}$ and $\mathcal{B}$ are "matched up", and the morphisms are "reversed". If F, G are given, we can use them to "dualize" properties of objects and morphisms from one category to the other.

Now every category $\mathcal{B}$ gives rise to an opposite category $\mathcal{B}^{\text{op}}$, with $\text{Obj}(\mathcal{B}^{\text{op}}) = \text{Obj}(\mathcal{B})$ and $\text{Mor}_{\mathcal{B}^{\text{op}}}(B, B') = \text{Mor}_{\mathcal{B}}(B', B)$. The obvious "do-nothing" functors from $\mathcal{B}$ to $\mathcal{B}^{\text{op}}$ and back clearly define a duality between $\mathcal{B}$ and $\mathcal{B}^{\text{op}}$. Taking this into account, a duality between $\mathcal{A}$ and $\mathcal{B}$ then amounts to an equivalence between $\mathcal{A}$ and $\mathcal{B}^{\text{op}}$. In this light, it may seem that the study of dualities has nothing to add to the study of equivalences. However, in practice, this is not so. When we work with the categories $\mathcal{A}$, $\mathcal{B}$ in a certain framework, the opposite categories $\mathcal{A}^{\text{op}}$, $\mathcal{B}^{\text{op}}$ usually have no concrete meanings, so it may not be desirable to work with them. Thus, we may still wish to study directly dualities between $\mathcal{A}$, $\mathcal{B}$ *without converting them into equivalences*.

The example of module categories is very much a case in point. For any ring R, the category $\mathfrak{M}_R$ is of great interest to module-theorists, but $\mathfrak{M}_R^{\text{op}}$ has little more than a formal meaning. Therefore, the study of dualities between categories of modules is by no means directly reducible to the study of equivalences between these categories. In fact, in the case of dualities, it will be seen that we can *never*

have a duality between a full category of modules, say $\mathfrak{M}_R$, with another one, say $\mathfrak{M}_S$, or ${}_S\mathfrak{M}$ (unless $R = 0$). Thus, it will only be fruitful to seek dualities between *subcategories* $\mathcal{A} \subseteq \mathfrak{M}_R$ and *subcategories* $\mathcal{B} \subseteq {}_S\mathfrak{M}$. This is a main difference between the study of dualities and the study of equivalences. (See Examples (19.28), (19.31), and (19.33) below.)

Before we begin our formal study of dualities, it behooves us to recall some familiar examples. The first example below is perhaps the champion of them all.

(19.27) Example (Galois Duality). Let K/k be a finite Galois field extension, with Galois group G. Let $\mathcal{A}$ be the category of fields L between k and K, with morphisms given by inclusion maps. Let $\mathcal{B}$ be the category of subgroups H of G, with morphisms also given by inclusion maps. Then the usual Galois correspondence $L \mapsto \text{Gal}(K/L)$ and $H \mapsto K^H$ leads to contravariant functors

$$F : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad F' : \mathcal{B} \to \mathcal{A}$$

which give a duality between $\mathcal{A}$ and $\mathcal{B}$. There is also a "profinite" version of this duality, where K/k is allowed to be an infinite Galois extension. Here, $\mathcal{B}$ is the category of *closed* subgroups of the profinite group G. For another variant of Galois duality, due essentially to A. Grothendieck, see Exercise 36 below.

(19.28) Example (Vector Space Duality). Let k be any division ring. For any right (left) k-vector space V, let $\hat{V}$ denote its first dual. This gives contravariant functors

$$F : \mathfrak{M}_k \to {}_k\mathfrak{M} \quad \text{and} \quad G : {}_k\mathfrak{M} \to \mathfrak{M}_k,$$

which give a duality between the subcategories $\mathcal{A}$, $\mathcal{B}$ of *finite-dimensional* (right, left) k-vector spaces. (The crucial fact here is that, for any such vector space V, we have a natural isomorphism $V \cong V^{\wedge\wedge}$.) We note gingerly that the duality *does not* work on the level of $\mathfrak{M}_k$ and ${}_k\mathfrak{M}$.

(19.29) Example (Pontryagin Duality). Let $\mathcal{A} = \mathcal{B}$ be the category of locally compact Hausdorff topological abelian groups. For $G \in \mathcal{A}$, let $\chi(G)$ be its character group $\text{Hom}_c(G, S^1)$ (consisting of *continuous* group homomorphisms from G to the circle group S^1). With the compact-open topology, $\chi(G)$ is again in $\mathcal{B}$. By the Pontryagin Duality Theorem, $\chi(\chi(G)) \cong G$ in $\mathcal{A}$, so (χ, χ) defines a duality between $\mathcal{A}$ and $\mathcal{B}$. If we let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the subcategory of discrete abelian groups, and $\mathcal{B}_0 \subseteq \mathcal{B}$ be the subcategory of compact abelian groups, it is further known that (χ, χ) defines a duality between $\mathcal{A}_0$ and $\mathcal{B}_0$. Finally, (χ, χ) gives a *self-duality* on the category of *finite* abelian groups G, for which the definition of χ simplifies to $\chi(G) = \text{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. (This last part admits a nice generalization to an arbitrary commutative ring: see Exercise 21.)

(19.30) Example (Gel'fand-Naimark Duality). Let $\mathcal{A}$ be the category of commutative C^*-algebras and $\mathcal{B}$ be the category of compact Hausdorff spaces. We

define contravariant functors

$$\Delta : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad C : \mathcal{B} \to \mathcal{A}$$

as follows. For $A \in \mathcal{A}$, $\Delta(A)$ is the maximal ideal space of A (identified with the set of $\mathbb{C}$-homomorphisms from A to $\mathbb{C}$), given the Gel'fand topology, and for $X \in \mathcal{B}$, $C(X)$ is the C^*-algebra of continuous $\mathbb{C}$-valued functions, with the sup-norm, and with involution given by complex conjugation. (Δ and C are defined on morphisms in the obvious way.) An elementary exercise in analysis shows that $\Delta(C(X))$ is homomorphic to X, and the Gel'fand-Naimark Theorem shows that there is a $*$-isomorphism $A \cong C(\Delta(A))$. (The natural map $A \to C(\Delta(A))$ is the "Gel'fand transform" which takes $a \in A$ to $\hat{a} \in C(\Delta(A))$, where $\hat{a}(\lambda) = \lambda(a)$ for any complex homomorphism $\lambda \in \Delta(A)$.) Thus, *the pair* (Δ, C) *gives a duality between* $\mathcal{A}$ *and* $\mathcal{B}$.

(19.31) Example (k-Duality over k-Algebras). Consider $\mathfrak{M}_R$ and ${}_R\mathfrak{M}$, where R is a finite-dimensional algebra over a field k. We have functors

$$F : \mathfrak{M}_R \to {}_R\mathfrak{M} \quad \text{and} \quad G : {}_R\mathfrak{M} \to \mathfrak{M}_R$$

defined by taking k-duals: $V \mapsto \hat{V}$. *The pair* (F, G) *gives a duality between* $\mathcal{A} = \mathfrak{M}_R^{fg}$ and $\mathcal{B} = {}_R^{fg}\mathfrak{M}$ since, for any R-module M with $\dim_k M < \infty$, the natural map $M \to M^{\wedge\wedge}$ is an R-isomorphism. It is of interest to recall that *F and G are both representable functors.* In fact, consider the module $\hat{R} = \mathrm{Hom}_k({}_R R_R, k)$, which has a natural structure as an (R, R)-bimodule. For any $M \in \mathfrak{M}_R$, Brauer's Equivalence Theorem (16.70) gives a natural left R-module isomorphism

$$\hat{M} \cong \mathrm{Hom}_R(M, {}_R(\hat{R})_R), \tag{19.32}$$

and hence $F \cong \mathrm{Hom}_R(-, (\hat{R})_R)$. Similarly, we have $G \cong \mathrm{Hom}_R(-, {}_R(\hat{R}))$. In other words, both duality functors F and G are "represented by" the canonical (R, R)-bimodule $\hat{R}$. (Recall (from (19.11)(3)) that $(\hat{R})_R$ and ${}_R(\hat{R})$ are injective cogenerators, respectively, in $\mathfrak{M}_R$ and ${}_R\mathfrak{M}$.)

(19.33) Example (QF Ring Duality). Consider $\mathfrak{M}_R$ and ${}_R\mathfrak{M}$, where R is a QF ring. We have functors $F : \mathfrak{M}_R \to {}_R\mathfrak{M}$ and $G : {}_R\mathfrak{M} \to \mathfrak{M}_R$ defined by taking R-duals, as in (15.12). We note again that both functors arise from a single bimodule, in this case, ${}_R R_R$. If we take $\mathcal{A} = \mathfrak{M}_R^{fg}$ and $\mathcal{B} = {}_R^{fg}\mathfrak{M}$, then (F, G) *gives a duality between* $\mathcal{A}$ *and* $\mathcal{B}$, in view of the reflexivity of f.g. modules over a QF ring (see (15.11)(2)).

What happens when R is not a QF ring? We can certainly define the same functors F, G as above by taking R-duals. Here we need to introduce the subcategories $\mathcal{A}_0$, $\mathcal{B}_0$ of *reflexive* modules, and F, G will give a duality between these.[101] In general, there is no lack of reflexive modules: $\mathcal{A}_0$ and $\mathcal{B}_0$ always con-

[101]This statement actually requires a short proof. For a more general formulation, see (19.38) and (19.40) below.

tain the f.g. projective modules (see Exercise (2.7)), and these are matched up under the above duality. However, various things may "go wrong" with the formation of R-duals. For instance, with the commutative local algebra $R = k[x, y]/(x, y)^2$ over a field k (with maximal ideal $\mathfrak{m} = (x, y)/(x, y)^2$), the unique simple module $V = R/\mathfrak{m}$ has the following higher R-duals (see Exer. (16.13)):

$$V^* \cong 2 \cdot V, \quad V^{**} \cong 4 \cdot V, \quad V^{***} \cong 8 \cdot V, \quad \text{etc.},$$

none of which is reflexive. The only obvious examples of reflexive modules here are the free modules R^n. With the example $R = \mathbb{Z}$, bad things happen the other way round: here, all cyclic modules $\mathbb{Z}/n\mathbb{Z}$ $(n > 1)$ have zero duals! In either example, the categories $\mathcal{A}_0$, $\mathcal{B}_0$ of reflexive modules are *not* closed under the formation of quotient objects — a serious drawback.

§19D. Morita Dualities: Morita I

Motivated by the classical examples and the discussions in §19C, we study in this subsection a special kind of duality, called *Morita dualities*. These are the ones given by a pair of functors F, G, both defined via a fixed bimodule U. We shall define precisely what is meant by a Morita duality, and characterize the bimodules U which give rise to such dualities. This constitutes "Morita I", which is essentially the analogue of the theorem under the same name for category equivalences (Thm. (18.24)), now developed in the context of duality.

Let R, S be given rings, and ${}_SU_R$ be an (S, R)-bimodule. We define contravariant functors

(19.34) $$F : \mathfrak{M}_R \to {}_S\mathfrak{M}, \quad G : {}_S\mathfrak{M} \to \mathfrak{M}_R \text{ by}$$

(19.35) $$F(M_R) = \operatorname{Hom}_R(M_R, {}_SU_R), \quad G({}_SN) = \operatorname{Hom}_S({}_SN, {}_SU_R),$$

with obvious definitions on the morphisms. The modules in (19.34) are called the *U-duals* of M and N. We shall view U as fixed throughout this subsection, so we often drop the reference to U and simply speak of the modules in (19.35) as the duals of M and N. To further simplify the notation, we shall denote them by M^* and N^*, without reference to U. (Various motivating examples have been given in §19C.)

Just as in the classical theory of vector spaces over fields, we can define, for any $M \in \mathfrak{M}_R$, an R-homomorphism $\theta_M : M \to M^{**}$, namely, $\theta_M(m) = \hat{m}$ where $\hat{m}(f) = f(m)$ for any $f \in M^*$. After some checking, we see that $\{\theta_M\}$ gives a natural transformation from the identity functor on $\mathfrak{M}_R$ to the double-dual functor **. And, of course, we have the same thing for the category ${}_S\mathfrak{M}$. Generalizing earlier definitions, we introduce:

(19.36) Definition. (1) $M \in \mathfrak{M}_R$ is called *U-torsionless* (or simply torsionless[102]) if θ_M is an injection. (2) $M \in \mathfrak{M}_R$ is called *U-reflexive* (or simply reflexive) if θ_M is a bijection. (Same definitions for left S-modules.) In the following, we shall write $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$ for the subcategories of reflexive modules in $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$. (All subcategories are understood to be full subcategories in our discussions.)

(19.37) Remarks. It is routine to check that $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$ are closed under finite direct sums and taking direct summands. On the other hand, the subcategories of torsionless modules are closed under arbitrary direct products and taking arbitrary submodules. Also, just as in the case of $U = {}_RR_R$, a module M_R is torsionless iff M can be embedded into a direct product of copies of U_R. In particular, *all* $M \in \mathfrak{M}_R$ are torsionless iff U_R is a cogenerator for $\mathfrak{M}_R$. The latter is, generally speaking, a reasonable assumption. However, we should not have any expectation that all $M \in \mathfrak{M}_R$ be *reflexive.* Finally, the subcategories of torsionless and reflexive modules are usually not closed with respect to the formation of quotient modules.

Consider the R-homomorphism $\theta_M : M \to M^{**}$, where $M \in \mathfrak{M}_R$. Applying the functor F to this morphism, we get a morphism $F(\theta_M) : M^{***} \to M^*$ in ${}_S\mathfrak{M}$. The following result, valid for any choice of the bimodule U, is possibly a bit surprising.

(19.38) Third Dual Theorem. *The S-homomorphism $\theta_{M^*} : M^* \to M^{***}$ is a monomorphism split by $F(\theta_M)$. In particular, M^* is always torsionless (for any $M \in \mathfrak{M}_R$), and if M is reflexive, then so is M^*.*

Proof. Let $f \in M^*$, so $f : M \to U_R$ is an R-homomorphism. By definition, $\hat{f} := \theta_{M^*}(f)$ is an R-homomorphism $M^{**} \to U$. This induces an R-homomorphism $g : M \to U$, upon composition with $\theta_M : M \to M^{**}$. To see that $f = g$, consider any $m \in M$. Writing $\hat{m} = \theta_M(m)$, we perform what is literally a "hat trick":

$$g(m) = \hat{f}(\hat{m}) = \hat{m}(f) = f(m). \tag{19.39}$$

This little piece of magic proves the first conclusion in (19.38), from which we see that M^* is always torsionless. Now suppose $M \in \mathfrak{M}_R[U]$. Then θ_M is an isomorphism, and the functor F must transform it into an isomorphism $F(\theta_M)$. From the first conclusion of (19.38), it follows that θ_{M^*} is an isomorphism, so $M^* \in {}_S\mathfrak{M}[U]$. □

(19.40) Corollary. *In the above notations, the functors F, G give a duality between $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$.*

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[102]We adopt the convention that, whenever some U is present, "torsionless" always means "U-torsionless", just as "dual" always means "U-dual", etc. This convention will be used freely throughout §19.

The duality in this corollary will be more interesting, of course, if $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$ are nice subcategories of $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$. The nicest kind of subcategories in (say) $\mathfrak{M}_R$ are those $\mathfrak{M} \subseteq \mathfrak{M}_R$ with the property that, for any exact sequence $0 \to K \to L \to M \to 0$ in $\mathfrak{M}_R$, we have $L \in \mathfrak{M}$ iff $K, M \in \mathfrak{M}$. Such subcategories $\mathfrak{M}$ are known as *Serre subcategories*. This prompts the following

(19.41) Definition. We say that $U = {}_SU_R$ *defines a Morita duality* (from R to S if the rings need to be mentioned) if both $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$ are Serre subcategories containing R_R and ${}_SS$, respectively. Note that, although the duality applies only to $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$, the functors F and G are defined as before between $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$.

Let us rehash two quick examples. For $R = S$ as in (19.31), and $U = {}_R(\hat{R})_R$, we have $\mathfrak{M}_R[U] = \mathfrak{M}_R^{fg}$ and ${}_R\mathfrak{M}[U] = {}_R^{fg}\mathfrak{M}$. (We assume the fact that $V \to V^{\wedge\wedge}$ is an isomorphism iff $\dim_k V < \infty$.) These are Serre subcategories containing R_R and ${}_RR$, respectively. Therefore, U defines a Morita duality. On the other hand, if $R = S = \mathbb{Z}$ and $U = \mathbb{Q}/\mathbb{Z}$, the first U-dual of $\mathbb{Z}$ is $\mathbb{Q}/\mathbb{Z}$, and the second U-dual of $\mathbb{Z}$ is $\mathrm{End}(\mathbb{Q}/\mathbb{Z}) \not\cong \mathbb{Z}$. Therefore, $\mathbb{Z}$ is *not* U-reflexive, so U does not define a Morita duality (although U defines exact functors F, G as in (19.34)).

We shall now seek necessary and sufficient conditions for a bimodule $U = {}_SU_R$ to define a Morita duality. To begin this task, we first compute the first and second duals of the regular modules R_R, ${}_SS$. Since the set-up is symmetric, it is enough to work with R_R (after which we'll assume the same results for ${}_SS$).

(19.42) Lemma. (1) *$(R_R)^* \cong {}_SU$, and $(R_R)^{**} \cong \mathrm{Hom}_S({}_SU, {}_SU_R)$.*

(2a) *$R \in \mathfrak{M}_R$ is torsionless iff U_R is faithful.*

(2b) *R_R is reflexive iff the natural map $R \to \mathrm{End}({}_SU)$ is an isomorphism.*

(3) *If R_R is reflexive, then so is ${}_SU$.*

Proof. (1) For the first dual, we have the standard isomorphism

$$R^* = \mathrm{Hom}_R(R_R, {}_SU_R) \cong {}_SU.$$

Taking the dual again yields $R^{**} \cong \mathrm{Hom}_S({}_SU, {}_SU_R)$.

(2) It is routine to check that, under the above identifications, $\theta_R : R \to R^{**}$ corresponds to the natural map $R \to \mathrm{End}({}_SU)$. This immediately yields (2a) and (2b).

(3) This follows from the last part of (19.38), since ${}_SU \cong (R_R)^*$. □

With the above lemma, we can now characterize the bimodules U which define Morita dualities. The following important result is to be viewed as an analogue of (19.24) which described equivalences of $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$ by progenerators. Because of this analogy, we shall refer to it again as "Morita I".

(19.43) Theorem (Morita I). *For a bimodule $U = {}_SU_R$, the following are equivalent:*

(1) *U defines a Morita duality (between R and S);*
(2) *every quotient of R_R, ${}_SS$, U_R and ${}_SU$ is reflexive;*
(3) *U_R and ${}_SU$ are injective cogenerators, and ${}_SU_R$ is faithfully balanced.*

If these conditions hold, any f.g. or f.cog. module M_R is reflexive.

Proof. (1)$\Longrightarrow$(2) follows directly from the definitions (in view of (19.42)(1)).

(2)$\Longrightarrow$(3). Part (2b) of (19.42) implies that ${}_SU_R$ is faithfully balanced. To show the injectivity of U_R, we apply Baer's Test. For any right ideal $I \subseteq R$, we need to show that $g : R^* \to I^*$ is *surjective.* Let $P = \text{im}(g)$. The exact sequence $0 \to I \to R \to R/I \to 0$ induces $0 \to (R/I)^* \to R^* \xrightarrow{g} P \to 0$, which in turn induces the bottom row of the following exact diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \theta_R} & & \downarrow{\scriptstyle \theta_{R/I}} \\ 0 & \longrightarrow & P^* & \longrightarrow & R^{**} & \longrightarrow & (R/I)^{**} \end{array}$$

Here, h is the composite of $I \to I^{**} \to P^*$. Since θ_R and $\theta_{R/I}$ are isomorphisms (by hypothesis), so is h. The hypothesis also gives $R^* \cong {}_SU \in {}_S\mathfrak{M}[U]$ and $P \in {}_S\mathfrak{M}[U]$. Therefore, $I \cong P^* \in \mathfrak{M}_R[U]$ by (19.38). Recalling the definition of h, we see that $I^{**} \to P^*$ is an isomorphism, and hence $P^{**} \to I^{***}$ is an isomorphism. Since $P \cong P^{**}$ and $I^* \cong I^{***}$, we conclude that $P \to I^*$ is an isomorphism. Thus, U_R is injective. For any simple module V_R, (2) implies that $V_R \in \mathfrak{M}_R[U]$. In particular, $V^* \neq 0$, so we have an embedding of V_R in U_R. By (19.9), we see that U_R is an injective cogenerator, and by symmetry, so is ${}_SU$.

(3)$\Longrightarrow$(1). From (19.42), we see that R_R, ${}_SS$ are reflexive. To check that $\mathfrak{M}_R[U]$ is a Serre subcategory, consider any exact sequence $0 \to K \to L \to M \to 0$ in $\mathfrak{M}_R$. Since U_R and ${}_SU$ are injective, we have an exact diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_K} & & \downarrow{\scriptstyle \theta_L} & & \downarrow{\scriptstyle \theta_M} & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & L^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 \end{array}$$

If $K, M \in \mathfrak{M}_R[U]$, θ_K, θ_M are isomorphisms. By the 5-Lemma, θ_L is also an isomorphism, so $L \in \mathfrak{M}_R[U]$. Conversely, assume θ_L is an isomorphism. Since U_R is a cogenerator, K and M are torsionless by (19.6); that is, θ_K, θ_M are *injective.* By a simple diagram chase, we see that θ_K, θ_M are also *surjective.* Therefore, $K, M \in \mathfrak{M}_R[U]$. By symmetry, it follows that ${}_S\mathfrak{M}[U]$ is also a Serre subcategory.

Finally, assume the above conditions hold, and let M_R be f.g. We have an epimorphism $R^n \to M$ for a suitable $n \geq 1$. Since R_R^n is reflexive, so is M. If M is f.cog. instead, fix an embedding $M \to \prod_i U$. Using the finite cogeneration of M, we can choose an embedding $M \to \prod_{i=1}^n U$. Since U_R is reflexive, so is M. ☐

The conditions listed in (3) of the theorem above are not all independent. First, of course, the fact that U_R and ${}_SU$ are cogenerators already implies that they are faithful. More importantly, Colby and Fuller have shown that, *if* ${}_SU_R$ *is faithfully balanced and* ${}_SU_R$ *are cogenerators, then they are necessarily injective cogenerators.* We shall not present this finer result, but point out that we have proved this in the special case $R = S$, $U = {}_RR_R$: see (19.18). Since, in this special case, U is automatically faithfully balanced, Morita I simplifies to the following.

(19.44) Corollary. *For any ring R, the following are equivalent:*

(1) *the bimodule $U = {}_RR_R$ defines a Morita duality (from R to R);*
(2) *every cyclic (right and left) R-module is reflexive;*
(2′) *every f.g. (right and left) R-module is reflexive;*
(3) *R is a cogenerator ring.*

Thus, a cogenerator ring R has a "self-duality" defined by the canonical bimodule ${}_RR_R$. For this reason, a cogenerator ring is sometimes called a "ring with perfect duality" (see, e.g., Kasch [82]).

§19E. Consequences of Morita I

We shall now record some properties of a Morita duality in general. The first couple of results in this direction do not require the full assumptions of a Morita duality, so we state precisely the hypotheses needed in each case. The annihilator notations used below should be self-explanatory (cf. (15.14), of which (1) of the following result is a generalization).

(19.45) Lemma. *Let $U = {}_SU_R$ and $M \in \mathfrak{M}_R$. Let N be an R-submodule of M and P be an S-submodule of M^*.*

(1) $(M/N)^* \cong \mathrm{ann}_{M^*}N$, *and, if U_R is injective,* $N^* \cong M^*/\mathrm{ann}_{M^*}N$.
(2) *Assume U_R is a cogenerator. Then* $\mathrm{ann}_M(\mathrm{ann}_{M^*}N) = N$.
(3) *Assume ${}_SU$ is a cogenerator and $M \in \mathfrak{M}_R[U]$. Then* $\mathrm{ann}_{M^*}(\mathrm{ann}_M P) = P$.

Proof. (1) is easy. For (2), we need only prove the inclusion "$\subseteq$". If $m \in M\backslash N$, there exists $f \in M^*$ with $f(N) = 0$ but $f(m) \neq 0$ (since U_R is a cogenerator). Hence $f \in \mathrm{ann}_{M^*}N$, and $m \notin \mathrm{ann}_M(\mathrm{ann}_{M^*}N)$. For (3), just identify M^{**} with M and apply (2) to M^*. □

(19.46) Proposition. *Assume that U_R and ${}_SU$ are cogenerators, and $M \in \mathfrak{M}_R[U]$. Then*

(1) $N \mapsto \mathrm{ann}_{M^*}N$ *gives an anti-isomorphism from the lattice of submodules of M to that of M^*.*

(2) *M is simple iff M^* is.*
(3) *M is noetherian (resp. artinian) iff M^* is artinian (resp. noetherian).*
(4) *M is f.g. (resp. f.cog.) iff M^* is f.cog. (resp. f.g.). (In particular, ${}_SU$ is f.cog.)*
(5) (Osofsky) *If $M = \bigoplus_{i\in I} M_i$, then $M_i = 0$ for almost all i.*

Proof. (1) follows from (2) and (3) of (19.45). The fact that we get a *lattice* anti-isomorphism follows from the obvious formula

$$\text{(19.47)} \qquad \operatorname{ann}_{M^*}\Big(\sum_j N_j\Big) = \bigcap_j \operatorname{ann}_{M^*}(N_j)$$

and the less obvious formula

$$\text{(19.48)} \qquad \operatorname{ann}_{M^*}\Big(\bigcap_j N_j\Big) = \sum_j \operatorname{ann}_{M^*}(N_j)$$

obtained easily by using (19.47) and (2), (3) of (19.45).

(2), (3), (4) These follow easily from (1) (together with (19.47) and (19.48) in the case of (4)).

(5) Suppose, on the contrary, that I is infinite and each $M_i \neq 0$. Since U_R is a cogenerator, there exists $f \in M^*$ with $f(M_i) \neq 0$ for each i. Let $N_j = \bigoplus_{i\neq j} M_i$. Then $\bigcap_j N_j = 0$ implies that

$$\text{(19.49)} \qquad M^* = \operatorname{ann}_{M^*}\Big(\bigcap_j N_j\Big) = \sum_j \operatorname{ann}_{M^*}(N_j) \quad \text{(by (19.48))}.$$

In particular, $f = f_{j_1} + \cdots + f_{j_n}$ where $f_{j_k}(N_{j_k}) = 0$. But then, for any $i \in I\backslash\{j_1, \ldots, j_n\}$, we have $M_i \subseteq N_{j_k}$ for all k, so $f(M_i) \subseteq \sum_k f_{j_k}(M_i) = 0$, a contradiction. □

The last conclusion (5) above can be proved under considerably weaker hypotheses. For the details, see Exercise 22.

(19.50) Corollary (Osofsky). *Suppose ${}_SU_R$ defines a Morita duality from R to S, and let $J = \operatorname{rad} R$. Then:*

(1) *for any $M \in \mathfrak{M}_R[U]$, $\operatorname{u.dim} M < \infty$ and $N = MJ^n/MJ^{n+1}$ is a f.g. semisimple R-module (for any $n \geq 1$).*
(2) *R is a semiperfect ring (i.e., R/J is semisimple, and idempotents in R/J can be lifted to R). Same for S.*

Proof. That u.dim $M < \infty$ follows from (6.4) and (19.46)(5) since submodules of M also belong to $\mathfrak{M}_R[U]$. Applying this to $M = U_R$, we see from (13.3) that $S \cong \operatorname{End}(U_R)$ is semiperfect, so by symmetry R is also semiperfect.[103] Now

[103]For another view of the semiperfectness of R and S, see Exercise 24.

consider N, which is a right module over R/J. Since R/J is semisimple, N is a semisimple module. It follows that N_R is f.g. since

$$M \in \mathfrak{M}_R[U] \Longrightarrow N \in \mathfrak{M}_R[U] \Longrightarrow \text{u.dim } N < \infty.$$

$\square$

Some special cases of (19.50) are noteworthy. For instance, in the case when R is a division ring and $U = {}_RR_R$, (1) above implies the known fact that *only finite-dimensional R-vector spaces can be reflexive.* The second conclusion (2) above implies, for instance, the following.

(19.51) Corollary. *Any cogenerator ring is a semiperfect ring.*

We stated this Corollary only because it follows for free from (19.50). In general, a stronger result is available: according to (19.25) (which we did not prove), *even a 1-sided PF ring is already a semiperfect ring.*

Our next result focuses on the relations between a pair of rings R, S which are linked by a Morita duality.

(19.52) Theorem. *Suppose ${}_SU_R$ defines a Morita duality from R to S. Then:*

(1) *R and S have isomorphic ideal lattices, both being anti-isomorphic to the lattice of (S, R)-submodules of U. The isomorphism between the ideal lattices of R and S preserves finite products (and therefore preserves prime, semiprime, and nilpotent ideals).*
(2) *Let $I \subseteq R$, $J \subseteq S$, and $V \subseteq U$ correspond under (1), and let $\overline{R} = R/I$, $\overline{S} = S/J$. Then ${}_{\overline{S}}V_{\overline{R}}$ defines a Morita duality from $\overline{R}$ to $\overline{S}$.*
(3) *$Z(R) \cong Z(S)$ as rings. (In particular, if there is a Morita duality from one commutative ring to another, then these rings are isomorphic.)*

Proof. (1) We have $R^* \cong U$ (as left S-modules), and a right ideal $I \subseteq R$ corresponds to an S-submodule $\text{ann}_U(I) \subseteq U$. It is easy to see that $I \subseteq R$ is an ideal iff $\text{ann}_U(I) \subseteq U$ is an (S, R)-submodule. Repeating this argument for ideals of S, we deduce the first part of (1). Now suppose I_1, I_2 are ideals of R corresponding to ideals J_1, J_2 of S. We have then $\text{ann}_U(I_i) = \text{ann}_U(J_i)$ $(i = 1, 2)$. For $u \in U$, we have

$$uI_1I_2 = 0 \iff J_2uI_1 = 0 \iff J_1J_2u = 0.$$

Hence $\text{ann}_U(I_1I_2) = \text{ann}_U(J_1J_2)$, and I_1I_2 corresponds to J_1J_2.

(2) Since $V = \text{ann}_U(I) = \text{ann}_U(J)$, we can view V as an $(\overline{S}, \overline{R})$-bimodule. Now view $\mathfrak{M}_{\overline{R}}$ and ${}_{\overline{S}}\mathfrak{M}$ as subcategories of $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$, respectively. It is easy to check that $M \in \mathfrak{M}_{\overline{R}}$ is V-reflexive iff it is U-reflexive as an R-module. (Note that $\text{Hom}_{\overline{R}}(M, V) \cong \text{Hom}_R(M, U)$.) From this, we deduce that $\mathfrak{M}_{\overline{R}}[V]$ and ${}_{\overline{S}}\mathfrak{M}[V]$ are Serre subcategories of $\mathfrak{M}_{\overline{R}}$ and ${}_{\overline{S}}\mathfrak{M}$, containing $\overline{R}$ and $\overline{S}$, respectively.

(3) follows from (18.41), since ${}_SU_R$ is faithfully balanced. □

(19.53) Remark. An alternative proof to (2) can also be given by checking directly that ${}_{\overline{S}}V$, $V_{\overline{R}}$ are injective cogenerators, and that ${}_{\overline{S}}V_{\overline{R}}$ is faithfully balanced. (For this check, Exercise (3.28) is relevant.)

For convenience of language, we shall say that a ring R *admits a Morita duality* if there exists a Morita duality from R to another ring S, defined by some bimodule ${}_SU_R$. (Of course, S is necessarily isomorphic to $\text{End}(U_R)$.) According to (19.50), the fact that R admits a Morita duality places rather severe restrictions on R and its right modules. In particular, by (19.50)(2), R must be semiperfect. Therefore, rings such as $\mathbb{Z}$, $k[x_1, \ldots, x_n]$ (k a field, $n \geq 1$) cannot possibly have a Morita duality. This is in stark contrast to the fact that there always exist Morita equivalences from a ring R to other rings, namely, the matrix rings $\mathbb{M}_n(R)$.

The problem of characterizing rings R admitting Morita dualities is largely solved. In the case of right artinian rings, this goes back to the work of Morita and Azumaya in the late 1950's. For general rings, this problem was successfully tackled by B. J. Müller in 1970. Some of Müller's (and Onodera's) main results in this direction will be presented in §19F. To conclude the present subsection, we would like to compile a good list of examples of rings admitting Morita dualities.

To begin with, any finite-dimensional algebra R over a field k admits a Morita self-duality (from R to R). The details were in (19.31). Secondly, any cogenerator ring R also admits a Morita self-duality (defined by ${}_RR_R$). This includes all Frobenius and quasi-Frobenius rings (in particular all semisimple rings and group algebras of finite groups over fields). To give an example of a Morita duality that is *not* a self-duality, we proceed as follows.

(19.54) Example. Suppose $R' \approx R$; that is, R, R' are Morita equivalent rings. If R admits a Morita duality, then so does R'. (In the terminology introduced in §18, *"admitting a Morita duality" is a Morita invariant property.*) Indeed, if ${}_SU_R$ defines a Morita duality from R to S, then, for any category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_{R'}$, we can define an (S, R')-bimodule structure on $U' = F(U)$ and check that ${}_SU'_{R'}$ defines a Morita duality from R' to S. It follows, for instance, that for any finite-dimensional k-algebras R, R' over a field k, *if $R' \approx R$, then there exists a Morita duality from R' to R.* More desirably, if there is a k-equivalence from R' to R, then there is a Morita k-duality from R' to R. (By "k-equivalence" and "k-duality", we simply mean that the defining functors are "k-functors" making everything in sight k-linear.)

For our next example, we consider the case of a commutative noetherian local ring $(R, \mathfrak{m})$. Here, the minimal injective cogenerator is just $U = E(R/\mathfrak{m})$; we have studied the structure of this module in §3I, where we called it informally the "standard module" of the local ring. For this module U, we have the important result of Matlis (3.84) which states that $\text{End}_R(U)$ is isomorphic to the $\mathfrak{m}$-adic

completion $\tilde{R} = \varprojlim R/\mathfrak{m}^i$ of R. Let us now explore the remarkable consequences of this theorem in the context of duality theory. It is of historical interest to observe that Matlis' paper on injective indecomposables and Morita's paper on equivalences and dualities appeared in the same year, 1958. Although Matlis did refer to a "forthcoming" work of Azumaya on duality for injective modules, it seems all but certain that neither Matlis nor Morita had seen the other's work before his own paper appeared.

(19.55) Example (Matlis). Let $(R, \mathfrak{m})$ be a commutative *complete* noetherian local ring (so that $R = \tilde{R}$) with $U = E(R/\mathfrak{m})$. Then, by Matlis' Theorem recalled above, ${}_RU_R$ is faithfully balanced. Since U_R and ${}_RU$ are injective cogenerators, Morita I implies that U *defines a Morita duality from* R *to* R. Let ${}^a_R\mathfrak{M} \subseteq {}_R\mathfrak{M}$ be the subcategory of artinian left R-modules. Since these modules are f.cog. and therefore U-reflexive, their U-duals are noetherian, or equivalently f.g. It follows that, under the Morita self-duality, $\mathfrak{M}_R^{fg}$ *corresponds exactly to* ${}^a_R\mathfrak{M}$. The duality functors can be described as follows. For $A \in \mathfrak{M}_R^{fg}$, let $A = R^n/N$. Identifying $(R^n)^*$ with U^n, we have from (19.45)(1):

$$A^* \cong \{(u_1, \ldots, u_n) \in U^n : \sum u_i r_i = 0 \ \forall (r_1, \ldots, r_n) \in N\}.$$

For $B \in {}^a_R\mathfrak{M}$, we may assume that $B \subseteq U^n$ (for some n). Identifying $(U^n)^*$ with R^n, we have, again from (19.45)(1):

$$B^* \cong R^n/\{(r_1, \ldots, r_n) \in R^n : \sum u_i r_i = 0 \ \forall (u_1, \ldots, u_n) \in B\}.$$

In view of (19.46), we have the following duality facts for our ring R.

(19.56) Corollary. *Let* $(R, \mathfrak{m})$ *be a commutative complete noetherian local ring, with* $U = E(R/\mathfrak{m})$. *Then* U *is an artinian* R*-module, and it defines a Morita duality from* R *to itself. Thus:*

(1) *The* U*-dual of a noetherian* R*-module is an artinian* R*-module.*
(2) *The* U*-dual of an artinian* R*-module is a noetherian* R*-module.*
(3) *If an* R*-module* M *is either noetherian or artinian, then it is* U*-reflexive; in other words, we have a natural isomorphism* $M^{**} \cong M$.
(4) *If an* R*-module* M *is both noetherian and artinian (i.e.,* $\text{length}(M) < \infty$*), then* $\text{length}_R(M) = \text{length}_R(M^*)$.

Note that this Corollary applies, in particular, to any commutative artinian local ring $(R, \mathfrak{m})$. (Such R is necessarily noetherian, and $\mathfrak{m}$ must be nilpotent, which implies easily that R is $\mathfrak{m}$-adically complete.) The duality facts in this special case are already quite important in commutative algebra and in algebraic geometry.

More examples of rings admitting Morita dualities will be given at the end of §19F.

§19F. Linear Compactness and Reflexivity

In this subsection, we shall explore a somewhat surprising connection between Morita duality theory and a topological/module-theoretic notion, that of linear compactness. The study of linearly compact modules goes as far back as to the work of Lefshetz in the 1940's, and was continued in papers of Dieudonné, Zelinsky, and others in the 1950's. An account of the basic properties of linearly compact modules and rings may be found in the exercises of Bourbaki's *Algèbre Commutative*, Ch. 3, §2.

While Bourbaki's inclusion of the material on linearly compact modules helped popularize this notion, the connection between linearly compact modules and Morita duality theory was apparently not noticed until the publication of B. J. Müller's paper [70]. In this paper, Müller not only proved the equivalence of linear compactness and reflexivity for modules under a Morita duality, but also found the criterion for any ring R to admit a Morita duality (into some other ring S) in terms of linear compactness. This subsection is intended as a more or less self-contained introduction to some of Müller's main results. Besides the original papers Müller [70] and Onodera [72], I have also consulted the very well-written recent Springer Lecture Notes volume of Xue [92]. In this volume, the reader will find many more current results on Morita duality theory which we will not have space to cover in our general introduction here.

For the most general study of linear compactness, we should work in the framework of *topological* modules. However, as far as the connection with reflexive modules is concerned, it turns out to be largely sufficient to use the discrete topology. In the interest of keeping the prerequisites minimal for our exposition, we shall therefore avoid altogether any use of topological modules (and their completions with respect to uniformities). In particular, this makes possible a purely algebraic treatment of the subject at hand, yet without any sacrifice of the central results. We shall now introduce the notion of linearly compact modules below, specifically for the case of modules with the discrete topology.

Consider any system of submodules $\{M_i\}_{i\in I}$ in a module M_R, and elements $m_i \in M$ indexed by the same set I. We say that the system $\{m_i,\ M_i\}_{i\in I}$ is *finitely solvable* if, for any finite subset $J \subseteq I$, there exists an element $m_J \in M$ such that $m_J \equiv m_j \pmod{M_j}$ for all $j \in J$. Similarly, we say that $\{m_i,\ M_i\}_{i\in I}$ is *solvable* if there exists $m \in M$ such that $m \equiv m_i \pmod{M_i}$ for *all* $i \in I$.

(19.57) Definition. A module M_R is said to be *linearly compact* (or l.c. for short) if every finitely solvable system $\{m_i, M_i\}_{i\in I}$ in M is solvable. (Stated in coset notation, M is l.c. if, for any system as above, $\bigcap_{j\in J}(m_j + M_j) \neq \emptyset$ for any finite $J \subseteq I$ implies that $\bigcap_{i\in I}(m_i + M_i) \neq \emptyset$.) For instance, any module of finite cardinality is l.c. We shall denote the subcategory of all linearly compact modules in $\mathfrak{M}_R$ by $\mathfrak{M}_R^{lc}$.

(19.58) Lemma. (Zelinsky) *$\mathfrak{M}_R^{lc}$ is a Serre subcategory of $\mathfrak{M}_R$.*

Proof. Let $0 \to K \to L \to M \to 0$ be exact in $\mathfrak{M}_R$. If L is l.c., a routine check shows that K and M are also l.c. If, conversely, K, M are l.c., it requires a little work to show that L is l.c. We shall not need this result below, so it will be relegated to Exercise 35. □

Let us say that a ring R is *right linearly compact* (right l.c.) if R_R is l.c. As usual, we also have the notion of R being left l.c. and (2-sided) l.c. Some examples follow.

(19.59) Example. The ring $\mathbb{Z}$ is not l.c. In fact, let $M_i = p_i\mathbb{Z}$, where $2 = p_1 < p_2 < \dots$ are the primes. Let $m_1 = 1$ and $m_2 = m_3 = \cdots = 0$. Clearly, the system $\{m_i, M_i\}_{i \geq 1}$ is finitely solvable, but not solvable. (The first n congruences $m \equiv m_i \pmod{M_i}$ can be solved by $m = p_2 p_3 \dots p_n$, since this product is odd. But $m \equiv m_i \pmod{M_i}$ for *all* i would imply $m = 0$ and m is odd!)

(19.60) Example. For any ring $k \neq 0$, the polynomial ring $R = k[x]$ is not (right) l.c. Here, let $M_i = x^i R$ $(i \geq 1)$ which are ideals, and let $m_i = 1 + x + \cdots + x^{i-1}$. The first n congruences $m \equiv m_i \pmod{M_i}$ can be solved by choosing $m = m_n$. But if there exists $f(x) = \sum a_i x^i \in R$ solving *all* congruences $f(x) \equiv m_i$ (mod M_i), then all $a_i = 1$ and we have a contradiction.

(19.61) Example. This example shows that there is a close connection between completeness and linear compactness. To be precise, let $(R, \mathfrak{m})$ be a commutative noetherian local ring. We claim that *R is l.c. iff $(R, \mathfrak{m})$ is complete.* To see this, first assume R is not complete. Then the natural map $\varepsilon : R \to \varprojlim R/\mathfrak{m}^i$ is not onto. This means there exists a compatible sequence $(\overline{m}_i)_{i \geq 1} \in \varprojlim R/\mathfrak{m}^i$ that is not in $\operatorname{im}(R)$. This amounts to the fact that $\{m_i, \mathfrak{m}^i\}_{i \geq 1}$ is not solvable. However, this system is *finitely* solvable, since $m \equiv m_i \pmod{\mathfrak{m}^i}$ for $1 \leq i \leq n$ can be solved by taking $m = m_n$. Thus, R is not l.c. By using a similar (but somewhat more elaborate) argument, one can show that $(R, \mathfrak{m})$ being complete also implies R is l.c. From this, we deduce, for instance, that the power series ring $R = k[[x_1, \dots, x_n]]$ is l.c. for any field k.

Next, let us work a little with modules.

(19.62) Proposition. *If M_R is l.c., then* $\operatorname{u.dim} M < \infty$.

Proof. Assume $\operatorname{u.dim} M = \infty$. Then there exists a submodule $N = \bigoplus_{i \in I} M_i$ where I is infinite and each $M_i \neq 0$. Since N is also l.c. (by (19.58)), we may as well assume $N = M = \bigoplus_{i \in I} M_i$. Let $0 \neq m_i \in M_i$ and let $N_j = \bigoplus_{i \neq j} M_i$. We see easily that $\{m_i, N_i\}$ is finitely solvable, but not solvable, a contradiction. (**Note.** $m \equiv m_i \pmod{N_i}$ simply means that the "i^{th} coordinate" of m is m_i with respect to the direct sum decomposition $M = \bigoplus_i M_i$.) □

(19.63) Corollary. *If* u. dim $R_R = \infty$, *then R is not right l.c. (This applies, for instance, to $R = k\langle x, y\rangle$, and to $R = k \times k \times \ldots$, for any nonzero ring k.)*

(19.64) Proposition. *Any artinian module M_R is l.c.*

Proof. Let $\{m_i,\ M_i\}_{i\in I}$ be a finitely solvable system in M. With a little abuse of notation, say $M_1 \cap \cdots \cap M_n$ is minimal among all finite intersections of the M_i 's. Fix $m \in M$ such that $m \equiv m_i \pmod{M_i}$ for $i = 1, \ldots, n$. We claim that m solves the above congruences *for all* $i \in I$. It suffices to check that $m \equiv m_{n+1}$ (mod M_{n+1}) (with another abuse of notation). First find $m' \in M$ such that $m' \equiv m_i$ (mod M_i) for $i = 1, 2, \ldots, n+1$. Then

$$m - m' \in M_1 \cap \cdots \cap M_n = M_1 \cap \cdots \cap M_n \cap M_{n+1}.$$

But then $m \equiv m' \equiv m_{n+1} \pmod{M_{n+1}}$, as claimed. □

(19.65) Corollary. *For any module M_R over a right artinian ring R, the following are equivalent:*

(1) *M is f.g.*
(2) *M is artinian.*
(3) *M is l.c.*

Proof. (1) $\Rightarrow$ (2) is well known over a right artinian ring R, and (2) $\Rightarrow$ (3) is true over any ring by (19.64). We finish by proving (3) $\Rightarrow$ (1). For this implication, we can get by with the weaker hypothesis that R is a *semiprimary* ring, that is, a ring R with $J :=$ rad R nilpotent and R/J semisimple. Say $J^n = 0$ and M_R is l.c. Since R/J is semisimple, $M/MJ \cong \bigoplus_{i\in I} V_i$ where the V_i 's are simple R-modules. But M/MJ is l.c. by (19.58), so I must be finite. Therefore, $M = N + MJ$ for a suitable f.g. submodule $N \subseteq M$. But then

$$M = N + (N + MJ)J = N + MJ^2 = \cdots = N + MJ^n = N,$$

so M_R is f.g., as desired. □

For further information relating l.c. modules to artinian modules, see Exercises 27-29. Since our main objective is not so much to study l.c. modules for their own sake, we shall not dwell on giving more examples, but go directly to the task of analyzing the relationship between *linear compactness* and *reflexivity.* The first important theorem in this direction is the following poetic rendition of Müller's result in Onodera [72].

(19.66) Theorem. *Let U_R be a cogenerator, $S =$ End(U_R), and $M \in \mathfrak{M}_R$. Then $M \in \mathfrak{M}_R^{lc}$ iff, for any S-submodule $N \subseteq M^*$ and any $g \in$ Hom$_S(N, {}_SU)$, there exists $m \in M$ such that $g = \hat{m}|N$. (Recall that $\hat{m} = \theta_M(m)$ where $\theta_M : M \to M^{**}$.)*

Proof. *First assume* $M \in \mathfrak{M}^{lc}_R$, and consider g as above. The idea is to "approximate" g on N by $\hat{m}_{f_1,\dots,f_n}$ for finitely many $f_1, \dots, f_n \in N$ and then use l.c. to express g as $\hat{m}|N$ for a suitable $m \in M$. Given $f_1, \dots, f_n \in N$, consider $(f_1, \dots, f_n) : M_R \to U^n_R$, with image V, say. *We claim that* $(g(f_1), \dots, g(f_n)) \in V$. To see this, it suffices to show (since U_R is a cogenerator) that

$$\forall s \in (U^n_R)^* : \quad s(V) = 0 \Longrightarrow s(g(f_1), \dots, g(f_n)) = 0.$$

Express s in the form $(s_1, \dots, s_n)$, where $s_i \in (U_R)^* = \mathrm{Hom}_R(U_R, U_R) = S$. The fact that $s(V) = 0$ means that

$$\forall m \in M, \quad 0 = s(f_1(m), \dots, f_n(m)) = \sum_i s_i(f_i(m)),$$

so $\sum s_i f_i = 0 \in S$. But then

$$s(g(f_1), \dots, g(f_n)) = \sum s_i(g(f_i)) = \sum g(s_i f_i) = 0,$$

since $g : M^* \to U$ is an S-homomorphism. This proves our claim, showing that

$$(g(f_1), \dots, g(f_n)) = (f_1(m), \dots, f_n(m))$$

for some $m = m_{f_1,\dots,f_n} \in M$. In particular, for every single element $f \in N$, we have an associated element $m_f \in M$ such that $g(f) = f(m_f)$. Now consider the system $\{m_f, \ker(f)\}_{f \in N}$. This system is *finitely solvable*: given $f_1, \dots, f_n \in N$, the congruences $m \equiv m_{f_i} \pmod{\ker(f_i)}$ are solved by taking $m = m_{f_1,\dots,f_n}$, since

$$f_i(m_{f_1,\dots,f_n} - m_{f_i}) = g(f_i) - g(f_i) = 0.$$

Since M is l.c., there exists $m \in M$ such that, *for all* $f \in N$, $m \equiv m_f$ (mod $\ker(f)$); that is, $f(m) = f(m_f) = g(f)$. Therefore, $g = \hat{m}|N$.

For the converse, let $\{m_i, M_i\}_{i \in I}$ be a *finitely solvable* system in M. We construct a map

(19.67) $\quad g : N = \sum \mathrm{ann}_{M^*}(M_i) \longrightarrow {}_S U \quad$ by $\quad g(\sum f_i) = \sum f_i(m_i) \in U$,

where $f_i \in \mathrm{ann}_{M^*}(M_i)$. To see that g is well-defined, suppose $\sum_{j \in J} f_j = 0$, where $J \subseteq I$ is finite. Take $m \in \bigcap_{j \in J}(m_j + M_j)$. Then $f_j(m) = f_j(m_j)$ (since $f_j(M_j) = 0$), and so

$$\sum_{j \in J} f_j(m_j) = \sum_{j \in J} f_j(m) = 0.$$

Clearly, from (19.67), g is an S-homomorphism. By assumption, $g = \hat{m}|N$ for some $m \in M$. For any $f_i \in \mathrm{ann}_{M^*}(M_i)$, (19.67) shows $f_i(m_i) = g(f_i) = f_i(m)$. Therefore, by (19.45)(2):

$$m - m_i \in \mathrm{ann}_M(\mathrm{ann}_{M^*}(M_i)) = M_i$$

for every $i \in I$, as desired. □

For left S-modules U and X, let us say that U is X-*injective* if, for any S-module $N \subseteq X$, any $g \in \mathrm{Hom}_S(N, U)$ can be extended to X. Using this convenient terminology, we may rephrase (19.66) as follows.

(19.66)′ Theorem. *In the setting of* (19.66), $M \in \mathfrak{M}_R^{lc}$ *iff* $M \in \mathfrak{M}_R[U]$ *and* ${}_SU$ *is* M^**-injective.*

Note that, if ${}_SU$ is injective, then it is X-injective for any ${}_SX$ (and conversely). In this case, the last condition above can be dropped. In particular, we obtain:

(19.68) Müller's First Theorem. *Suppose a bimodule* ${}_SU_R$ *defines a Morita duality from* R *to* S. *Then* $\mathfrak{M}_R[U] = \mathfrak{M}_R^{lc}$ *(and similarly* ${}_S\mathfrak{M}[U] = {}_S^{lc}\mathfrak{M}$*). In particular, any f.g. or f.cog. right* R*-module is l.c.*

This is a nice theorem in that it gives a new characterization for the U-reflexive modules (in terms of the solution of congruences). It is rather surprising because the characterization given is completely independent of the bimodule U which defines the duality! The theorem tells us that, if a Morita duality exists at all (from R to some other ring), then the reflexive modules with respect to the duality are already "predetermined": they must coincide with the linearly compact modules, independently of U! As a special case, consider a right artinian ring R. If a Morita duality exists (from R to some S), then the reflexive right R-modules (with respect to this duality) are precisely the f.g. ones, according to (19.65). Thus, for instance, for a QF ring R (with the self-duality given by ${}_RR_R$), we know that the R-reflexive modules must be precisely the f.g. R-modules.

In his paper [72], Onodera has given various applications of his result (19.66). Particularly noteworthy among these is his 1-sided characterization of a cogenerator ring, which can be stated as follows.

(19.69) Theorem. (Onodera) *A ring* R *is a cogenerator ring iff* R_R *is a l.c. cogenerator.*

Since this deeper result will not be needed in the text, we shall not go into its proof here. Instead, we move on to give a new characterization for a bimodule ${}_SU_R$ to define a Morita duality, in terms of linear compactness. The advantage of this characterization is that it involves only properties of (right) R-modules, and not S-modules. It is Onodera's refinement of a result of Müller; Müller's result will be retrieved in (19.71) below.

(19.70) Theorem. *Let* $U_R \in \mathfrak{M}_R$ *and* $S = \text{End}(U_R)$. *Then* ${}_SU_R$ *defines a Morita duality from* R *to* S *iff* U_R *is a f.cog. injective cogenerator, and* $R_R, U_R \in \mathfrak{M}_R^{lc}$.

Proof. The "only if" part follows from Morita I, (19.46)(4), and (19.68). Now assume the conditions on R_R, U_R. To show that ${}_SU_R$ defines a Morita duality, it suffices (by Morita I) to check that:

(a) ${}_SU_R$ *is faithfully balanced, and*
(b) ${}_SU$ *is an injective cogenerator.*

Since U_R is a cogenerator, (19.66)′ gives $\mathfrak{M}_R^{lc} \subseteq \mathfrak{M}_R[U]$. Therefore, by (19.58):

(c) *Quotient modules of* R_R, U_R *are in* $\mathfrak{M}_R[U]$.

The fact that $R_R \in \mathfrak{M}_R[U]$ means that $R \cong \text{End}(_SU)$ (by (19.42)(2b)), so (a) follows. Since U_R is injective and f.cog., Exercise 6 yields $U_R = E(U_1) \oplus \cdots \oplus E(U_n)$ where the U_i 's are simple right R-modules. Let $e_i \in S$ be the projection of U onto $E(U_i)$. By (13.3), S is a semiperfect ring, and $1 = e_1 + \cdots + e_n$ is a decomposition of $1 \in S$ into orthogonal primitive idempotents. Let $J = \text{rad } S$. By *FC*–(25.3), any simple left S-module is isomorphic to some Se_i/Je_i. Since $J \cdot U_i = 0$ by (13.1) (and Exercise (6.12)(2)), we can define $\alpha_i : Se_i/Je_i \to {}_SU$ by writing $U_i = u_iR$ and taking

$$\alpha_i(se_i + Je_i) = se_iu_i \quad \text{for any } s \in S.$$

This is clearly a (nonzero) S-homomorphism, and therefore embeds the (typical) simple S-module Se_i/Je_i into $_SU$. It only remains for us to prove that $_SU$ is *injective*, for then (b) will follow, by (19.9). To this end, we apply (19.66)$'$ to $M = U_R \in \mathfrak{M}_R^{lc}$. Since $(U_R)^* \cong {}_SS$, (19.66)$'$ implies that $_SU$ is S-injective. By Baer's Criterion, it follows that $_SU$ is injective, as desired. □

Now let $\{V_i : i \in I\}$ be a complete set of simple right R-modules and let $U^0 = E(\oplus_i V_i)$ be the minimal injective cogenerator over R. By specializing (19.70) to U^0, we deduce the following.

(19.71) Müller's Second Theorem. *A ring* R *admits a Morita duality iff* R_R *and* U_R^0 *are l.c. In this case, for* $T = \text{End}(U_R^0)$, $_TU_R^0$ *defines a Morita duality from* R *to* T.

Proof. Suppose $_SU_R$ defines a Morita duality, from R to some ring S. Being U-reflexive, R_R, $U_R \in \mathfrak{M}_R^{lc}$ by (19.68). On the other hand, U_R is an injective cogenerator by Morita I, so U_R^0 embeds in U_R by (19.13). Therefore, $U_R^0 \in \mathfrak{M}_R^{lc}$. Conversely, assume that R_R, $U_R^0 \in \mathfrak{M}_R^{lc}$ and let $T = \text{End}(U_R^0)$. By (19.62), we have u.dim $U_R^0 < \infty$, so $|I| < \infty$. Then $U^0 \cong \bigoplus_i E(V_i)$, and U_R^0 is f.cog. by Exercise 7. From (19.70), it follows that $_TU_R^0$ defines a Morita duality from R to T. □

(19.72) Example. A simple illustration for (19.71) is given by the case when R is a commutative *complete* noetherian local ring, say with maximal ideal $\mathfrak{m}$. Suppose we have verified, one way or another, that the minimal injective cogenerator $U_R^0 = E(R/\mathfrak{m})$ is an artinian R-module; then by (19.64), it is l.c. By (19.61), R_R is also l.c., so by (19.71), U^0 defines a Morita duality from R to $\text{End}(U_R^0)$. Of course, we already know this from (19.55); in fact, according to Matlis' Theorem (3.84), $\text{End}(U_R^0)$ is canonically isomorphic to R. On the other hand, if R is *not* complete, then by (19.61) R_R is not l.c., and (19.71) implies that R admits no Morita duality into any ring whatsoever.

Next, we shall apply (19.70) and (19.71) to right artinian rings. For these rings, we recover some of the classical duality results of Morita and Azumaya. The first of these substantially simplifies Morita I.

(19.73) Corollary. *Let R be a right artinian ring, $U_R \in \mathfrak{M}_R$, and $S = \operatorname{End}(U_R)$. Then ${}_SU_R$ defines a Morita duality from R to S iff U_R is a f.g. injective cogenerator.*

Proof. This follows from (19.4), (19.65), and (19.70). □

(19.74) Corollary. *Let R be a right artinian ring, with simple right modules $V_1, \ldots, V_n$. Then the following are equivalent:*

(1) *R admits a Morita duality;*
(2) *each $E(V_i)$ is f.g.;*
(3) *each f.cog. right R-module is f.g.* (cf. (19.4)).

Under these conditions, $U^0 = E(V_1) \oplus \cdots \oplus E(V_n)$ defines a Morita duality from R to $T = \operatorname{End}(U_R^0)$.

Proof. Since R_R is artinian, it is l.c. By (19.65), U_R^0 is l.c. iff it is f.g. Therefore, the equivalence (1)$\Longleftrightarrow$(2) and the last statement of the Corollary both follow from (19.71).

(1)$\Longrightarrow$(3). Let M_R be f.cog. Then $M \in \mathfrak{M}_R^{lc} = \mathfrak{M}_R^{fg}$ by (19.68) and (19.65).

(3)$\Longrightarrow$(2) follows from the fact that $E(V_i)$ is always f.cog. (cf. Exercise 7). □

In the case when $(R, \mathfrak{m})$ is a *commutative local* artinian ring, we have $U^0 = E(R/\mathfrak{m})$ and $\operatorname{End}(U_R^0) \cong R$. Here, of course, we get back the special case of the Matlis duality given in (19.56).

We mention without proof another interesting class of right artinian rings admitting Morita duality, due to K. Fuller.

(19.75) Example. Let A be a right artinian ring, and $e \in A$ be an idempotent such that eA is injective (as well as projective). *Then the right artinian ring eAe* (see *FC*–(21.13)) *admits a Morita duality.* In fact, there exists a suitable idempotent $f \in A$ such that, for the rings $R = eAe$, $S = fAf$, the bimodule $U = {}_S(fAe)_R$ defines a Morita duality from R to S. The details of the verification can be found in Fuller [69]; see also Xue [92]. This example applies well, for instance, to *any* idempotent e in a QF ring A.

We have seen, from Exercise (3.34), that there exist 2-sided artinian rings R for which some $E(V_i)$ is *not* f.g., in the notation of (19.74) (although $E(V_i)$ is always f.cog.). By (19.74), therefore, such artinian rings R have *no* Morita duality into any ring whatsoever.

In the case of *commutative* artinian rings, however, the situation is much more amenable, as the following result shows.

(19.76) Corollary. *Let R be a commutative artinian ring. Then R has a Morita self-duality.*

Proof. Recall from the proof of (2) $\Longrightarrow$ (3) in (15.27) that $R \cong R_1 \times \cdots \times R_n$, where the R_i 's are (commutative) artinian local rings. As we have observed in the paragraph following the proof of (19.74), each R_i has a Morita duality into itself. Therefore, the same follows for R. $\square$

The above result and its proof suggest that the theory of Morita duality for commutative rings should be considerably simpler than the corresponding theory for general (semiperfect) rings. Indeed, in the case of commutative rings, it turns out that Müller's Second Theorem can be further simplified. We mention without proof the following recent result of Ánh [90], which affirms earlier conjectures of Zelinsky, Müller, and Vámos.

(19.77) Theorem. *If R is a commutative ring, then R admits a Morita duality iff R_R is l.c. In this case, a suitable bimodule ${}_RU_R$ defines a Morita self-duality from R to R.*

§19G. Morita Dualities: Morita II

To conclude our discussion of duality theory, we shall prove in this subsection a theorem of Morita which shows that, for suitable subcategories $\mathcal{A} \subseteq \mathfrak{M}_R$, $\mathcal{B} \subseteq {}_S\mathfrak{M}$ (for two rings R, S), any duality between $\mathcal{A}$, $\mathcal{B}$ is essentially a Morita duality in the sense of Definition (19.41). We shall call this theorem "Morita II" again, in analogy with the corresponding theorem (18.26) classifying equivalences of module categories. In retrospect, this new "Morita II" gives the ultimate justification for the earlier emphasis we placed on Morita dualities in §§19D–F. Our proof of "Morita II" given below is a somewhat more informal version of the one given in Anderson-Fuller [92: pp. 273–275].

(19.78) Theorem (Morita II for Dualities). *Let $\mathcal{A} \subseteq \mathfrak{M}_R$, $\mathcal{B} \subseteq {}_S\mathfrak{M}$ be (full) subcategories of $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$ with the following properties:*

(i) *$R_R \in \mathcal{A}$ and ${}_SS \in \mathcal{B}$.*
(ii) *$\mathcal{A}$, $\mathcal{B}$ are closed under quotients. More precisely, if A'_R is isomorphic to a quotient of some $A \in \mathcal{A}$, then $A' \in \mathcal{A}$ (and the same for $\mathcal{B}$).*

Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be a duality between $\mathcal{A}$ and $\mathcal{B}$ (in the sense of (19.26)). Then there exists a bimodule ${}_SU_R$ such that

(1) *$F \cong \mathrm{Hom}_R(-, U)$ and $G \cong \mathrm{Hom}_S(-, U)$.*
(2) *$\mathcal{A} \subseteq \mathfrak{M}_R[U]$ and $\mathcal{B} \subseteq {}_S\mathfrak{M}[U]$.*
(3) *U defines a Morita duality from R to S (in the sense of (19.41)).*

Proof. There is a considerable amount of formal details in this proof. In order to better focus on the key ideas, we shall present the main steps in the proof, skipping

some of the formal verifications (such as proving that certain isomorphisms are natural).

Let $U = F(R_R) \in \mathcal{B}$ and $V = G({}_SS) \in \mathcal{A}$. For any $r \in R$, left multiplication by r is an endomorphism of R_R, so it gives rise to an endomorphism of ${}_SU$ (written on the right of ${}_SU$). This makes U an (S, R)-bimodule, and similarly, V is an (S, R)-bimodule. In view of the duality given by F, G, we have, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

(19.79) $\quad \mathrm{Hom}_S({}_SS_S,\ F(A)) \cong \mathrm{Hom}_R(A,\ {}_SG(S)_R)$ (left S-isomorphism),

(19.80) $\quad \mathrm{Hom}_R({}_RR_R,\ G(B)) \cong \mathrm{Hom}_S(B,\ {}_SF(R)_R)$ (right R-isomorphism).

Applying the former to $A = R_R$, we have:

(19.81) $$U \cong \mathrm{Hom}_S(S,\ U) \cong \mathrm{Hom}_R(R,\ V) \cong V.$$

This is a left S-isomorphism by (19.79), but the "naturality" of the isomorphism in A in (19.79) can be used to show that (19.81) is an (S, R)-isomorphism. Therefore, for any $A \in \mathcal{A}$:

$$_S(F(A)) \cong \mathrm{Hom}_S({}_SS_S,\ F(A)) \cong \mathrm{Hom}_R(A,\ {}_SV_R) \cong \mathrm{Hom}_R(A,\ {}_SU_R).$$

After checking that this S-isomorphism is natural in A, we see that $F \cong \mathrm{Hom}_R(-,\ {}_SU_R)$, and similarly, using (19.80), we can show that $G \cong \mathrm{Hom}_S(-,\ {}_SU_R)$. This gives (1) in the theorem.

From now on, we can think of F and G as given by the "U-dual" functors. In particular, we shall use the star notation (cf. §19D) for F and G whenever it is more convenient to do so. However, we only know from the given hypotheses that there *are* natural equivalences

(19.82) $$\varphi : 1_{\mathcal{A}} \to G \circ F \quad \text{and} \quad \psi : 1_{\mathcal{B}} \to F \circ G.$$

In order to prove (2), we still have to show that, for $B \in \mathcal{B}$, the canonical "hat" map (cf. §19D):

(19.83) $$\theta_B : B \to B^{**} = FG(B), \quad b \mapsto \hat{b}$$

given by $\hat{b}(g) = g(b)$ ($b \in B$, $g \in B^*$) is an isomorphism (and similarly for $A \in \mathcal{A}$). Since B is the U-dual of some R-module by (19.82), the injectivity of θ_B follows from (19.38). The proof of the *surjectivity* of θ_B is, however, nontrivial, and requires careful work.

We first try to define an R-homomorphism $\gamma : B^* \to B^*$, as follows. By (19.82), we know there exists *some* natural S-homomorphism $\psi_B : B \to B^{**}$. Let us write $\tilde{b}$ for the image of $b \in B$ under this isomorphism. For $f \in B^*$, we then define $\gamma(f)$ by

(19.84) $$\gamma(f)(b) = \tilde{b}(f) \in U.$$

It is easy to check that $\gamma(f) \in B^*$, and that $\gamma : B^* \to B^*$ is an R-homomorphism. The crucial claim is that

(19.85) *γ is a surjective map.*

Assuming this claim, the proof for the surjectivity of θ_B can be given as follows. For any $\sigma \in B^{**}$, consider the composition

$$\sigma' = \sigma \circ \gamma : \ B^* \xrightarrow{\ \gamma\ } B^* \xrightarrow{\ \sigma\ } U.$$

Since $\sigma' \in B^{**}$, $\sigma' = \tilde{b}$ for some $b \in B$. For any $g \in B^*$, find $f \in B^*$ such that $\gamma(f) = g$ (by (19.84)). Then

$$\sigma(g) = \sigma(\gamma(f)) = \tilde{b}(f) = \gamma(f)(b) = g(b) = \hat{b}(g),$$

so $\sigma = \hat{b}$, as desired.

To complete the proof, *we must show how to construct the f above, for a given* $g \in B^*$. First note that, since $U = {}_SU_R$, $U_R^* = \operatorname{Hom}_R(U_R, U_R)$ is an (S, S)-bimodule, so $\operatorname{Hom}_S(U_R^*, B)$ is a left S-module. We construct an S-isomorphism $\tau_B : B \to \operatorname{Hom}_S(U_R^*, B)$ so as to make the following diagram commutative:

(19.86)
$$\begin{array}{ccc} B & \overset{\tau_B}{\dashrightarrow} & \operatorname{Hom}_S(U_R^*, B) \\ {\scriptstyle \psi_B}\downarrow & & \downarrow{\scriptstyle \cong} \\ B^{**} = \operatorname{Hom}_R(B^*, U_R) & \xrightarrow{\ \cong\ } & \operatorname{Hom}_R(B^*, U_R^{**}) \end{array}$$

Here, the isomorphism on the right comes from the duality pair (F, G), and the bottom isomorphism is induced by $\varphi_U : U_R \cong U_R^{**}$. Assuming the fact that τ_B is *natural* in B, we see that:

(19.87) *The functor* $\operatorname{Hom}_S(U_R^*, -) : \mathcal{B} \to \mathcal{B}$ *is naturally equivalent to* $1_{\mathcal{B}}$.

Given $g \in B^*$, there is a unique S-homomorphism h making the following diagram commutative:

(19.88)
$$\begin{array}{ccc} b \in B & \underset{\tau_B}{\overset{\cong}{\longrightarrow}} & \operatorname{Hom}_S(U_R^*, {}_SB) \ni \beta \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ U & \underset{\varphi_U}{\overset{\cong}{\longrightarrow}} & U_R^{**} = \operatorname{Hom}_S(U_R^*, {}_SU) \end{array}$$

(Here, we need the fact that φ_U is not only an R-isomorphism, but also an S-isomorphism: this follows from the naturality of φ.) Let $b \in B$ and $\beta = \tau_B(b)$, as in (19.88). By (19.87), the map h induced by a unique $f : {}_SB \to {}_SU$. Thus, $\varphi_U(g(b)) = h(\beta) = f \circ \beta$. But by the commutativity of (19.86), $\varphi_U \circ \tilde{b} = \beta^*$ (recall that $\tilde{b}$ is by definition $\psi_B(b)$). More explicitly, we have the commutative diagram

$$\begin{array}{ccc} B^* & \xrightarrow{\ \tilde{b}\ } & U \\ & {\scriptstyle \beta^*}\searrow & \downarrow{\scriptstyle \varphi_U} \\ & & U^{**} \end{array}$$

For $f \in B^*$, this gives

$$\varphi_U(\tilde{b}(f)) = \beta^*(f) = f \circ \beta = \varphi_U(g(b)).$$

Thus, $g(b) = \tilde{b}(f) = \gamma(f)(b)$ (cf. (19.84)), and we have $g = \gamma(f)$, proving (19.85). This completes the proof of $\mathcal{B} \subseteq {}_S\mathfrak{M}[U]$, and $\mathcal{A} \subseteq \mathfrak{M}_R[U]$ follows from symmetry. Since R_R, $U_R \in \mathcal{A}$ and ${}_SS$, ${}_SU \in \mathcal{B}$, and $\mathcal{A}$, $\mathcal{B}$ are closed under quotients, we see that the quotients of the above four modules are all U-reflexive. Therefore, "Morita I" (Theorem (19.43)) implies that U defines a Morita duality from R to S. □

Exercises for §19

0. Prove (19.4): a module M_R is artinian iff every quotient of M is f.cog.

1. Give a direct proof for the equivalence of the conditions (1) and (2) in (19.1), without using the notion of the socle.

2. Let $N \subseteq M$ be R-modules. If M is f.cog., it is clear that N is also f.cog. Show that the converse holds if $N \subseteq_e M$.

3. For any module M, show that the following are equivalent:

   (1) M is semisimple and f.g.;
   (2) M is semisimple and f.cog.;
   (3) rad $M = 0$ and M is f.cog.

   Show that these statements imply, but are *not* equivalent to:

   (4) rad $M = 0$ and M is f.g.

4. *True or False*: For any exact sequence $0 \to K \to L \to M \to 0$, if K, M are f.cog., so is L ?

5. Show that $M = M_1 \oplus \cdots \oplus M_n$ is f.cog. iff each M_i is.

6. Show that any f.cog. module has finite uniform dimension. Is the converse true, at least for injective modules?

7. (Vámos) For any module M_R, show that the following are equivalent:

   (1) M_R is f.cog.;
   (2) $E(M) \cong E(V_1) \oplus \cdots \oplus E(V_n)$ for suitable simple modules $V_1, \ldots, V_n$;
   (3) $M \subseteq E(V_1) \oplus \cdots \oplus E(V_r)$ for suitable simple modules $V_1, \ldots, V_r$.

8. (Matlis) For any commutative noetherian ring R, show that a module M_R is artinian iff it is f.cog. Your proof should work as long as R has the property that its localizations at maximal ideals are all noetherian. (**Hint.** Reduce to the case when R is local, noetherian, and complete, and use (19.56).)

9. A module M_R is said to be *cofaithful* if R_R embeds into M^n for some $n < \infty$.

(1) Show that a cofaithful module M_R is always faithful.
(2) If R is commutative, show that a f.g. M_R is faithful iff it is cofaithful.
(3) (Beachy) Show that R_R is f.cog. iff all faithful right R-modules are cofaithful. (In particular, the latter condition holds over any right artinian ring, and any cogenerator ring.)

9′. If R is a semiprime ring such that R_R is f.cog., show that R is a semisimple ring.

10. For any ring R, $R' := \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is an (R, R)-bimodule. Show that $(R')_R$ is an injective cogenerator in $\mathfrak{M}_R$ (and similarly for ${}_R(R')$).

11. Show that $R = \prod_{j=1}^{n} R_j$ is a cogenerator ring iff each R_j is.

12. Give another proof for the fact that any QF ring R is a cogenerator ring, without using (15.11)(1). **(Hint.** Recall Exercise (15.13)(1) and use (19.8).)

13. Show that over $R = \mathbb{Z}$, a module U_R is a cogenerator iff every nonzero M_R admits a nonzero homomorphism into U. **(Hint.** Every nonzero epimorphic image of the Prüfer group C_{p^∞} is isomorphic to C_{p^∞}.)

14. Prove the "only if" part of (19.22).

15. Keep the notations in (19.22), and assume that ${}_SM$ and M_S are faithful, $M = E(V)$ for some module V_S, and that the map $S \to \mathrm{End}(M_S)$ is onto. Show that, upon viewing V as a right ideal in $R = S \oplus M$, we have $R = E(V_R)$. (A special case of this appeared in the arguments in (19.24).)

16A. Suppose R_R is a cogenerator.
(1) If R has no nontrivial idempotents, show that R is a right self-injective local ring.
(2) If R is a domain, show that R is a division ring.

16B. For any ring R, show that $E(R_R)$ is an (injective) cogenerator iff R is a right Kasch ring.

16C. (Kato) For any right self-injective ring R, show that the following are equivalent:
(1) R_R is a cogenerator;
(2) R is right Kasch;
(3) $\mathrm{ann}_r(\mathrm{ann}_\ell(A)) = A$ for any right ideal $A \subseteq R$.
(Recall that, for a right self-injective ring R, these (and other) equivalent conditions define the notion of a right PF ring: see (19.25).)

16D. Let R be a commutative ring that is subdirectly irreducible; i.e., R has a smallest nonzero ideal. If R is self-injective, show that R is a local cogenerator ring.

The next four exercises purport to show that "R_R being injective" and "R_R being a cogenerator" are, in general, independent conditions. (These two

conditions together define the notion of a right PF ring.) Exercises 18-20 are adapted from Osofsky [66]; see also Kasch [82: pp. 323-333].

17. Let $R = \prod_{j \in J} A_j$, where the A_j 's are division rings, and J is infinite. By (3.11B), R_R is injective. Without assuming (19.18), show that R_R is not a cogenerator, and that R is not right Kasch.

18. Let k be a field, and S be the commutative k-algebra $k \oplus \bigoplus_{i \geq 1} ke_i$ constructed in Exercise (8.17) (with $e_i e_j = \delta_{ij} e_i$). Let $V_i = kv_i$ $(i \geq 0)$ be the simple right S-modules constructed in that exercise, with the S-action

$$v_i\, e_j = \delta_{ij}\, v_i \quad (i \geq 0,\ j \geq 1).$$

Let M be the right S-module $\bigoplus_{i \geq 0} V_i = kv_0 \oplus kv_1 \oplus \cdots$, and define a left S-action on M by

$$e_j\, v_i = \delta_{j-1,i}\, v_i \quad (i \geq 0,\ j \geq 1).$$

(1) Check that, under the above actions, M is an (S, S)-bimodule, faithful on both sides.
(2) Let $R = S \oplus M$ be the trivial extension of ${}_S M_S$ by S. Show that R is right Kasch but not left Kasch.
(3) Use (19.22) to show that R is neither right nor left self-injective.

19. Keep the above notations and work in the ring R. Show that:
(1) $e_i R = ke_i + V_{i-1}$ $(i \geq 1)$, with $e_i R / V_{i-1} \cong V_i$ as right R-modules;
(2) $\mathrm{soc}(e_i R) = V_{i-1} \subseteq_e e_i R$;
(3) For any right ideal $A \subseteq R$ and any $i, j \geq 1$, show that any R-homomorphism $f : A \longrightarrow e_i R$ can be extended to an R-homomorphism $g : A + e_j R \longrightarrow e_i R$;
(4) Using (3) and Baer's Criterion, show that $e_i R = E((V_{i-1})_R)$ for any $i \geq 1$.

20. For the ring R in the last two exercises, show that:
(1) R_R is a cogenerator, but ${}_R R$ is not a cogenerator;
(2) R is neither right nor left PF.

21. Let U_R be the minimal injective cogenerator over a commutative ring R. View U as a bimodule ${}_R U_R$ and let $F : \mathfrak{M}_R \to {}_R\mathfrak{M}$, $G : {}_R\mathfrak{M} \to \mathfrak{M}_R$ be the U-dual functors, denoted as usual by $*$. Show that for any simple V_R, V^* is a simple (left) R-module. Using this, show that F, G define a self-duality on the Serre subcategory of R-modules of finite length. (This generalizes the usual self-duality of finite abelian groups noted in (19.29). However, ${}_R U_R$ may not define a Morita duality since ${}_R U_R$ may not be faithfully balanced.)

22. Let U be an (S, R)-bimodule, and let $*$ denote the U-dual as usual. Let $P_R = \bigoplus_{i \in I} P_i$. We identify P^* with $\prod P_i^*$, and $(\bigoplus P_i^*)^*$ with $\prod P_i^{**}$. Let $\varepsilon : \bigoplus P_i^* \to \prod P_i^*$ be the inclusion map.

(1) Show that $P \in \mathfrak{M}_R[U]$ iff each $P_i \in \mathfrak{M}_R[U]$, ε^* is injective, and $\mathrm{im}(\varepsilon^*) = \bigoplus P_i^{**}$ (in $\prod P_i^{**}$).

(2) Show that ε^* is injective iff $\left(\prod P_i^* / \bigoplus P_i^*\right)^* = 0$.

(3) Show that, if ${}_SU$ is injective or a cogenerator, then $P \in \mathfrak{M}_R[U]$ implies that $P_i = 0$ for almost all i.

(The theme of this exercise is due to B. Osofsky. The formulation above was suggested to me by I. Emmanouil; see also Kasch [82: p. 329], for the case $U = {}_RR_R$.)

23. Show that the hypothesis on U in (3) of the exercise above cannot be dropped. (**Hint.** Recall Exercise (2.8′).)

24. Suppose ${}_SU_R$ defines a Morita duality from R to S. Show that any $M \in \mathfrak{M}_R^{fg}$ has a projective cover (in the sense of FC–(24.9)). In view of FC–(24.16), this yields another proof for the fact that R is semiperfect.) (**Hint.** Deduce the existence of projective cover for $M \in \mathfrak{M}_R^{fg}$ from the existence of injective hull for $M^* \in {}_S\mathfrak{M}[U]$.)

25. Suppose ${}_SU_R$ defines a Morita duality from R to S. If R is right artinian, show that S is left artinian, and U gives a duality between $\mathfrak{M}_R^{fg} = \mathfrak{M}_R[U]$ and ${}_S^{fg}\mathfrak{M} = {}_S\mathfrak{M}[U]$. (**Hint.** Show that ${}_SN$ f.g. $\Longrightarrow$ ${}_SN$ f.cog.; then use (19.4).)

26. Suppose ${}_SU_R$ defines a Morita duality from R to S. If R is QF, show that S is also QF.

27. (Partial converse to (19.64).) Let R be a ring such that every nonzero right R-module has a simple submodule. Show that if N_R is l.c., then it is artinian.

28. Let R be a *left* perfect ring (i.e., $R/\mathrm{rad}(R)$ is semisimple, and for any sequence $\{a_1, a_2, \dots\} \subseteq \mathrm{rad}(R)$, $a_1a_2\cdots a_n = 0$ for some n).

(1) Show that every nonzero right R-module has a simple submodule.

(2) For any M_R, show that $\mathrm{soc}(M) \subseteq_e M$.

(3) (Generalization of (19.65)) Deduce from the last exercise that a right R-module is l.c. iff it is artinian.

29. Let R be a *right* perfect ring (i.e., $R/\mathrm{rad}(R)$ is semisimple, and for any sequence $\{a_1, a_2, \dots\} \subseteq \mathrm{rad}(R)$, $a_n \cdots a_2a_1 = 0$ for some n). Show that if a module M_R is l.c., then it is noetherian. Deduce that, over a perfect ring, a (left or right) module is l.c. iff it has finite length.

30. (Osofsky, Sandomierski) Let R be a right l.c. ring (e.g., any ring R that admits a Morita duality into some other ring S). If R is 1-sided perfect, show that R must be right artinian.

31. (Osofsky) Let R be a cogenerator ring. If R is 1-sided perfect, show that R is QF.

32. Suppose ${}_SU_R$ defines a Morita duality from R to S. Assuming Onodera's 1-sided characterization of cogenerator rings (19.69), show that if R is a cogenerator ring, then so is S.

33. (Leptin) Let $N \subseteq M$ be R-modules where N is l.c. Let $\{A_i : i \in I\}$ be an inverse system of submodules in M, in the sense that, for any finite $J \subseteq I$, there exists $j \in I$ such that $A_j \subseteq A_i$ for all $i \in J$. Show that $\bigcap_{i \in I}(N + A_i) = N + \bigcap_{i \in I} A_i$.

34. (Sandomierski) Let $N \subseteq M$ be R-modules where N is l.c. Show that the family $\mathcal{F}$ of submodules $A \subseteq M$ such that $N + A = M$ has a minimal member. (Such a minimal member is called an *addition complement* of N in M.) (**Hint.** Zorn's Lemma applies to $\mathcal{F}$ thanks to the last exercise.)

35. The proof for half of Zelinsky's result (19.58) was left out of the text. Supply this missing proof.

36. (Essentially Grothendieck) Let K/k be a finite Galois field extension with Galois group G. Let $\mathcal{B}$ be the category of finite G-sets, and $\mathcal{A}$ be the category of finite-dimensional commutative étale k-algebras that are split over K (i.e. algebras A such that $A \otimes_k K \cong K \times \cdots \times K$). Show that there are natural contravariant functors $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{A}$ defining a duality between $\mathcal{A}$ and $\mathcal{B}$, such that the transitive G-sets in $\mathcal{B}$ correspond to field extensions of k which are embeddable into K. (**Hint.** The field K, being both a k-étale algebra and a G-set, plays the role of U in this duality. For $A \in \mathcal{A}$ and $B \in \mathcal{B}$, define $F(A) = A^* = \mathrm{Hom}_{k\text{-alg}}(A, K) \in \mathcal{B}$, and $F'(B) = B^* = \mathrm{Hom}_G(B, K) \in \mathcal{A}$. Then verify that $A^{**} \cong A$ and $B^{**} \cong B$. There is a profinite version of this duality also.)

References

(This list of references contains only items explicitly referred to in the text. It is not a guide to the literature in ring theory. Readers interested in further reading in the subject should consult the exhaustive list of books on ring theory in the reference section of Rowen's two-volume work listed below. Those interested in the general literature in ring theory will be amply rewarded by consulting L. Small's compilation [Small: 81, 86] of reviews of ring theory papers appearing in the *Mathematical Reviews* before 1984.)

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